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A note on intuitionistic supra fuzzy soft topological spaces

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Abstract: The goal of this study is to introduce an intuitionistic supra fuzzy soft topological space and an intuitionistic fuzzy soft bitopological space. Moreover, we give the definition of an intuitionistic supra fuzzy soft closure space and investigate relations between intuitionistic supra fuzzy soft topological space and intuitionistic supra fuzzy soft closure space. We also obtain intuitionistic supra fuzzy soft topological space induced by an intuitionistic fuzzy soft bitopological space.
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1 Introduction and Preliminaries

In 1999, Molodtsov [17] proposed a completely new concept called soft set theory to model uncertainty, which associates a set with a set of parameters. Later, Maji et al. [16] introduced the concept of fuzzy soft set which combines fuzzy sets and soft sets. All over the globe, (fuzzy) soft set theory is a topic of interest for many authors working in diverse areas due to its rich potential for applications in several directions. Nowadays the scholars study the theoretics and also applications of (fuzzy) soft sets in algebra and topology (see [2, 7, 8, 16, 18]).

The topology of fuzzy sets was first defined by C. L. Chang [9]. Later Kubiak [15] and Sostak [20] independently extended this notion and introduced the fuzzy topology. Atanassov [5] introduced the idea of intuitionistic fuzzy set. Çoker et al. [12, 13] introduced the idea of the topology of intuitionistic fuzzy sets. Samanta et al. [19] gave the definition of the intuitionistic gradation of openness. Then, intuitionistic fuzzy topological structures are observed by some authors [6, 10, 11]. Moreover, Ghanim et al. [14] introduced the gradation of supra openness as an extension of supra fuzzy topology in a sense of Abd-Elmonsef and Ramadan [1]. In [3], Abbas defined an intuitionistic supra fuzzy closure space and an intuitionistic fuzzy bitopological space and also investigated some of their properties. In this study, we softify the notions of intuitionistic fuzzy topology and intuitionistic fuzzy bitopology defined by Abbas [3]. In this direction, we introduce an intuitionistic supra fuzzy soft topological space and an intuitionistic fuzzy soft bitopological space. We obtain the intuitionistic supra fuzzy soft topological space induced by an intuitionistic fuzzy soft bitopological space.

Throughout this study, X refers to an initial universe, L and M denote the completely distributive lattice and the complete lattice with the least elements $0_L, 0_M$ and the greatest elements $1_L, 1_M$, respectively and there is an order reversing involution $'$ on L, M. Let E and K be arbitrary nonempty sets viewed on the sets of parameters. A lattice M is called order dense if for each $a, b \in M$ such that $a < b$ there exists $c \in M$ such that $a < c < b$.

Definition 1.1 [18] f is called an L-fuzzy soft set on X, where f is a mapping from E into L^X, i.e., $f_e \triangleq f(e)$ is an L-fuzzy set on X for each $e \in E$.

The family of all L-fuzzy soft sets on X is denoted by $(L^X)^E$.

Definition 1.2 [4, 18] Let f and g be two L-fuzzy soft sets on X, then

(1) we say that f is an L-fuzzy soft subset of g and write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are called equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) the union of f and g is an L-fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(3) the intersection of f and g on X is an L-fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(4) the complement of an L-fuzzy soft set f is denoted by f', where f'^X is a mapping given by $f'_e = (f_e)'$, for each $e \in E$.

(5) f is called a null L-fuzzy soft set and denoted by 0_X, if $f_e(x) = 0$, for each $e \in E, x \in X$.

(6) f is called an absolute L-fuzzy soft set and denoted by 1_X, if $f_e(x) = 1$, for each $e \in E, x \in X$.

We refer to [7, 8, 16] for all the basic definitions and notations related to fuzzy soft sets.

2 Intuitionistic supra fuzzy soft topological spaces

Definition 2.1 An intuitionistic supra fuzzy soft topology on X is an ordered pair (τ, τ^*) of maps from K to $M^{(L^X)^E}$ such that for each $k \in K$,

(IS1) $\tau_k(f) \leq (\tau_k^*(f))'$, for each $f \in (L^X)^E$.

(IS2) $\tau_k(0_X) = \tau_k(1_X) = 1_M$, $\tau_k^*(0_X) = \tau_k^*(1_X) = 0_M$.

(IS3) $\tau_k(\bigsqcup_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \tau_k(f_i)$ and $\tau_k^*(\bigsqcup_{i\in\Delta} f_i) \leq \bigvee_{i\in\Delta} \tau_k^*(f_i)$, for each $\{f_i\}_{i\in\Delta} \subseteq (L^X)^E$.

The triplet (X,τ,τ^*) is called an intuitionistic supra fuzzy soft topological space. An intuitionistic supra fuzzy soft topology (τ,τ^*) is called an intuitionistic fuzzy soft topology on X iff

(IT) $\tau_k(f\sqcap g)\geq \tau_k(f)\wedge\tau_k(g)$ and $\tau^*_k(f\sqcap g)\leq \tau^*_k(f)\vee\tau^*_k(g)$, for each $f,g\in(L^X)^E$.

