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A Note on the extended modal operator $G_{\alpha,\beta}$

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Abstract: In the paper the product of finite and infinite in number extended modal operators of the kind G_{α_s,β_s} are considered and studied.
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Let E be a given universe. The class of all *IFS* over E we shall denote by IFS(E) and the class of all *FS* over E we shall denote by FS(E).

Let $\alpha, \beta \in [0,1]$ and $A \in$ IFS(E), i.e.

$$A \stackrel{\text{def}}{=} \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where:

$$\mu_A : E \to [0,1]\ ;\ \nu_A : E \to [0,1]$$

are the defining mappings for A.

In [1] two extended modal operators $F_{\alpha,\beta}$: IFS(E) $\to$ IFS(E) and $G_{\alpha,\beta}$: IFS(E) $\to$ IFS(E) are introduced and some of their basic properties are established. The operator $F_{\alpha,\beta}$ is given by

$$F_{\alpha,\beta}(A) \stackrel{\text{def}}{=} \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \tag{1}$$

and the operator $G_{\alpha,\beta}$ is given by

$$G_{\alpha,\beta}(A) \stackrel{\text{def}}{=} \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle | x \in E\}. \tag{2}$$

But there are two essential differences between the operators $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$ that we must note. The first is that for $F_{\alpha,\beta}$ we have to impose the additional condition $\alpha + \beta \leq 1$, while for $G_{\alpha,\beta}$ such condition is not necessary. The second difference is that $G_{\alpha,\beta}$ maps each *IFS* (which is not *FS*) into *IFS* (which is not *FS*), while for the operator $F_{\alpha,\beta}$ for $\alpha + \beta = 1$ the *IFS* is always mapped into *FS*.

Let $n \geq 1$ be an integer and $\alpha_i, \beta_i \in [0,1], i = 1, \ldots, n$. Using the principle of mathematical induction, it is trivial to verify that:

$$G_{\alpha_n,\beta_n}(\ldots(G_{\alpha_1,\beta_1}(A))\ldots) = \{\langle x, \mu_A(x)\prod_{i=1}^{n}\alpha_i, \nu_A(x)\prod_{i=1}^{n}\beta_i\rangle | x \in E\}. \tag{3}$$

The above equality suggests us to introduce the product of extended modal operators $G_{\alpha_i,\beta_i}, (i = 1, \ldots, n)$, putting

$$G_{\alpha_n,\beta_n}(\ldots(G_{\alpha_1,\beta_1})\ldots) \stackrel{\text{def}}{=} G_{\prod\limits_{i=1}^{n}\alpha_i, \prod\limits_{i=1}^{n}\beta_i}$$

It is clear from (3) that the mentioned product is a sequential superposition of operators $G_{\alpha_1,\beta_1}, \ldots, G_{\alpha_n,\beta_n}$ and it represents an operator G_{e_n,f_n} : IFS$(E) \to$ IFS(E) with:

$$e_n \stackrel{\text{def}}{=} \prod_{i=1}^{n}\alpha_i;\ f_n \stackrel{\text{def}}{=} \prod_{i=1}^{n}\beta_i. \tag{4}$$

So we observe the following proposition:

Proposition 1. *The finite product of extended modal operators G_{α_i,β_i}, $i = 1, \ldots, n$ (i.e. $G_{\alpha_n,\beta_n}(\ldots(G_{\alpha_1,\beta_1})\ldots)$) is an extended modal operator of the same type (i.e. G_{e_n,f_n}).*

The same proposition stays valid if we change the letter "G" with the letter "F" but in that case e_n and f_n are given by

$$e_n = \alpha_1 + \sum_{k=2}^{n}\alpha_k\prod_{j=1}^{k-1}(1-\alpha_j-\beta_j)$$

$$f_n = \beta_1 + \sum_{k=2}^{n}\beta_k\prod_{j=1}^{k-1}(1-\alpha_j-\beta_j).$$

Here we must note the following important fact. In [1] several other extended modal operators (besides F and G) are considered but for none of them an analogous to the above proposition is fulfilled.

In [2] the finite product F_{e_n,f_n} is studied and extended to an infinite product $F_{e,f}$, where:

$$e \stackrel{\text{def}}{=} \lim_{n\to\infty} e_n;\ f \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n, \tag{5}$$

and some very useful results are obtained. For instance, new criteria (different from the classical ones) which use *IFS* and *FS* for convergence and divergence of infinite products are proposed.

In the present paper we shall study the situation with the infinite product $G_{e,f}$ with e and f given by (5) and e_n and f_n given by (4).

Remark 1. *The infinite products $\prod_{i=1}^{\infty}\alpha_i$ and $\prod_{i=1}^{\infty}\beta_i$ converge if and only if the sequences $\{e_n\}$ and $\{f_n\}$ tend to limits different from 0. If at least one of these sequences tends to 0, then the corresponding infinite product diverges.*

First we note that $\{e_n\}$ and $\{f_n\}$ are monotonously decreasing sequences bounded from below by 0, because of the reccurent relations:

$$e_{n+1} = \alpha_{n+1}.e_n;\ f_{n+1} = \beta_{n+1}.f_n, n = 1, 2, 3, \ldots$$

and the inequalities $0 \le \alpha_i \le 1$ and $0 \le \beta_i \le 1$, $i = 1, 2, \ldots$

Therefore, the numbers e and f certainly exist and they belong to the interval [0,1], but in the case when at least one of these numbers is zero the corresponding infinite product diverges.

