Twelfth Int. Conf. on IFSs, Sofia, 17-18 May 2008
NIFS Vol. 14 (2008), 1, 33-37

Operation ÷ over Intuitionistic Fuzzy Sets

Anton Cholakov
Student of Sofia University, Bulgaria

A lot of operations have been defined over Intuitionistic Fuzzy Sets (see [1]). In this article we will define one new operation.

(Throughout this article we will mark the class of all IFSs with *IFS* (in italic))

Definition 1: If we have an universe E and two IFSs over it $A = \{(x, \mu_A(x), \nu_A(x)) | x \in E\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) | x \in E\}$, then we will assign the set

$$C = \{(x, \mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x), \mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)) \mid x \in E\}$$

as a result of the operation between A and B:

$$A \div B = C.$$

Proposition 1: $A \div B$ is an IFS.

Proof: We will prove that the sum of the membership and non-membership of the result is not greater than 1.

$$(\mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x)) + (\mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)) =$$
$$= \mu_A(x)\mu_B(x) + \mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x) =$$
$$= \mu_A(x)(\mu_B(x) + \nu_B(x)) + \nu_A(x)(\mu_B(x) + \nu_B(x)) =$$
$$= (\mu_A(x) + \nu_A(x))(\mu_B(x) + \nu_B(x))$$

From $A, B \in IFS$ follows that $\mu_A(x) + \nu_A(x) \le 1$ and $\mu_B(x) + \nu_B(x) \le 1$.

Therefore $(\mu_A(x) + \nu_A(x))(\mu_B(x) + \nu_B(x)) \le 1$ so $A \div B$ is an IFS.

We will now examine some of the properties of the operation:

Proposition 2: Operation ÷ is commutative.

Proof:

$$A \div B = \{(x, \mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x), \mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)) \mid x \in E\},$$
$$B \div A = \{(x, \mu_B(x)\nu_A(x) + \nu_B(x)\mu_A(x), \mu_B(x)\mu_A(x) + \nu_B(x)\nu_A(x)) \mid x \in E\}.$$

Therefore $A \div B = B \div A$.

Proposition 3: Operation ÷ is associative.

Proof:

$$(A \div B) \div C = \{(x, \mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x), \mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)) \mid x \in E\} \div C =$$
$$= \{(x, \mu_A(x)\nu_B(x)\nu_C(x) + \nu_A(x)\mu_B(x)\nu_C(x) + \mu_A(x)\mu_B(x)\mu_C(x) + \nu_A(x)\nu_B(x)\mu_C(x),$$
$$\mu_A(x)\nu_B(x)\mu_C(x) + \nu_A(x)\mu_B(x)\mu_C(x) + \mu_A(x)\mu_B(x)\nu_C(x) + \nu_A(x)\nu_B(x)\nu_C(x)) \mid x \in E\},$$

$$A \div (B \div C) = A \div \{(x, \mu_B(x)\nu_C(x) + \nu_B(x)\mu_C(x), \mu_B(x)\mu_C(x) + \nu_B(x)\nu_C(x)) \mid x \in E\} =$$
$$= \{(x, \mu_A(x)\mu_B(x)\mu_C(x) + \mu_A(x)\nu_B(x)\nu_C(x) + \nu_A(x)\mu_B(x)\nu_C(x) + \nu_A(x)\nu_B(x)\mu_C(x),$$
$$\mu_A(x)\mu_B(x)\nu_C(x) + \mu_A(x)\nu_B(x)\mu_C(x) + \nu_A(x)\mu_B(x)\mu_C(x) + \nu_A(x)\nu_B(x)\nu_C(x)) \mid x \in E\}.$$

Therefore $(A \div B) \div C = A \div (B \div C)$.

Theorem 1: According to the set of all IFSs (or *IFS*) the operation ÷ forms a monoid.

Proof: From Proposition 1 it follows that (*IFS*, ÷) is a groupoid. From Proposition 2 it follows that (*IFS*, ÷) is a semi-group. From Propostion 2 it follows that ÷ is associative.
We will show the existence of a neutral element:
Let us suppose that X is a neutral element, i.e.

$$A \div X = A.$$

Therefore
$\{(x, \mu_A(x)\nu_X(x) + \nu_A(x)\mu_X(x), \mu_A(x)\mu_X(x) + \nu_A(x)\nu_X(x)) \mid x \in E\} = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\}$
and we have the following system of equations:

$$\begin{cases} \mu_A(x)\nu_X(x) + \nu_A(x)\mu_X(x) = \mu_A(x) \\ \mu_A(x)\mu_X(x) + \nu_A(x)\nu_X(x) = \nu_A(x) \end{cases}$$

1st case: $\mu_A(x) = 0$. Hence
$\nu_A(x)\nu_X(x) = \nu_A(x)$ i.e. $\nu_X(x) = 1$ and $\mu_X(x) = 0$.

