NIFS 6 (2000) 1, 7-14

Conditional Probability and Independence of Intuitionistic Fuzzy Events

Przemysław Grzegorzewski

Systems Research Institute, Polish Academy of Sciences,
ul. Newelska 6, 01-447 Warsaw, Poland,
and F. Math. Inf. Sci., Warsaw Univ. of Technology,
Plac Politechniki 1, 00-661 Warsaw, Poland

Abstract

The notion of independence of events and the concept of conditional probability is generalized on the intuitionistic fuzzy events.

Key words: intuitionistic fuzzy sets, intuitionistic fuzzy event, independence of events, conditional probability, probability of intuitionistic fuzzy event.

1 Introduction

Probability theory provides very powerful tools for dealing with uncertainty. However, in classical probability all random events should be precisely defined. Unfortunately this assumption appears too rigid in many real life problems. Very often, especially when using natural language, people deal with imprecisely defined notions, like: "high income", "cloudy sky", "low temperature", etc. For the traditional probability theory such expressions are ill defined and they are beyond the scope of that theory.

To handle situations like described above Zadeh (1965) introduced the concept of fuzzy set. Also Zadeh (1968) was the first who defined a fuzzy event, suggested how to compute probabilities of such events and who showed some basic properties of the probabilities of fuzzy events.

In conventional fuzzy set a membership function assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The degree of nonbelongingness is just automatically the complement to 1 of the membership degree. However, a human being who expresses the degree of membership of given element in a fuzzy set very often does not express corresponding degree of nonmembership as the complement to 1. This reflects a well known psychological fact that the linguistic negation not always identifies with logical negation.

Thus Atanassov (1986) introduced the concept of an intuitionistic fuzzy set which is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness, respectively. This idea, which is a natural generalization of usual fuzzy set, seems to be useful in modeling many real life situations, like negotiation processes, etc. (see Atanassov 1986, 1994a, 1994b, 1999; Szmidt and Kacprzyk, 1996a, 1996b, 1996c, 1997, 1998a, 1998b).

Szmidt and Kacprzyk (1999a, 1999b, 1999c) suggested how to define the probability of an intuitionistic fuzzy event. Contrary to the probability of a fuzzy event by Zadeh, the probability of an intuitionistic fuzzy event is no longer a given real number but is described by an interval.

One of the most important notions in the probability theory is the concept of independence and conditional probability. In this paper we suggest the definition of independent intuitionistic fuzzy events. We also propose the concept of the conditional probability of intuitionistic fuzzy events. Our definitions are not straightforward generalizations of Zadeh's definitions since now we deal with intervals. However, our approach is consistent with Zadeh's approach when considered intuitionistic fuzzy events become fuzzy events.

2 Intuitionistic fuzzy events

Let X denote a universe of discourse. Then a fuzzy set A in X is defined as a set of ordered pairs, i.e.

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$ is the membership function of A and $\mu_A(x)$ is the grade of belongingness of x into A. Thus automatically the grade of nonbelongingness of x into A is equal to $1-\mu_A(x)$. However, in real life the linguistic negation not always identifies with logical negation (see Pacholczyk, 1999). This situation is very common in natural language processing, computing with words, etc. Therefore Atanassov (1986) suggested a generalization of usual fuzzy set, called intuitionistic fuzzy set.

An intuitionistic fuzzy set A in X is given by an ordered triple

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{2}$$

where $\mu_A, \nu_A : X \to [0,1]$ such that

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \qquad \forall x \in X. \tag{3}$$

For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represents the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively.

It is easily seen that a $\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in X\}$ is equivalent to (1), i.e. each fuzzy set is a particular case of the intuitionistic fuzzy set.

We will denote a family of fuzzy sets in X by $FS(X)$, while $IFS(X)$ stands for the family of all intuitionistic fuzzy sets in X.

For each element $x \in X$ we can compute, so called, the intuitionistic index of x in A defined as follows

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{4}$$

which measures the degree of hesitancy of whether x belongs to A. Thus function π_A is sometimes called the hesitancy margin of the intuitionistic set A. It is seen immediately that $\pi_A(x) \in [0,1]\ \forall x \in X$. If $A \in FS(X)$ then $\pi_A(x) = 0\ \forall x \in X$.