The triplet (X,τ,τ^*) is called an intuitionistic fuzzy soft topological space. The values $\tau_k(f)$ and $\tau^*_k(f)$ denote the degree of openness and the degree of nonopenness of $f\in(L^X)^E$ with respect to $k\in K$.

The triplet $(X,(\tau,\tau^*),(\mathcal{U},\mathcal{U}^*))$ is called an intuitionistic fuzzy soft bitopological space where (τ,τ^*) and $(\mathcal{U},\mathcal{U}^*)$ are intuitionistic fuzzy soft topologies on X.

Let (τ,τ^*) and $(\mathcal{U},\mathcal{U}^*)$ be intuitionistic supra fuzzy soft topologies on X. We say that (τ,τ^*) is finer than $(\mathcal{U},\mathcal{U}^*)$ (or $(\mathcal{U},\mathcal{U}^*)$ is coarser than (τ,τ^*)) iff $\mathcal{U}_k(f)\leq\tau_k(f)$ and $\mathcal{U}^*_k(f)\geq\tau^*_k(f)$ for each $k\in K, f\in(L^X)^E$.

Definition 2.2 A mapping $\mathcal{C}: K\times(L^X)^E\times M_0\longrightarrow(L^X)^E$ is called an intuitionistic supra fuzzy soft closure operator on X if and only if $\mathcal{C}$ satisfies the following conditions, for each $k\in K$, $r,s\in M_0$ (where $M_0=M-\{0_M\}$) and $f,g\in(L^X)^E$.

(C1) $\mathcal{C}(k,0_X,r)=0_X$.

(C2) $f\sqsubseteq\mathcal{C}(k,f,r)$.

(C3) $\mathcal{C}(k,f,r)\sqcup\mathcal{C}(k,g,r)\sqsubseteq\mathcal{C}(k,f\sqcup g,r)$.

(C4) $\mathcal{C}(k,f,r)\sqsubseteq\mathcal{C}(k,f,s)$ if $r\leq s$.

(C5) $\mathcal{C}(k,\mathcal{C}(k,f,r),r)=\mathcal{C}(k,f,r)$.

The pair $(X,\mathcal{C})$ is called an intuitionistic supra fuzzy soft closure space. The intuitionistic supra fuzzy soft closure space $(X,\mathcal{C})$ is called an intuitionistic fuzzy soft closure space iff

(C) $\mathcal{C}(k,f,r)\sqcup\mathcal{C}(k,g,r)=\mathcal{C}(k,f\sqcup g,r)$, for each $f,g\in(L^X)^E$.

Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be intuitionistic supra fuzzy soft closure operators on X. We say that $\mathcal{C}^1$ is finer than $\mathcal{C}^2$ (or $\mathcal{C}'2$ is coarser than $\mathcal{C}^1$) iff $\mathcal{C}^1(k,f,r)\sqsubseteq\mathcal{C}^2(k,f,r)$ for each $k\in K, f\in(L^X)^E$ and $r\in M_0$.

Theorem 2.3 Let (X,τ,τ^*) be an intuitionistic (resp., intuitionistic supra) fuzzy soft topological space. Then for each $k\in K, r\in M_0$ and $f\in(L^X)^E$, we define an operator $\mathcal{C}_{\tau,\tau^*}: K\times(L^X)^E\times M_0\to(L^X)^E$ as follows:

$$\mathcal{C}_{\tau,\tau^*}(k,f,r)=\sqcap\{g\in(L^X)^E\mid f\sqsubseteq g,\tau_k(g'^*_k(g')\leq r'\}.$$

Then $(X,\mathcal{C}_{\tau,\tau^*})$ is an intuitionistic (resp., intuitionistic supra) fuzzy soft closure space.

Proof. Let (X,τ,τ^*) be an intuitionistic supra fuzzy soft topological space. Then (C1), (C2) and (C4) are trivial from the definition of $\mathcal{C}_{\tau,\tau^*}$.

(C3) Since $f,g\sqsubseteq f\sqcup g$, we have $\mathcal{C}_{\tau,\tau^*}(k,f,r)\sqsubseteq\mathcal{C}_{\tau,\tau^*}(k,f\sqcup g,r)$ and $\mathcal{C}_{\tau,\tau^*}(k,g,r)\sqsubseteq\mathcal{C}_{\tau,\tau^*}(k,f\sqcup g,r)$. Hence, $\mathcal{C}_{\tau,\tau^*}(k,f,r)\sqcup\mathcal{C}_{\tau,\tau^*}(k,g,r)\sqsubseteq\mathcal{C}_{\tau,\tau^*}(k,f\sqcup g,r)$.

