Notes on Intuitionistic Fuzzy Sets
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Note on one inequality and its application in intuitionistic fuzzy sets theory. Part 2

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Abstract: In the paper, the inequality $\frac{\mu^{\frac{1}{\nu}}}{\nu} + \frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{1}{2\mu\nu} - 1$ is introduced and proved. The same inequality is valid for $\mu = \mu_A(x), \nu = \nu_A(x)$, where μ_A and ν_A are the membership and the non-membership functions of an arbitrary intuitionistic fuzzy set A over a fixed universe E and $x \in E$.
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1 Introduction

The Intuitionistic Fuzzy Sets (IFSs) are introduced by K. Atanassov in [1,2] as follows. Let E be a universal set, $\mu_A, \nu_A : E \to I := [0, 1]$ be mappings and for each $x \in E$:

$$\mu_A(x) + \nu_A(x) \leq 1.$$

Then the set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called an IFS.

Mappings μ_A and ν_A are called membership and non-membership functions for the element $x \in E$ to the set $A \subseteq E$.

When for each $x \in E$, it holds that $\mu_A(x) + \nu_A(x) = 1$, then the set A is transformed to the ordinary fuzzy (Zadeh's) set [4].

2 Main result

We formulate and prove the following statement.

Theorem. *Let* $\mu, \nu \in (0,1)$ *be real numbers satisfying inequality*

$$\mu + \nu \leq 1.$$

Then the inequality

$$\frac{\mu^{\frac{1}{\nu}}}{\nu} + \frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{1}{2\mu\nu} - 1 \tag{1}$$

holds and in it the equality is possible only when $\mu = \nu = \frac{1}{2}$.

We recall that in [3] the inequality

$$\mu^{\frac{1}{\nu}} + \nu^{\frac{1}{\mu}} \leq \frac{1}{2}$$

was proved under the same conditions as in the above Theorem. We recall also the well-known Young's inequality

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB, \tag{2}$$

which is valid when A, B, p, q are positive real numbers and $\frac{1}{p} + \frac{1}{q} = 1$. The equality is possible if and only if (iff) $A = B$.

In the paper, we shall use a particular case of (2), when $\frac{1}{p} = 1 - x, \frac{1}{q} = x, A = x, B = 1 - x$, where $x \in (0,1)$. In this case, (2) yields

$$(1-x)x^{\frac{1}{1-x}} + x(1-x)^{\frac{1}{x}} \geq x(1-x). \tag{3}$$

3 Proof of the Theorem

From (3) we obtain

$$x^{\frac{1}{1-x}} - x.x^{\frac{1}{1-x}} - (1-x)(1-x)^{\frac{1}{x}} + (1-x)^{\frac{1}{x}} \geq x(1-x).$$

Hence,

$$x(1-x) + x.x^{\frac{1}{1-x}} + (1-x)(1-x)^{\frac{1}{x}} \leq x^{\frac{1}{1-x}} + (1-x)^{\frac{1}{x}}. \tag{4}$$

But in [3], the inequality

$$x^{\frac{1}{1-x}} + (1-x)^{\frac{1}{x}} \leq \frac{1}{2}$$

was proved and the equality is possible iff $x = \frac{1}{2}$.

From the above inequality and (4) we obtain

$$x(1-x) + x.x^{\frac{1}{1-x}} + (1-x)(1-x)^{\frac{1}{x}} \leq \frac{1}{2}.$$

Dividing the both sides of the last inequlity by $x(1-x)$, we obtain

$$\frac{x^{\frac{1}{1-x}}}{1-x} + \frac{(1-x)^{\frac{1}{x}}}{x} \leq \frac{1}{2x(1-x)} - 1. \tag{5}$$

The right-hand side of (5) may be represent in each one of the following forms:

$$\frac{x^2+(1-x)^2}{2x(1-x)}; \quad \frac{\frac{x}{1-x}+\frac{1-x}{x}}{2}; \quad \frac{\frac{1}{x}+\frac{1}{1-x}}{2}-1.$$

Therefore, the following inequalities are true:

$$\frac{x^{\frac{1}{1-x}}}{1-x}+\frac{(1-x)^{\frac{1}{x}}}{x} \leq \frac{1}{2x(1-x)}-1, \tag{6}$$

$$\frac{x^{\frac{1}{1-x}}}{1-x}+\frac{(1-x)^{\frac{1}{x}}}{x} \leq \frac{x^2+(1-x)^2}{2x(1-x)}, \tag{7}$$

$$\frac{x^{\frac{1}{1-x}}}{1-x}+\frac{(1-x)^{\frac{1}{x}}}{x} \leq \frac{\frac{x}{1-x}+\frac{1-x}{x}}{2}, \tag{8}$$

$$\frac{x^{\frac{1}{1-x}}}{1-x}+\frac{(1-x)^{\frac{1}{x}}}{x} \leq \frac{\frac{1}{x}+\frac{1}{1-x}}{2}-1. \tag{9}$$