In his paper Atanassov (1986, 1989)defined basic operations on intuitionistic fuzzy sets. If $A, B \in IFS(X)$ then

$$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}, \tag{5}$$

$$A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}, \tag{6}$$

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle : x \in X\}, \tag{7}$$

$$A \cdot B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle : x \in X\}, \tag{8}$$

$$\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}. \tag{9}$$

The basic concept of probability theory is probability space $(X, \mathcal{A}, P)$, where X is a sample space, $\mathcal{A}$ is a $\sigma-$ field of subsets of X and P is a real-valued function which assigns to every event A in $\mathcal{A}$ its probability $P(A)$. By using the concept of fuzzy set Zadeh (1968) extended the notions of an event and its probability to fuzzy context. We shall assume for simplicity that X is a set in $\mathcal{R}^n$, $\mathcal{A}$ is the smallest Borel $\sigma-$field on X and P is a probability distribution on $(X, \mathcal{A})$ Then, according to Zadeh's definition, a fuzzy event in X is a fuzzy set A in X whose membership function μ_A is Borel measurable and the probability of such fuzzy event is given by

$$P(A) = \int_X \mu_A(x)dP. \tag{10}$$

The existence of that Lebesgue-Stielties integral is assured by the assumption that μ_A is Borel measurable. In his paper Zadeh also showed some properties of the probabilities defined on fuzzy events.

With reference to Zadeh's paper Szmidt and Kacprzyk (1999a, 1999b, 1999c) proposed the definition of intuitionistic fuzzy event and its probability. Hence, by an intuitionistic fuzzy event A they mean an intuitionistic fuzzy subset of the universe of discourse whose membership function μ_A and hesitancy margin π_A are Borel measurable. Then they defined the notion of the probability a intuitionistic fuzzy set. However, they considered finite universe of discourse only. Below we present more general definition formulated in the spirit of definition given by Szmidt and Kacprzyk (1999a, 1999b, 1999c).

Definition 1 *Let A denote an intuitionistic fuzzy event in X, where X is a set in $\mathcal{R}^n$ and let P denote a probability measure over X. Then the probability of the intuitionistic fuzzy event A is a number $p(A)$ from the interval*

$$[p_{\min}(A), p_{\max}(A)], \tag{11}$$

where

$$p_{\min}(A) = \int_X \mu_A(x)dP, \tag{12}$$

$$p_{\max}(A) = \int_X (\mu_A(x) + \pi_A(x))dP = \int_X (1 - \nu_A(x))dP \tag{13}$$

are called the minimal and maximal probabilities of the intuitionistic fuzzy event A, respectively.

Therefore we have

$$p(A) \in \left[\int_X \mu_A(x)dP, \int_X (\mu_A(x) + \pi_A(x))\,dP\right] \tag{14}$$

or

$$p(A) \in \left[\int_X \mu_A(x)dP, \int_X (1 - \nu_A(x))dP\right] \tag{15}$$

The minimal probability $p_{\min}(A)$ gives "sure" probability that the event A will occur. The maximal probability $p_{\max}(A)$ gives the highest possible probability that the event A will occur. It is achieved only if the hesitancy margin function support occurrence of the event A. The difference between the maximal and minimal probabilities, i.e. $p_{\max}(A) - p_{\min}(A)$, reflects the unsureness of occurrence of the intuitionistic fuzzy event A. Therefore, if $\pi_A(x) = 0\ \forall x \in X$, i.e. if A is a classical fuzzy set, then (14) obviously reduces to (15), i.e. to the probability of a fuzzy set in the sense of Zadeh's definition.

3 Conditional probabilities

One of the most important notions in the probability theory is the concept of the conditional probability. If $(X, \mathcal{A}, P)$ is a probability space and $B \in \mathcal{A}$ such that $P(B) > 0$ then the conditional probability of A given B is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}. \tag{16}$$

Two events A and B are independent iff

$$P(A \cap B) = P(A)P(B). \tag{17}$$

Zadeh (1968) proposed a following definition of independence of fuzzy events: let A and B be two fuzzy events in a probability space $(X, \mathcal{A}, P)$, where X is a set in $\mathcal{R}^n$. Then A and B are called independent fuzzy events if

$$P(AB) = P(A)P(B), \tag{18}$$

i.e. if

$$\int_{X\times Y}\mu_A(x)\mu_B(y)dP(x,y)=\int_X\mu_A(x)dP(x)\int_Y\mu_B(x)dP(y), \tag{19}$$

where μ_A and μ_B are membership functions of fuzzy events A and B, respectively. In his paper Zadeh also defined the conditional probability of fuzzy events. Namely, the conditional probability of fuzzy event A given B, such that $P(B) > 0$, is given by a following formula

$$P(A\mid B)=\frac{P(AB)}{P(B)}, \tag{20}$$

where $P(AB)$ is defined as in (19). It is easily seen that if A and B are independent fuzzy events then $P(A \mid B) = P(A)$, similarly as in the case of crisp events.