Therefore the neutral element $X = \{(x, 0, 1) \mid x \in E\} \equiv \overline{0}$ (of [1]).

2nd case: $\mu_A(x) \neq 0$. Hence

$$\mu_X(x) = \frac{\nu_A(x)(1-\nu_A(x))}{\mu_A(x)},$$

$$\mu_A(x)\nu_X(x) + \nu_A(x)\frac{\nu_A(x)(1-\nu_A(x))}{\mu_A(x)} = \mu_A(x) \mid .\ \mu_A(x),$$

$$\mu_A^2(x)\nu_X(x) + \nu_A^2(x) - \nu_A^2(x)\nu_X(x) = \mu_A^2(x),$$

$$\nu_X(x)(\mu_A^2(x) - \nu_A^2(x)) = \mu_A^2(x) - \nu_A^2(x),$$

$$\nu_X(x) = 1 \text{ and } \mu_X(x) = 0.$$

Hence the neutral element is again the above $X = \{(x, 0, 1) \mid x \in E\} \equiv \overline{0}$, i.e. (*IFS*, ÷) is a monoid.

Now we will show that there is no opposite element and thus so ÷ does not form a group.

Let us suppose that X is the opposite element of the element A, i.e.

$$A \div X = \overline{0},$$

i.e.

$$\{(x, \mu_A(x)\nu_X(x) + \nu_A(x)\mu_X(x), \mu_A(x)\mu_X(x) + \nu_A(x)\nu_X(x)) \mid x \in E\} = \{(x, 0, 1) \mid x \in E\}.$$

Therefore we have the following system of equations:

$$\mu_A(x)\nu_X(x) + \nu_A(x)\mu_X(x) = 0$$
$$\mu_A(x)\mu_X(x) + \nu_A(x)\nu_X(x) = 1$$

1st case: $\mu_A(x) = 0$. Hence

$\nu_A(x)\mu_X(x) = 0$

$\nu_A(x)\nu_X(x) = 1$, but $\nu_A(x) \leq 1$ and $\nu_X(x) \leq 1$ therefore $\nu_A(x) = \nu_X(x) = 1$

and thus so $\mu_A(x) = \mu_X(x) = 0$.
Therefore the opposite element of $\{(x, 0, 1) \mid x \in E\}$ is $\{(x, 0, 1) \mid x \in E\}$ (i.e. the same).
2nd case: $\mu_A(x) \neq 0$. Hence

$\nu_X(x) = -\dfrac{\nu_A(x)\mu_X(x)}{\mu_A(x)}$, but $\nu_X(x) \geq 0$ therefore $\nu_A(x)\mu_X(x) = 0$.

2.1 $\nu_A(x) = 0$ i.e. $\mu_A(x)\mu_X(x) = 1$, but $\mu_A(x) \leq 1$ and $\mu_X(x) \leq 1$, therefore $\mu_A(x) = \mu_X(x) = 1$.

Therefore $\nu_X(x) = 0$

and so the opposite element of $\{(x, 1, 0) | x \in E\}$ is $\{(x, 1, 0) | x \in E\}$ (e.i. the same).
2.2 $\nu_A(x) \neq 0$ i.e. $\mu_X(x) = 0$.

Therefore $\nu_A(x)\nu_X(x) = 1$, but $\nu_A(x) \leq 1$ and $\nu_X(x) \leq 1$so $\nu_A(x) = \nu_X(x) = 1$.

Therefore $\mu_A(x) = 0$, which is contrary with the condition of 2nd case $\mu_A(x) \neq 0$.
Therefore our supposition is wrong and so there is no opposite element in the common case. Therefore (*IFS*, $\div$) is a monoid and not a group.

Note: Actually, we can notice that the $\{(x, 0, 1) | x \in E\}$ set, in $\div$ operation with $A = \{(x, \mu_A(x), \nu_A(x)) | x \in E\}$, actually keeps the other set A intact.
For example $A \div \{(x, 0, 1) | x \in E\} =$
$= \{(x, \mu_A(x), \nu_A(x)) | x \in E\} \div \{(x, 0, 1) | x \in E\} = \{(x, \mu_A(x), \nu_A(x)) | x \in E\}$.