(C5) Suppose that there exist $f\in(L^X)^E, e\in E, x\in X$ and $r\in M_0$ such that

$$\mathcal{C}_{\tau,\tau^*}(k,\mathcal{C}_{\tau,\tau^*}(k,f,r),r)_e(x)\not\leq\mathcal{C}_{\tau,\tau^*}(k,f,r)_e(x).$$

By the definition of $\mathcal{C}_{\tau,\tau^*}(k,f,r)$, there exists $g \in (L^X)^E$ with $g \sqsupseteq f$ and $\tau_k(g') \geq r$, $\tau^*_k(g') \leq r'$ such that $\mathcal{C}_{\tau,\tau^*}(k, \mathcal{C}_{\tau,\tau^*}(k,f,r),r)_e(x) \not\sqsubseteq g_e(x)$ and $g_e(x) \sqsupseteq \mathcal{C}_{\tau,\tau^*}(k,f,r)_e(x)$.

On the other hand $\mathcal{C}_{\tau,\tau^*}(k,f,r) \sqsubseteq g$ and $\tau_k(g'^*_k(g') \leq r'$, by the definition of

$$\mathcal{C}_{\tau,\tau^*}(k, \mathcal{C}_{\tau,\tau^*}(k,f,r),r),$$

we have $\mathcal{C}_{\tau,\tau^*}(k, \mathcal{C}_{\tau,\tau^*}(k,f,r),r) \sqsubseteq g$.

It is a contradiction. Thus, $\mathcal{C}_{\tau,\tau^*}(k, \mathcal{C}_{\tau,\tau^*}(k,f,r),r) = \mathcal{C}_{\tau,\tau^*}(k,f,r)$. Hence $\mathcal{C}_{\tau,\tau^*}$ is an intuitionistic supra fuzzy soft closure operator on X.

Let (X,τ,τ^*) be an intuitionistic fuzzy soft topological space. From (C2), we have

$f \sqsubseteq \mathcal{C}_{\tau,\tau^*}(k,f,r), \tau_k((\mathcal{C}_{\tau,\tau^*}(k,f,r))') \geq r$ and $\tau^*_k((\mathcal{C}_{\tau,\tau^*}(k,f,r))') \leq r'$.

$g \sqsubseteq \mathcal{C}_{\tau,\tau^*}(k,g,r), \tau_k((\mathcal{C}_{\tau,\tau^*}(k,g,r))') \geq r$ and $\tau^*_k((\mathcal{C}_{\tau,\tau^*}(k,g,r))') \leq r'$.

It implies $f \sqcup g \sqsubseteq \mathcal{C}_{\tau,\tau^*}(k,f,r) \sqcup \mathcal{C}_{\tau,\tau^*}(k,g,r)$ such that

$$\begin{aligned}
\tau_k((\mathcal{C}_{\tau,\tau^*}(k,f,r) \sqcup \mathcal{C}_{\tau,\tau^*}(k,g,r))') &= \tau_k(\mathcal{C}_{\tau,\tau^*}(k,f,r)' \sqcap \mathcal{C}_{\tau,\tau^*}(k,g,r)') \\
&\geq \tau_k(\mathcal{C}_{\tau,\tau^*}(k,f,r)') \wedge \tau_k(\mathcal{C}_{\tau,\tau^*}(k,g,r)') \geq r. \\
\tau^*_k((\mathcal{C}_{\tau,\tau^*}(k,f,r) \sqcup \mathcal{C}_{\tau,\tau^*}(k,g,r))') &= \tau^*_k(\mathcal{C}_{\tau,\tau^*}(k,f,r)' \sqcap \mathcal{C}_{\tau,\tau^*}(k,g,r)') \\
&\leq \tau^*_k(\mathcal{C}_{\tau,\tau^*}(k,f,r)'^*_k(\mathcal{C}_{\tau,\tau^*}(k,g,r)') \leq r'.
\end{aligned}$$

Hence, $\mathcal{C}_{\tau,\tau^*}(k,f,r) \sqcup \mathcal{C}_{\tau,\tau^*}(k,g,r) \sqsupseteq \mathcal{C}_{\tau,\tau^*}(k,f \sqcup g,r)$. Therefore, $\mathcal{C}_{\tau,\tau^*}$ is an intutionistic fuzzy soft closure operator on X. □

Theorem 2.4 Let $(X,\mathcal{C})$ be an intutionistic (resp., intuitionstic supra) fuzzy soft closure space. Define the mappings $\tau_{\mathcal{C}}, \tau^*_{\mathcal{C}} : K \to M^{(L^X)^E}$ on X by follows:

$$(\tau_{\mathcal{C}})_k(f) = \bigvee\{r \in M_0 \mid \mathcal{C}(k,f',r) = f'\},$$

$$(\tau^*_{\mathcal{C}})_k(f) = \bigwedge\{r' \in M_0 \mid \mathcal{C}(k,f',r) = f'\}.$$

Then the following properties are satisfied:

(1) If M is an order dense chain, (τ_C, τ^*_C) is an intuitionistic (resp., intuitionistic supra) fuzzy soft topology on X.