In the present section we propose how to compute conditional probabilities for intuitionistic fuzzy events. At first we turn to the notion of independence of two intuitionistic fuzzy events.

Definition 2 *Let A and B denote two intuitionistic fuzzy events in $X \subseteq \mathcal{R}^n$ such that $p(A) \in [p_{\min}(A), p_{\max}(A)]$, $p(B) \in [p_{\min}(B), p_{\max}(B)]$. These two events are called independent if and only if*

$$[p_{\min}(A\cdot B),p_{\max}(A\cdot B)]=[p_{\min}(A)p_{\min}(B),p_{\max}(A)p_{\max}(B)]. \tag{21}$$

Note that using well known formula for arithmetic multiplication of closed intervals

$$[a,b]\circ[c,d]=[\min\{ac,ad,bc,bd\},\max\{ac,ad,bc,bd\}] \tag{22}$$

and applying a following notation for probability intervals:

$$\mathcal{P}(A)=[p_{\min}(A),p_{\max}(A)], \tag{23}$$

we can rewrite (21) as

$$\mathcal{P}(A\cdot B)=\mathcal{P}(A)\circ\mathcal{P}(B), \tag{24}$$

which resembles definition (17) for crisp events. However, we employ the product rather $A \cdot B$ than $A \cap B$ because of the following: let P_1 and P_2 denote probability measures on $X_1 = \mathcal{R}^{n_1}$ and $X_2 = \mathcal{R}^{n_2}$, respectively, while $P = P_1 \times P_2$ is a product measure on $X = \mathcal{R}^{n_1+n_2}$. Suppose A and B are intuitionistic fuzzy events in X_1 and X_2, respectively. Moreover, let $\mu_A(x_1) = \mu_A(x_1, x_2)$, $\mu_B(x_2) = \mu_B(x_1, x_2)$ and $\nu_A(x_1, x_2) = \nu_A(x_1)$, $\nu_B(x_2) = \nu_B(x_1, x_2)$. Then we get

$$\begin{aligned}\mathcal{P}(A\cdot B)&=[p_{\min}(A\cdot B),p_{\max}(A\cdot B)]\\ &=\left[\int_X\mu_A(x_1)\mu_B(x_2)dP,\int_X(1-\nu_A(x_1)-\nu_B(x_2)+\nu_A(x_1)\nu_B(x_2))dP\right]\end{aligned}$$

$$
\begin{aligned}
&= \left[\int_X \mu_A(x_1)\mu_B(x_2)d(P_1 \times P_2),\right.\\
&\quad \left.\int_X (1-\nu_A(x_1)-\nu_B(x_2)+\nu_A(x_1)\nu_B(x_2))d(P_1 \times P_2)\right]\\
&= \left[\int_X \mu_A(x_1)\mu_B(x_2)d(P_1 \times P_2), \int_X (1-\nu_A(x_1))(1-\nu_B(x_2))d(P_1 \times P_2)\right]\\
&= [p_{\min}(A)p_{\min}(B), p_{\max}(A)p_{\max}(B)]\\
&= \left[\int_{X_1} \mu_A(x_1)dP_1, \int_{X_1} (1-\nu_A(x_1))dP_1\right]\\
&\quad \circ \left[\int_{X_2} \mu_B(x_2)dP_2, \int_{X_2} (1-\nu_B(x_2))dP_2\right]\\
&= [p_{\min}(A), p_{\max}(A)] \circ [p_{\min}(B), p_{\max}(B)] = \mathcal{P}(A) \circ \mathcal{P}(B).
\end{aligned}
$$

Thus our intuitionistic fuzzy events A and B are independent in the sense of (24). It is worth noting that this natural example would not hold if in the definition (21) of independence of intuitionistic fuzzy events the intersection $A \cap B$ were used instead of $A \cdot B$.

Now we can define the conditional probability of intuitionistic fuzzy events.

Definition 3 *Let A and B denote two intuitionistic fuzzy events in $X \subseteq \mathcal{R}^n$. Then the conditional probability of A given B is a number $p(A \mid B)$ from the interval*

$$[p_{\min}(A \mid B), p_{\max}(A \mid B)], \tag{25}$$

which is a solution of the following equation

$$[p_{\min}(A \cdot B), p_{\max}(A \cdot B)] = [p_{\min}(A \mid B), p_{\max}(A \mid B)] \circ [p_{\min}(B), p_{\max}(B)]. \tag{26}$$

Using interval notation (23) we may rewrite equation (26) as

$$\mathcal{P}(A \cdot B) = \mathcal{P}(A \mid B) \circ \mathcal{P}(B), \tag{27}$$

which resembles well known formula for the probability of the union of crisp events $P(A \cap B) = P(A \mid B)P(B)$ obtained by the transformation of (16) or for the probability of the union of fuzzy events $P(AB) = P(A)P(B)$ obtained from (20).