The following result is valid (see [3]):

Proposition 2. *A necessary and sufficient condition for the convergence (the divergence) of each one of the infinite products* $\prod_{i=1}^{\infty} \alpha_i$ *and* $\prod_{i=1}^{\infty} \beta_i$ *is the convergence (the divergence) of each one of the infinite series:* $\sum_{n=1}^{\infty} \varrho_n, \sum_{n=1}^{\infty} \theta_n$, *where* $\varrho_n = 1 - \alpha_n$ *and* $\theta_n = 1 - \beta_n$, $n = 1, 2, 3, \ldots$

Let us remind the following well-known:

Proposition 3. *A necessary condition for the convergence of the infinite series* $\sum_{n=1}^{\infty} \varrho_n$ *(resp.* $\sum_{n=1}^{\infty} \theta_n$*) is* $\varrho_n \to 0$ *(resp.* $\theta_n \to 0$*).*

Remark 2. *If* $\varrho_n = \frac{1}{n}$, *then* $\varrho_n \to 0$, *but* $\sum_{n=1}^{\infty} \varrho_n$ *represents the Harmonic series that diverges.*

Therefore, for $\alpha_i \overset{\text{def}}{=} 1 - \frac{1}{i}, i = 1, 2, 3, \ldots$, the infinite sequence $\{e_n\}$ tends to 0, i.e. the product $\prod_{i=1}^{\infty} \alpha_i$ diverges.

As a corollary from Proposition 3 we obtain:

Proposition 4. *A necessary condition for the convergence of each one of the infinite products:* $\prod_{i=1}^{\infty} \alpha_i$; $\prod_{i=1}^{\infty} \beta_i$, *is* $\alpha_i \to 1$; $\beta_i \to 1$.

Remark 1 and Proposition 2 yield:

Corollary 1. *A necessary and sufficient condition for the existence of* $G_{0,f}$, *with* $f \neq 0$, *is* $e_n \to 0$ *and the convergeness of the series* $\sum_{n=1}^{\infty} \theta_n$ *with* $\theta_n = 1 - \beta_n$, $n = 1, 2, 3, \ldots$

Corollary 2. *A necessary and sufficient condition for the existence of* $G_{e,0}$, *with* $e \neq 0$, *is* $f_n \to 0$ *and the convergeness of the series* $\sum_{n=1}^{\infty} \varrho_n$ *with* $\varrho_n = 1 - \alpha_n$, $n = 1, 2, 3, \ldots$

Proposition 2 yields:

Corollary 3. *A necessary and sufficient condition for the existence of* $G_{e,f}$ *with* $e.f \neq 0$ *is the simultanous convergeness of the infnite series* $\sum_{n=1}^{\infty} \varrho_n, \sum_{n=1}^{\infty} \theta_n$, *where* $\varrho_n = 1 - \alpha_n$ *and* $\theta_n = 1 - \beta_n$, $n = 1, 2, 3, \ldots$

Remark 2 and Proposition 4 yield:

Corollary 4. *A necessary condition for the existence of* $G_{0,f}$, *with* $f \neq 0$, *is* $e_n \to 0$ *and* $\beta_n \to 1$.

Corollary 5. *A necessary condition for the existence of* $G_{e,0}$, *with* $e \neq 0$, *is* $f_n \to 0$ *and* $\alpha_n \to 1$.

Corollary 6. *A necessary condition for the existence of* $G_{e,f}$ *with* $e.f \neq 0$ *is* $\alpha_n \to 1$ *and* $\beta_n \to 1$.

Since $\alpha_i \leq 1$ and $\beta_i \leq 1$, $i = 1, 2, 3, \ldots$, we may rewrite Corollary 6 in the following equivalent form:

Corollary 7. *A necessary condition for the existence of* $G_{e,f}$ *with* $e.f \neq 0$ *is* $\alpha_n + \beta_n \to 2$.

An interesting case occurs when when we suppose that $\alpha_i + \beta_i \leq 1$ for $i = 1, 2, 3, \ldots$ In this case if we assume the existence of $G_{e,f}$ with $e.f \neq 0$, we will obtain that $\alpha_n \to 1$ and $\beta_n \to 1$, because of Corollary 6. But the last contradicts to the fact that $\alpha_i + \beta_i \leq 1$, $i = 1, 2, 3, \ldots$ Therefore, in this case at least one of the infinite sequences $\{e_n\}$ and $\{f_n\}$ tends to 0.

So we prove:

Corollary 8. *A necessary condition for the existence of $G_{e,f}$, when $\alpha_i + \beta_i \leq 1$ for $i = 1, 2, 3, \ldots$, is at least one of the infinite sequences $\{e_n\}$ and $\{f_n\}$ to tend to 0.*

Finally, we must note that the boundary operator $G_{e,f}$ maps each *IFS* that is not *FS* into *IFS* that is not *FS*. Moreover, if $A \in \text{FS(E)}$ then $G_{e,f}(A) \in \text{FS(E)}$ if and only if $\alpha_i = 1,\ \beta_i = 1,\ i = 1, 2, \ldots$

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