(2) $\mathcal{C}_{\tau_{\mathcal{C}},\tau^*_{\mathcal{C}}}$ is finer than $\mathcal{C}$.

Proof. (IS1) It is obvious from the definition.

(IS2) Let $(X,\mathcal{C})$ be an intuitionistic supra fuzzy soft closure space. Since for all $r \in M_0, k \in K, \mathcal{C}(k,0_X,r) = 0_X$ and $\mathcal{C}(k,1_X,r) = 1_X$, we have (IS2).

(IS2) Suppose that there exists $f = \bigsqcup_{i\in\Gamma} f_i \in (L^X)^E$ such that $(\tau_{\mathcal{C}})_k(f) \not\geq \bigwedge_{i\in\Gamma}(\tau_{\mathcal{C}})_k(f_i)$ and $(\tau^*_{\mathcal{C}})_k(f) \not\leq \bigvee_{i\in\Gamma}(\tau^*_{\mathcal{C}})_k(f_i)$. Since M is an order dense chain, there exists $r_0 \in M_0$ such that $(\tau_{\mathcal{C}})_k(f) < r_0 < \bigwedge_{i\in\Gamma}(\tau_{\mathcal{C}})_k(f_i)$ and $(\tau^*_{\mathcal{C}})_k(f) > r'_0 > \bigvee_{i\in\Gamma}(\tau^*_{\mathcal{C}})_k(f_i)$.

For all $i \in \Gamma$, there exist $r_i \in M_0$ with $\mathcal{C}(k,f'_i,r_i) = f'_i$ such that $r_0 < r_i \leq (\tau_{\mathcal{C}})_k(f_i)$ and $r'_0 > r'_i \geq (\tau^*_{\mathcal{C}})_k(f_i)$.

On the other hand, since $\mathcal{C}(k,f'_i,r_0) \sqsubseteq \mathcal{C}(k,f'_i,r_i) = f'_i$, by (C2) of Definition 2.9, we have $\mathcal{C}(k,f'_i,r_0) = f'_i$. It implies for all $i \in \Gamma$, $\mathcal{C}(k,f',r_0) \sqsubseteq \mathcal{C}(k,f'_i,r_0) = f'_i$. It follows that

$\mathcal{C}(k, f', r_0) \sqsubseteq \sqcap_{i\in\Gamma} f'_i = f'$. Thus, $\mathcal{C}(k, f', r_0) = f'$, that is $(\tau_{\mathcal{C}})_k(f) \geq r_0$ and $(\tau^*_{\mathcal{C}})_k(f) \leq r'_0$. It is a contradiction. Hence, $(\tau_{\mathcal{C}}, \tau^*_{\mathcal{C}})$ is an intuitionistic supra fuzzy soft topology on X.

Let $(X, \mathcal{C})$ be an intuitionistic fuzzy soft closure space. Suppose there exist $f_1, f_2 \in (L^X)^E$ such that $(\tau_{\mathcal{C}})_k(f_1 \sqcap f_2) \not\geq (\tau_{\mathcal{C}})_k(f_1) \wedge (\tau_{\mathcal{C}})_k(f_2)$ and $(\tau^*_{\mathcal{C}})_k(f_1 \sqcap f_2) \not\leq (\tau^*_{\mathcal{C}})_k(f_1) \vee (\tau^*_{\mathcal{C}})_k(f_2)$.

Since M is an order-dense chain, there exists $r \in M_0$ such that $(\tau_{\mathcal{C}})_k(f_1 \sqcap f_2) < r < (\tau_{\mathcal{C}})_k(f_1) \wedge (\tau_{\mathcal{C}})_k(f_2)$ and $(\tau^*_{\mathcal{C}})_k(f_1 \sqcap f_2) > r'^*_{\mathcal{C}})_k(f_1) \vee (\tau^*_{\mathcal{C}})_k(f_2)$.

For each $i \in \{1, 2\}$, there exist $r_i \in M_0$ with $\mathcal{C}(k, f'_i, r_i) = f'_i$ such that $r < r_i \leq (\tau_{\mathcal{C}})_k(f_i)$ and $(\tau^*_{\mathcal{C}})_k(f_i) \leq r'_i < r'$.

On the other hand since $\mathcal{C}(k, f'_i, r) = f'_i$ from (C2) and (C4) of Definition 2.2, we have for each $i \in \{1, 2\}$, $\mathcal{C}(kif'_1 \sqcup f'_2, r) = f'_1 \sqcup f'_2$. It follows $(\tau_{\mathcal{C}})_k(f_1 \sqcap f_2) \geq r$ and $(\tau^*_{\mathcal{C}})_k(f_1 \sqcap f_2) \leq r'$. It is a contradiction. Hence, for all $f, g \in (L^X)^E, k \in K$, $(\tau_{\mathcal{C}})_k(f_1 \sqcap f_2) \geq (\tau_{\mathcal{C}})_k(f_1) \wedge (\tau_{\mathcal{C}})_k(f_2)$ and $(\tau^*_{\mathcal{C}})_k(f_1 \sqcap f_2) \leq (\tau^*_{\mathcal{C}})_k(f_1) \vee (\tau^*_{\mathcal{C}})_k(f_2)$.