On the contrary, the $\{(x, 1, 0) | x \in E\}$ set, in $\div$ operation with $A = \{(x, \mu_A(x), \nu_A(x)) | x \in E\}$, returns the opposite of A i.e. $\neg A = \{(x, \nu_A(x), \mu_A(x)) | x \in E\}$.

Also notable sets are $\{(x, 0, 0) | x \in E\}$, which in $\div$ operation with every set returns itself $\{(x, 0, 0) | x \in E\}$, and $\{(x, 0.5, 0.5) | x \in E\}$, which in $\div$ operation with $A = \{(x, \mu_A(x), \nu_A(x)) | x \in E\}$, returns

$$\{(x, \frac{\mu_A(x)+\nu_A(x)}{2}, \frac{\mu_A(x)+\nu_A(x)}{2}) | x \in E\}.$$

Proposition 4: Operation $\div$ is distributive only with the @ operation and specifically $(A @ B) \div C = (A \div C) @ (B \div C)$ and that $(A \div B) @ C \neq (A @ C) \div (B @ C)$.
Proof: We will prove the first one:

$(A @ B) \div C = \{(x, \frac{\mu_A(x), \mu_B(x)}{2}, \frac{\nu_A(x), \nu_B(x)}{2}) | x \in E\} \div C =$

$= \{(x, \frac{\mu_A(x), \mu_B(x)}{2}\nu_C(x) + \frac{\nu_A(x), \nu_B(x)}{2}\mu_C(x), \frac{\mu_A(x), \mu_B(x)}{2}\mu_C(x) + \frac{\nu_A(x), \nu_B(x)}{2}\nu_C(x)) | x \in E\}$

$(A \div C) @ (B \div C) =$
$= \{(x, \mu_A(x)\nu_C(x) + \nu_A(x)\mu_C(x), \mu_A(x)\mu_C(x) + \nu_A(x)\nu_C(x)) | x \in E\} @ \{(x, \mu_B(x)\nu_C(x) + \nu_B(x)\mu_C(x), \mu_B(x)\mu_C(x) + \nu_B(x)\nu_C(x)) | x \in E\} =$

$= \{(x, \frac{\mu_A(x)\nu_C(x)+\nu_A(x)\mu_C(x)+\mu_B(x)\nu_C(x)+\nu_B(x)\mu_C(x)}{2},$

$\frac{\mu_A(x)\mu_C(x)+\nu_A(x)\nu_C(x)+\mu_B(x)\mu_C(x)+\nu_B(x)\nu_C(x)}{2}) | x \in E\}$.

The results are equal and therefore $(A @ B) \div C = (A \div C) @ (B \div C)$.
To prove that $(A \div B) @ C \neq (A @ C) \div (B @ C)$ it is sufficient to examine the case $A = B = C = \{(x, 1, 0) | x \in E\}$:
$(A \div B) @ C = \{(x, 0.5, 0.5) | x \in E\}$,
$(A @ C) \div (B @ C) = \{(x, 0, 1) | x \in E\}$.

There are no other distributive relations between $\div$ and either of these: $+$, $.$, $\cap$, $\cup$. To prove that it is sufficient to check:

$(\{(x, 0.5, 0) \mid x \in E\} \div \{(x, 1, 0) \mid x \in E\}) \bullet \{(x, 0.5, 0) \mid x \in E\} \neq$

$\neq (\{(x, 0.5, 0) \mid x \in E\} \bullet \{(x, 0.5, 0) \mid x \in E\}) \div (\{(x, 0.5, 0) \mid x \in E\} \bullet \{(x, 0.5, 0) \mid x \in E\})$

and:

$(\{(x, 0.5, 0) \mid x \in E\} \bullet \{(x, 1, 0) \mid x \in E\}) \div \{(x, 0.5, 0) \mid x \in E\} \neq$

$\neq (\{(x, 0.5, 0) \mid x \in E\} \div \{(x, 0.5, 0) \mid x \in E\}) \bullet (\{(x, 0.5, 0) \mid x \in E\} \div \{(x, 0.5, 0) \mid x \in E\})$,

where in the place of the $\bullet$ operation is put either of these: $+$, $.$, $\cap$, $\cup$.

Proposition 5: The $\div$ operation has the following distributive properties in relation with modal operators $\Box$ (necessity) and $\Diamond$ (possibility):

$$\Box(A \div B) \subset \Box A \div \Box B$$
$$\Diamond(A \div B) \supset \Diamond A \div \Diamond B$$

Proof:

$\Box(A \div B) = \{(x, \mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x),\ 1 - (\mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x))) \mid x \in E\}$,

$\Box A \div \Box B = \{(x, \mu_A(x), 1 - \mu_A(x)\} \div \{x, \mu_B(x), 1 - \mu_B(x)) \mid x \in E\} =$

$= \{(x, \mu_A(x) - \mu_A(x)\mu_B(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \mu_A(x)\mu_B(x) + 1 - \mu_B(x) - \mu_A(x) + \mu_A(x)\mu_B(x)) \mid x \in E\}$.