An immediate consequence of our definition is the following: let A and B be two independent intuitionistic fuzzy events, such that $p(B) \in [p_{\min}(B), p_{\max}(B)]$ and $p_{\min}(B) > 0$. Then according to (24)

$$\mathcal{P}(A \cdot B) = \mathcal{P}(A) \circ \mathcal{P}(B) = \mathcal{P}(A \mid B) \circ \mathcal{P}(B), \tag{28}$$

hence

$$\mathcal{P}(A \mid B) = \mathcal{P}(A). \tag{29}$$

This is an analog to the property of crisp events $P(A \mid B) = P(A)$ being the natural interpretation of independence, i.e. it states that occurrence or non-occurrence of event B has no effect on the probability of event A. Note that this would not be true if the conditional probability were defined in the spirit of (16) and (20) as interval quotient $\mathcal{P}(A \mid B) = \mathcal{P}(A \cdot B)/\mathcal{P}(B)$.

4 Conclusions

In the paper we try to handle two types of uncertainty: randomness - described by the probability theory, and imprecision - expressed here by intuitionistic fuzzy set theory. Both sources of uncertainty play a central role in decision making. In this paper we proposed the definition of independent intuitionistic fuzzy events and the conditional probability of intuitionistic fuzzy events. Our approach is consistent with Zadeh's approach when considered intuitionistic fuzzy events become fuzzy events. Since the concept of independence and conditional probabilities are of the great importance in the classical probability theory it seems that further investigations in this area is still required.

References

[1] Atanassov K. (1986), *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems **20**, 87–96.

[2] Atanassov K. (1989), *More on Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems **33**, 37–46.

[3] Atanassov K. (1994a), *New Operations Defined over the Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems **61**, 137–142.

[4] Atanassov K. (1994b), *Operators over Interval Valued Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems **64**, 159–174.

[5] Atanassov K. (1999), *Intuitionistic Fuzzy Sets: Theory and Applications*, Physica-Verlag.

[6] Pacholczyk D. (1999), *An Approach to Linguistic Negation of Fuzzy Property Combination*, Proceedings of the 7th European Congress on Intelligent Techniques and Soft Computing EUFIT'99, Aachen, on CD-Rom.

[7] Szmidt E., Kacprzyk J. (1996a), *Intuitionistic Fuzzy Sets in Decision Making*, Notes on IFS Vol. **2**, No 1, 15–32.

[8] Szmidt E., Kacprzyk J. (1996b), *Group Decision Making via Intuitionistic Fuzzy Sets in Decision Making*, Proceeding of FUBEST'96, Sofia, 107–112.

[9] Szmidt E., Kacprzyk J. (1996c), *Remarks on Some Applications on Intuitionistic Fuzzy Sets in Decision Making*, Notes on IFS Vol. **2**, No 3, 22–31.

[10] Szmidt E., Kacprzyk J. (1997), *Intuitionistic Fuzzy Sets for More Realistic Group Decision Making*, Proceedings of TRANSITIONS'97, Warsaw, pp. 430–433.

[11] Szmidt E., Kacprzyk J. (1998a), *Group Decision Making under Intuitionistic Fuzzy Preference Relations*, Proceedings of the 7th Int. Conference IMPU'98, Paris, pp. 172–178.

[12] Szmidt E., Kacprzyk J. (1998b), *Applications of Intuitionistic Fuzzy Sets in Decision Making*, Proceedings of the 8th Congreso EUSFLAT'98, Pampelona, pp. 150–158.

[13] Szmidt E., Kacprzyk J. (1999a), *A Concept of a Probability of an Intuitionistic Fuzzy Event*, Proceedings of the 1999 IEEE International Fuzzy Systems Conference, Seoul, pp. 1346–1349.

[14] Szmidt E., Kacprzyk J. (1999b), *Intuitionistic Fuzzy Events and their Real-valued Probabilities*, Proceedings of the 7th European Congress on Intelligent Techniques and Soft Computing EUFIT'99, Aachen, on CD-Rom.

[15] Szmidt E., Kacprzyk J. (1999c), *Probability of Intuitionistic Fuzzy Events and Their Applications in Decision Making*, Proceedings of the 9th Congreso EUSFLAT'99, pp. 457–460.

[16] Zadeh L.A. (1968), *Probability Measures of Fuzzy Events*, J. Math. Anal. Appl. **23**, 421–427.