Therefore, $(\tau_{\mathcal{C}}, \tau^*_{\mathcal{C}})$ is an intuitionistic fuzzy soft topology on X.

(2) Since $f \sqsubseteq \mathcal{C}(k, f, r), (\tau_{\mathcal{C}})_k(\mathcal{C}(k, f, r)') \geq r$ and $(\tau^*_{\mathcal{C}})_k(\mathcal{C}(k, f, r)') \leq r'$ from (C5) of Definition 2.2, we have $\mathcal{C}_{\tau_{\mathcal{C}},\tau^*_{\mathcal{C}}}(k, f, r) \sqsubseteq \mathcal{C}(k, f, r)$. Thus, $\mathcal{C}_{\tau_{\mathcal{C}},\tau^*_{\mathcal{C}}}$ is finer than $\mathcal{C}$. □

3 Intuitionistic fuzzy soft bitopological spaces

Theorem 3.1 Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be an intuitionistic fuzzy soft bitopological space. For each $k \in K, r \in M_0$ and $f \in (L^X)^E$, define a mapping $\mathcal{C}_{12} : K \times (L^X)^E \times M_0 \to (L^X)^E$ as follows:

$$\mathcal{C}_{12}(k, f, r) = \mathcal{C}_{\tau_1,\tau^*_1}(k, f, r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, f, r).$$

Then $(X, \mathcal{C}_{12})$ is an intuitionistic supra fuzzy soft closure space.

Proof. (C1), (C2) and (C4) are easily proved.

(C3) We prove it from the followings: for all $f, g \in (L^X)^E, k \in K$ and $r \in M_0$,

$$\begin{aligned}
&\mathcal{C}_{12}(k, f, r) \sqcup \mathcal{C}_{12}(k, g, r)\\
&= (\mathcal{C}_{\tau_1,\tau^*_1}(k, f, r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, f, r)) \sqcup (\mathcal{C}_{\tau_1,\tau^*_1}(k, g, r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, g, r))\\
&\leq (\mathcal{C}_{\tau_1,\tau^*_1}(k, f, r) \sqcup \mathcal{C}_{\tau_1,\tau^*_1}(k, g, r)) \sqcap (\mathcal{C}_{\tau_2,\tau^*_2}(k, f, r) \sqcup \mathcal{C}_{\tau_2,\tau^*_2}(k, g, r))\\
&= \mathcal{C}_{\tau_1,\tau^*_1}(k, f \sqcup g, r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, f \sqcup g, r)\\
&= \mathcal{C}_{12}(k, f \sqcup g, r).
\end{aligned}$$

(C5) We prove it from the followings: for all $f \in (L^X)^E, k \in K$ and $r \in M_0$,

$$\begin{aligned}
&\mathcal{C}_{12}(k, \mathcal{C}_{12}(k, f, r), r)\\
&= \mathcal{C}_{\tau_1,\tau^*_1}(k, \mathcal{C}_{12}(k, f, r), r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, \mathcal{C}_{12}(k, f, r), r)\\
&\leq \mathcal{C}_{\tau_1,\tau^*_1}(k, \mathcal{C}_{\tau_1,\tau^*_1}(k, f, r), r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, \mathcal{C}_{\tau_2,\tau^*_2}(k, f, r), r)\\
&= \mathcal{C}_{\tau_1,\tau^*_1}(k, f, r) \sqcap \mathcal{C}_{\tau_2,\tau^*_2}(k, f, r)\\
&= \mathcal{C}_{12}(k, f, r).
\end{aligned}$$

Lemma 3.2 Let (X, τ, τ^*) be an intuitionistic (resp., intuitionistic supra) fuzzy soft topological space. For each $k \in K, r \in M_0$ and $f \in (L^X)^E$, define a mapping $\mathcal{I}_{\tau,\tau^*} : K \times (L^X)^E \times M_0 \to (L^X)^E$ as follows:

$$\mathcal{I}_{\tau,\tau^*}(k, f, r) = \bigsqcup\{g \in (L^X)^E \mid g \sqsubseteq f, \tau_k(f) \geq r, \tau_k^*(f) \leq r'\}.$$

Then $\mathcal{I}_{\tau,\tau^*}(k, f', r) = (\mathcal{C}_{\tau,\tau^*}(k, f, r))'$.