We will subtract the membership and non-membership of the second result from the first and compare it with 0 to see which one is greater:

$(\mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x)) - (\mu_A(x) - \mu_A(x)\mu_B(x) + \mu_B(x) - \mu_A(x)\mu_B(x)) =$

$= \mu_A(x)(\mu_B(x) + \nu_B(x) - 1) + \mu_B(x)(\mu_A(x) + \nu_A(x) - 1) \leq 0$,

$(1 - (\mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x))) - (\mu_A(x)\mu_B(x) + 1 - \mu_B(x) - \mu_A(x) + \mu_A(x)\mu_B(x)) =$

$= - \mu_A(x)\nu_B(x) - \nu_A(x)\mu_B(x) - \mu_A(x)\mu_B(x) + \mu_B(x) + \mu_A(x) - \mu_A(x)\mu_B(x) =$

$= \mu_A(x)(1 - (\mu_B(x) + \nu_B(x))) + \mu_B(x)(1 - (\mu_A(x) + \nu_A(x))) \geq 0$.

This proves that $\Box(A \div B) \subset \Box A \div \Box B$.

$\Diamond(A \div B) = \{(x, 1 - (\mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)), \mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)) \mid x \in E\}$,

$\Diamond A \div \Diamond B = \{(x, 1 - \nu_A(x), \nu_A(x)\} \div \{x, 1 - \nu_B(x), \nu_B(x)) \mid x \in E\} =$

$= \{(x, \nu_B(x) - \nu_A(x)\nu_B(x) + \nu_A(x) - \nu_A(x)\nu_B(x), 1 - \nu_B(x) - \nu_A(x) + \nu_A(x)\nu_B(x) + \nu_A(x)\nu_B(x)) \mid x \in E\}$.

Again we will subtract the membership and non-membership of the second result from the first and compare it with 0:

$(1 - (\mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x))) - (\nu_B(x) - \nu_A(x)\nu_B(x) + \nu_A(x) - \nu_A(x)\nu_B(x)) =$

$= 1 - \mu_A(x)\mu_B(x) - \nu_B(x) - \nu_A(x) + \nu_A(x)\nu_B(x) = \nu_B(x)(\nu_A(x) -1) - (\nu_A(x) -1) - \mu_A(x)\mu_B(x) =$

$= (1 - \nu_A(x))(1 - \nu_B(x)) - \mu_A(x)\mu_B(x) \geq 0$

$(\mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)) - (1 - \nu_B(x) - \nu_A(x) + \nu_A(x)\nu_B(x) + \nu_A(x)\nu_B(x)) =$

$= \mu_A(x)\mu_B(x) - 1 + \nu_B(x) + \nu_A(x) - \nu_A(x)\nu_B(x) =$

$= - (1 - \mu_A(x)\mu_B(x) - \nu_B(x) - \nu_A(x) + \nu_A(x)\nu_B(x)) \leq 0$ (from the previous inequality)

Therefore $\Diamond(A \div B) \supset \Diamond A \div \Diamond B$.

Finally, we will check some interesting results, using the negation of

$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\}$:
$\neg A = \{(x, \nu_A(x), \mu_A(x)) \mid x \in E\}$.

Proposition 6: $A \div B = \neg A \div \neg B$.
Proof:
$\neg A \div \neg B = \{(x, \nu_A(x), \mu_A(x)\} \div \{x, \nu_B(x), \mu_B(x)) \mid x \in E\} =$
$= \{(x, \nu_A(x)\mu_B(x) + \mu_A(x)\nu_B(x), \nu_A(x)\nu_B(x) + \mu_A(x)\mu_B(x)) \mid x \in E\}$.
It is obvious, that this is equal to $A \div B$.

Proposition 7: $\neg(A \div B) \neq \neg A \div \neg B$.
Proof:
$\neg(A \div B) = \{(x, \mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x), \mu_A(x)\nu_B(x) + \nu_A(x)\mu_B(x)) \mid x \in E\}$
It is obvious that the two are not , therefore $\neg(A \div B) \neq \neg A \div \neg B$.

Reference

[1] Atanassov, K. Intuitionistic Fuzzy Sets, Springer Physica – Verlag, Heidelberg, 1999.