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REMARK ON THE TAUTOLOGIES OF THE INTUITIONISTIC FUZZY LOGICS

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ABSTRACT: Some assertions of the intuitionistic fuzzy logic are proved and the necesity of an axiomatic of this logic is discussed.

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The below remark was generated from the Esenin-Volpin's paper [1]. There the axioms $A \& \neg A \supset B$ and $A \vee \neg A \supset (\neg\neg A \supset A)$ are discussed and the author asserts that the second one is (in some sense) weaker that the first one. In the frameworks of the standard (first order) logic, both expressions are tautologies.

First, using the definitions of the Intuitionistic Fuzzy Logic (IFL) [2-5], we shall show that these expressions are Intuitionistic Fuzzy Tautologies (IFT), but the first of them is not a tautology; only the second one is a tautology. After this, we shall discuss some problems of the IFL.

Let S be the set of propositional forms (c.f., [6]: each proposition is a propositional form; if A is a propositional form then $\neg A$ is a propositional form; if A and B are propositional forms, then $A \& B$, $A \vee B$, $A \supset B$ are propositional forms) and let $V: S \to [0, 1] \times [0, 1]$, be defined for every $p \in S$ as $V(p) = \langle \mu(p), \gamma(p) \rangle$, where $\mu(p)$ and $\gamma(p)$ are the degrees of validity and non-validity of p, respectively, and let $\mu(p) + \gamma(p) \le 1$ (see [2]). The evaluation of the negation $\neg p$ of the proposition p will be defined through: $V(\neg p) = \langle \gamma(p), \mu(p) \rangle$. When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended also for the operations "&", "$\vee$" and "$\supset$" through the definition:

$$V(p \& q) = \langle \min(\mu(p), \mu(q)), \max(\gamma(p), \gamma(q)) \rangle,$$
$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\gamma(p), \gamma(q)) \rangle,$$
$$V(p \supset q) = \langle \max(\gamma(p), \mu(q)), \min(\mu(p), \gamma(q)) \rangle.$$

(a first version of the implication called (max-min)-implication)

$V(p \supset q) = \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)),\ \gamma(q).sg(\mu(p) - \mu(q)).$
$sg(\gamma(q) - \gamma(p))\rangle,$

where:

$$sg(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0. \end{cases}$$

(a second version of the implication called sg-implication)

By analogy with the operations over IFSs it will be convenient to define for the propositions $p, q \in S$:

$$\neg V(p) = V(\neg p),$$
$$V(p) \wedge V(q) = V(p \,\&\, q),$$
$$V(p) \vee V(q) = V(p \veebar q),$$
$$V(p) \rightarrow V(q) = V(p \supset q).$$

(for both variants of the implication).

Let everywhere below $p = q$ iff $V(p) = V(q)$.

The propositional form A is an tautology iff $V(A) = \langle 1, 0\rangle$ and it is an IFT iff if $V(A) = \langle a, b\rangle$, then $a \ge b$.

For the first expression we can see, that when $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$ and $a > b > c$, then

$V(A \,\&\, \neg A \supset B) = (\langle a, b\rangle \rightarrow \langle b, a\rangle) \rightarrow \langle c, d\rangle$

$= \langle 1 - (1 - b).sg(a - b),\ a.sg(a - b).sg(a - b)\rangle \rightarrow \langle c, d\rangle$

(from the obvious equality $sg(x) = sg(x)^n$ for $n \ge 1$)

$= \langle 1 - (1 - b).sg(a - b),\ a.sg(a - b)\rangle \rightarrow \langle c, d\rangle$

$= \langle 1 - (1 - c).sg(1 - (1 - b).sg(a - b) - c),\ d.sg(1 - (1 - b).$
$sg(a - b) - c).sg(d - a.sg(a - b))\rangle$

(from $a > b$)

$= \langle 1 - (1 - c).sg(b - c),\ d.sg(b - c).sg(d - a)\rangle$

(from $b > c$)

$= \langle c,\ d.sg(d - a)\rangle.$

From $1 \ge a > b > c$ it follows that $A \,\&\, \neg A \supset B$ is not a tautology. If $1 \ge d > a$, then $V(A \,\&\, \neg A \supset B) = \langle c, d\rangle = V(B)$. In the opposite case, $V(A \,\&\, \neg A \supset B) = \langle c, 0\rangle$.

If $a \le b$ and $c < 1$, we can see directly that

$V(A \,\&\, \neg A \supset B)$

$= \langle 1 - (1 - c).sg(1 - (1 - b).sg(a - b) - c),\ d.sg(1 - (1 - b).$
$sg(a - b) - c).sg(d - a.sg(a - b))\rangle$

$= \langle 1 - (1 - c).sg(1 - c),\ 0\rangle = \langle 1, 0\rangle,$

i.e., $A \,\&\, \neg A \supset B$ can be a truth.