Proof. For all $f \in (L^X)^E, k \in K$ and $r \in M_0$, following equality is valid:

$$\begin{aligned}(\mathcal{C}_{\tau,\tau^*}(k, f, r))' &= (\sqcap\{g \in (L^X)^E \mid g \sqsupseteq f, \tau_k(g') \geq r, \tau_k^*(g') \leq r'\})' \\ &= \sqcup\{g' \mid g \sqsupseteq f, \tau_k(g') \geq r \text{ and } \tau_k^*(g') \leq r'\} \\ &= \sqcup\{g' \mid g' \subseteq f', \tau_k(g_k'^*(g') \leq r'\} \\ &= \mathcal{I}_{\tau,\tau^*}(k, f', r).\end{aligned}$$

This completes the proof. □

Theorem 3.3 Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be an intuitionistic fuzzy soft bitopological space. For each $f \in (L^X)^E, k \in K$ and $r \in M_0$, define a mapping $\mathcal{I}_{12} : K \times (L^X)^E \times M_0 \to (L^X)^E$ as follows:

$$\mathcal{I}_{12}(k, f, r) = \mathcal{I}_{\tau_1,\tau_1^*}(k, f, r) \sqcup \mathcal{I}_{\tau_2,\tau_2^*}(k, f, r).$$

Then $\mathcal{I}_{12}(k, f', r) = (\mathcal{C}_{12}(k, f, r))'$.

Proof. For all $f \in XE, k \in K$ and $r \in M_0$, the following equality is valid:

$$\begin{aligned}(\mathcal{C}_{12}(k, f, r))' &= (\mathcal{C}_{\tau_1,\tau_1^*}(k, f, r) \sqcap \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r))' \\ &= (\mathcal{C}_{\tau_1,\tau_1^*}(k, f, r))' \sqcup (\mathcal{C}_{\tau_2,\tau_2^*}(k, f, r))' \\ &= \mathcal{I}_{\tau_1,\tau_1^*}(k, f', r) \sqcup \mathcal{I}_{\tau_2,\tau_2^*}(k, f', r) \\ &= \mathcal{I}_{12}(k, f', r).\end{aligned}$$

This completes the proof. □

From Theorem 2.4 and Theorem 3.3, we obtain the following corollary:

Corollary 3.4 Let $(X, \mathcal{C}_{12})$ be an intuitionistic supra fuzzy soft closure space. Define mappings $\tau_{\mathcal{C}_{12}}, \tau^*_{\mathcal{C}_{12}} : K \to M^{(L^X)^E}$ by follows:

$$(\tau_{\mathcal{C}_{12}})_k(f) = \bigvee\{r \in M_0 \mid \mathcal{C}_{12}(k, f', r) = f'\} = \bigvee\{r \in M_0 \mid \mathcal{I}_{12}(k, f, r) = f\}.$$

$$(\tau^*_{\mathcal{C}_{12}})_k(f) = \bigwedge\{r' \in M_0 \mid \mathcal{I}_{12}(k, f, r) = f\}.$$

Then $(\tau_{\mathcal{C}_{12}}, \tau^*_{\mathcal{C}_{12}})$ is an intuitionistic supra fuzzy soft topology on X.

Lemma 3.5 Let (X, τ, τ^*) be an intuitionistic fuzzy soft topological space. For each $f \in (L^X)^E, k \in K$ and $r \in M_0$, we have

$$\bigsqcup_{s \not\geq r} \mathcal{C}_{\tau,\tau^*}(k, f, s) = \mathcal{C}_{\tau,\tau^*}(k, f, r).$$

Proof. Straightforward. □

Lemma 3.6 Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be an intuitionistic fuzzy soft bitopological space. For each $f \in (L^X)^E, k \in K$ and $r \in M_0$, we have

$$\bigsqcup_{s \not\geq r} \mathcal{C}_{12}(k, f, s) = \mathcal{C}_{12}(k, f, r).$$

Proof. Since $\mathcal{C}_{\tau_1,\tau_1^*}(k, f, s) \sqsubseteq \mathcal{C}_{\tau_1,\tau_1^*}(k, f, r)$ and $\mathcal{C}_{\tau_2,\tau_2^*}(k, f, s) \sqsubseteq \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r)$ for all $s \not\geq r$,

$$\bigsqcup_{s \not\geq r} (\mathcal{C}_{\tau_1,\tau_1^*}(k, f, s) \sqcap \mathcal{C}_{\tau_2,\tau_2^*}(k, f, s)) \sqsubseteq \mathcal{C}_{\tau_1,\tau_1^*}(k, f, r) \sqcap \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r).$$

It follows: $\bigsqcup_{s \not\geq r} \mathcal{C}_{12}(k, f, s) \sqsubseteq \mathcal{C}_{12}(k, f, r)$.