Let $\mu, \nu > 0$ be real numbers and $\mu+\nu=1$. Then from (6)–(9) we obtain the following inequalities:

$$\frac{\mu^{\frac{1}{\nu}}}{\nu}+\frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{1}{2\mu\nu}-1$$

(i.e., (1)),

$$\frac{\mu^{\frac{1}{\nu}}}{\nu}+\frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{\mu^2+\nu^2}{2\mu\nu}, \tag{10}$$

$$\frac{\mu^{\frac{1}{\nu}}}{\nu}+\frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{\frac{\mu}{\nu}+\frac{\nu}{\mu}}{2}, \tag{11}$$

$$\frac{\mu^{\frac{1}{\nu}}}{\nu}+\frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{\frac{1}{\mu}+\frac{1}{\nu}}{2}-1. \tag{12}$$

We note that (1) and (10)–(12) become to equalities iff $\mu=\nu=\frac{1}{2}$.

Also, the right-hand sides of (1) and (10)–(12) coincides, since $\mu+\nu=1$.

Therefore, the Theorem is proved for the case, when $\mu+\nu=1$. Now, we shall prove it for the case $\mu+\nu<1$. In this case, the right-hand sides of (1) and (10)–(12) do not coincide. Really, when $\mu+\nu<1$, (1) takes the form

$$\frac{\mu^{\frac{1}{\nu}}}{\nu}+\frac{\nu^{\frac{1}{\mu}}}{\mu} \leq \frac{1}{2\mu\nu}-1, \tag{13}$$

i.e., the equality in (1) is impossible.

To prove (13), we first re-write it in the form

$$\mu^{1+\frac{1}{\nu}}+\nu^{1+\frac{1}{\mu}} \leq \frac{1}{2}-\mu\nu. \tag{14}$$

Second, we choose such $\nu^* \in (0,1)$ that $\mu+\nu^*=1$. Hence, $\nu<\nu^*$, $\frac{1}{\nu^*}<\frac{1}{\nu}$ and the inequality

$$\mu^{1+\frac{1}{\nu^*}}+(\nu^*)^{1+\frac{1}{\mu}} \leq \frac{1}{2}-\mu\nu^*$$

holds.

To prove (14) (and therefore, the Theorem), it is enough to establish the inequalities:

$$\mu^{1+\frac{1}{\nu}} + \nu^{1+\frac{1}{\mu}} < \mu^{1+\frac{1}{\nu^*}} + (\nu^*)^{1+\frac{1}{\mu}} \tag{15}$$

and

$$\frac{1}{2} - \mu\nu^* < \frac{1}{2} - \mu\nu. \tag{16}$$

Using the fact that $\mu \in (0, 1)$ and $\frac{1}{\nu^*} < \frac{1}{\nu}$, we obtain

$$\mu^{1+\frac{1}{\nu}} < \mu^{1+\frac{1}{\nu^*}}. \tag{17}$$

From $\nu < \nu^*$ we obtain

$$\nu^{1+\frac{1}{\mu}} < (\nu^*)^{1+\frac{1}{\mu}}. \tag{18}$$

Then, adding (17) and (18), we prove (15). The inequality (16) is obvious, since it is equivalent to $-\nu^* < -\nu$, i.e., to $\nu^* > \nu$ that is true.

Inequality (1) admits the intuitionistic fuzzy set's interpretation. Namely, let E be a universe, $\mu : E \to (0, 1), \nu : E \to (0, 1)$ are mappings and

$$A = \{\langle x, \mu(x) + \nu(x)\rangle | x \in E\}$$

be an intuitionistic fuzzy set see $[1, 2]$ with μ being the membership function and ν being the non-membership function. Then setting $\mu = \mu(x), \nu = \nu(x)$ for any fixed $x \in E$, we obtain that functions μ and ν satisfy (1).

4 Conclusion

In the present paper, we represented a new inequality which has an intuitionistic fuzzy set interpretation. It can be seen easily, that the case $\mu + \nu = 1$ has an interpretation for Zadeh's fuzzy sets [4].

References

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