<u>LEMMA 1</u>: For every propositional form A

$A \veebar \neg A \supset (\neg\neg A \supset A)$

is a tautology for a sg-implication.

Proof: Let $V(A) = \langle a, b\rangle$. Then

$V(A \veebar \neg A \supset (\neg\neg A \supset A))$

$= (\langle a, b\rangle \vee \langle b, a\rangle) \rightarrow (\langle a, b\rangle \rightarrow \langle a, b\rangle)$

$= \langle \max(a, b), \min(a, b)\rangle \rightarrow \langle 1 - (1 - a).sg(a - a), b.sg(a - a).sg(b - b)\rangle$

$= \langle \max(a, b), \min(a, b)\rangle \rightarrow \langle 1, 0\rangle$

$= \langle 1 - (1 - 1).sg(\max(a, b) - 1), 0.sg(\max(a, b) - 1).sg(0 - \min(a, b))\rangle$

$= \langle 1, 0\rangle$.

<u>LEMMA 2</u>: For every two propositional forms A and B:

(a) $A \,\&\, \neg A \supset B$

(b) $A \veebar \neg A \supset (\neg\neg A \supset A)$

are IFTs for a (max-min)-implication.

Proof: Let again $V(A) = \langle a, b\rangle$ and $V(B) = \langle c, d\rangle$.

(a) $V(A \,\&\, \neg A \supset B) = (\langle a, b\rangle \wedge \langle b, a\rangle) \rightarrow \langle c, d\rangle$

$= \langle \min(a, b), \max(a, b)\rangle \rightarrow \langle c, d\rangle$

$= \langle \max(a, b, c), \min(a, b, d)\rangle$.

Obviously, $\max(a, b, c) - \min(a, b, d) \geq 0$, i.e., (a) is an IFT.

(b) $V(A \veebar \neg A \supset (\neg\neg A \supset A))$

$= (\langle a, b\rangle \vee \langle b, a\rangle) \rightarrow (\langle a, b\rangle \rightarrow \langle a, b\rangle)$

$= \langle \max(a, b), \min(a, b)\rangle \rightarrow \langle \max(a, b), \min(a, b)\rangle$

$= \langle \max(a, b, \min(a, b)), \min(a, b), \max(a, b))\rangle$

Obviously, $\max(a, b, \min(a, b)) - \min(a, b), \max(a, b)) \geq 0$, i.e., (b) is an IFT.

From the above lemmas we see that in the frameworks of the IFL there are situations for which the Esenin-Volpin's assertion is valid - the second expression is a tautology while for the first one this is not always true.

Very interesting is the question for the double negation. In the first order logic $\neg\neg A$ coincides with A for every propositional form. The same is valid in the IFL, but in the ordinary intuitionistic logic this is not valid.

In the present moment the question for an axiomatic system of the IFL is open. IFL can use the most part of the first order logic axioms. They are valid as tautologies for the sg-implication, or as IFTs for the (max-min)-implication (see [2, 4, 5]), but there are no special axioms for the modal-type of IFL operators. On the other hand, the axioms of the standard modal logics (see [7]) are valid as theorems in the frameworks of the IFL. Therefore, in future it is necessary to construct axiomatic system for the IFL which to include special axioms related to the new modal-type of operators (D, F, G, H, H^*, J, J^*). Some of the first order axioms must be changed with axioms, related some of the above operators. For example, the axiom $\neg\neg A \supset A$ (see [6]) can be changed with the the axiom $H_{\alpha,\beta}(A) \supset A$, or with the axiom $H^*_{\alpha,\beta}(A) \supset A$ for every $\alpha \le 1$ and for every $\beta \ge 0$. We must note that $V(H^*_{1,0}(A) = V(A) = V(\neg\neg A)$ and $V(H^*_{0,1}(A) = \langle 0, 1\rangle \equiv$ "false".

Easily we can see that this new axiom (in every one of its two above forms) can be checked in the frameworks of the IFL and it will be as a tautology, as well as an IFT. This axiom can be included in the axioms of IF modal logic (cf. [3]). We must note that in the frames of the ordinary fuzzy logic there are some axiomatic systems (see [8-10]), but they cannot reflect the model-type of the operators of the IFL. These axiomatic systems can be used as a basis of the axiomatic system of the IFL (in general). It is possible, the IFL, in the part of the propositional calculus can use the ordinary first o rder axiomatic system, but in its parts, related to extensions, like modal, temporal and other IFLs, it is needed with a special axiomatic system.

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