Suppose there exist $x \in X$ and $e \in E$ such that

$$\bigvee_{s \not\geq r} (\mathcal{C}_{\tau_1,\tau_1^*}(k, f, s) \wedge \mathcal{C}_{\tau_2,\tau_2^*}(k, f, s))_e(x) \not\geq \mathcal{C}_{\tau_1,\tau_1^*}(k, f, r)_e(x) \wedge \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r)_e(x).$$

Since $\bigsqcup_{s_i \not\geq r} \mathcal{C}_{\tau_1,\tau_1^*}(k, f, s_i) = \mathcal{C}_{\tau_1,\tau_1^*}(k, f, r)$ for $i \in \{1, 2\}$, there exist r_i with $r_i \not\geq r$ for $i \in \{1, 2\}$ such that $\bigvee_{s \not\geq r} (\mathcal{C}_{\tau_1,\tau_1^*}(k, f, s) \wedge \mathcal{C}_{\tau_2,\tau_2^*}(k, f, s))_e(x) \not\geq \mathcal{C}_{\tau_1,\tau_1^*}(k, f, r_1)_e(x) \wedge \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r_2)_e(x)$.

Put $r^* = r_1 \vee r_2$. Then $r^* \not\geq r$ and

$$\begin{aligned}
&\mathcal{C}_{\tau_1,\tau_1^*}(k, f, r_1)_e(x) \wedge \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r_2)_e(x) \\
&\leq \mathcal{C}_{\tau_1,\tau_1^*}(k, f, r^*)_e(x) \wedge \mathcal{C}_{\tau_2,\tau_2^*}(k, f, r^*)_e(x) \\
&\leq \bigvee_{s \not\geq r} \mathcal{C}_{12}(k, f, s)_e(x).
\end{aligned}$$

It is a contradiction. Hence, $\bigsqcup_{s \not\geq r} \mathcal{C}_{12}(k, f, s) \sqsupseteq \mathcal{C}_{12}(k, f, r)$. □

Theorem 3.7 Let $(X, (\tau^1, \tau^{1*}), (\tau^2, \tau^{2*}))$ be an intuitionistic fuzzy soft bitopological space. Let $(X, \mathcal{C}_{12})$ be an intuitionistic supra fuzzy soft closure space. Define mappings $\tau_s, \tau_s^* : K \to M^{(L^X)^E}$ by follows:

$$(\tau_s)_k(f) = \bigvee \{\tau_k^1(f_1) \wedge \tau_k^2(f_2) \mid f = f_1 \sqcup f_2\},$$

where $\vee$ is taken over all families $\{f_1, f_2 \mid f = f_1 \sqcup f_2\}$, and

$$(\tau_s^*)_k(f) = \bigwedge \{\tau_k^{1*}(f_1) \vee \tau_k^{2*}(f_2) \mid f = f_1 \sqcup f_2\},$$

where $\wedge$ is taken over all families $\{f_1, f_2 \mid f = f_1 \sqcup f_2\}$, Then the following are satisfied:
(1) If M is an order dense chain, then $(\tau_s, \tau_s^*) = (\tau_{\mathcal{C}_{12}}, \tau_{\mathcal{C}_{12}}^*)$ is the coarsest intuitionistic supra fuzzy soft topology on X which is finer than (τ^1, τ^{1*}) and (τ^2, τ^{2*}).
(2) $\mathcal{C}_{12} = \mathcal{C}_{\tau_s,\tau_s^*} = \mathcal{C}_{\tau_{\mathcal{C}_{12}},\tau_{\mathcal{C}_{12}}^*}$.
Proof. (1) Suppose that there exist $f \in (L^X)^E$ and $k \in K$ such that $(\tau_{\mathcal{C}_{12}})_k(f) \not\leq (\tau_s)_k(f)$ and $(\tau_{\mathcal{C}_{12}}^*)_k(f) \not\geq (\tau_s^*)_k(f)$.

By the definition of $(\tau_{\mathcal{C}_{12}}, \tau^*_{\mathcal{C}_{12}})$ from Corollary 3.4, there exists $r_0 \in M_0$ with $\mathcal{C}_{12}(k, f', r_0) = f'$ such that $(\tau_{\mathcal{C}_{12}})_k(f) \geq r_0 > (\tau_s)_k(f)$ and $(\tau^*_{\mathcal{C}_{12}})_k(f) \leq r'_0 < (\tau^*_s)_k(f)$.

On the other hand, since $\mathcal{C}_{12}(k, f', r_0) = f'$, we have

$$\begin{aligned} f = \mathcal{C}_{12}(k, f', r_0)' &= (\mathcal{C}_{\tau^1,\tau^{1*}}(k, f', r_0) \sqcap \mathcal{C}_{\tau^2,\tau^{2*}}(k, f', r_0))' \\ &= (\mathcal{C}_{\tau^1,\tau^{*1}}(k, f', r_0))' \sqcup (\mathcal{C}_{\tau^2,\tau^{2*}}(k, f, r_0))' \\ &= \mathcal{I}_{\tau^1,\tau^{1*}}(k, f, r_0) \sqcup \mathcal{I}_{\tau^2,\tau^{2*}}(k, f, r_0). \end{aligned}$$

Since

$$\begin{aligned} \tau^1_k(\mathcal{I}_{\tau^1,\tau^{1*}}(k, f, r_0)) &= \tau^1_k(\bigsqcup\{g \mid g \sqsubseteq f, \tau^1_k(g) \geq r_0, \tau^{1*}_k(g) \leq r'_0\}) \\ &\geq \bigwedge\{\tau^1_k(g) \mid g \sqsubseteq f, \tau^1_k(g) \geq r_0, \tau^{1*}_k(g) \leq r'_0\} \geq r_0. \\ \tau^{1*}_k(\mathcal{I}_{\tau^1,\tau^{1*}}(k, f, r_0)) &= \tau^{1*}_k(\bigsqcup\{g \mid g \sqsubseteq f, \tau^1_k(g) \geq r_0, \tau^{1*}_k(g) \leq r'_0\}) \\ &\leq \bigvee\{\tau^{*1}_k(g) \mid g \sqsubseteq f, \tau^1_k(g) \geq r_0, \tau^{1*}_k(g) \leq r'_0\} \leq r'_0. \end{aligned}$$

and, similarly $\tau^2_k(\mathcal{I}_{\tau^2,\tau^{2*}}(k, f, r_0)) \geq r_0$ and $\tau^{2*}_k(\mathcal{I}_{\tau^2,\tau^{2*}}(k, f, r_0)) \leq r'_0$, so we have $(\tau_s)_k(f) \geq r_0$ and $(\tau^*_s)_k(f) \leq r'_0$. It is a contradiction. Hence, $\tau_{\mathcal{C}_{12}} \leq \tau_s$ and $\tau^*_{\mathcal{C}_{12}} \geq \tau^*_s$.

Suppose that there exist $k \in K$ and $h \in (L^X)^E$ such that $(\tau_{\mathcal{C}_{12}})_k(h) \not\geq (\tau_s)_k(h)$ and $(\tau^*_{\mathcal{C}_{12}})_k(h) \not\leq (\tau^*_s)_k(h)$.

Since M is an order-dense chain, there exist $r_1 \in M_0$ such that $(\tau_{\mathcal{C}_{12}})_k(h) < r_1 < (\tau_s)_k(h)$ and $(\tau^*_{\mathcal{C}_{12}})_k(h) > r'_1 > (\tau^*_s)_k(h)$.

By the definition of (τ_s, τ^*_s), there exist $h_1, h_2 \in (L^X)^E$ with $h = h_1 \sqcup h_2$ such that $(\tau_{\mathcal{C}_{12}})_k(h) < r_1 \leq \tau^1_k(h_1) \sqcap \tau^2_k(h_2) \leq (\tau_s)_k(h)$, $(\tau^*_{\mathcal{C}_{12}})_k(h) > r'_1 \geq \tau^{1*}_k(h_1) \sqcup \tau^{2*}_k(h_2) \geq (\tau^*_s)_k(h)$.

On the other hand, since $r_1 \leq \tau^i_k(h_i)$ and $r'_1 \geq \tau^{i*}_k(h_i)$ for all $i = 1, 2$, we get $\mathcal{C}_{\tau^i,\tau^{i*}}(k, h'_i, r_1) = h'_i$. It follows that

$$\begin{aligned} \mathcal{C}_{12}(k, h', r_1) &= \mathcal{C}_{\tau^1,\tau^{1*}}(k, h', r_1) \sqcap \mathcal{C}_{\tau^2,\tau^{2*}}(k, h', r_1) \\ &\sqsubseteq \mathcal{C}_{\tau^1,\tau^{1*}}(k, h'_1, r_1) \sqcap \mathcal{C}_{\tau^2,\tau^{2*}}(k, h'_2, r_1) \\ &= h'_1 \sqcap h'_2 = h'. \end{aligned}$$

Hence, $(\tau_{\mathcal{C}_{12}})_k(h) \geq r_1$ and $(\tau^*_{\mathcal{C}_{12}})_k(h) \leq r'_1$. It is a contradiction. Therefore, $\tau_{\mathcal{C}_{12}} \geq \tau_s$ and $\tau^*_{\mathcal{C}_{12}} \leq \tau^*_s$. Thus $(\tau_{\mathcal{C}_{12}}, \tau^*_{\mathcal{C}_{12}}) = (\tau_s, \tau^*_s)$ is an intuitionistic supra fuzzy soft topology on X from Corollary 3.4.

Finally we will show that (τ_s, τ^*_s) is the coarsest intuitionistic supra fuzzy soft topology on X which is finer than (τ^1, τ^{1*}) and (τ^2, τ^{2*}).

For $f = f \sqcup 0_X$ and $i \neq j \in \{1, 2\}$, we have $(\tau_s)_k(f) \geq \tau^i_k(f) \wedge \tau^i_k(0_X) = \tau^i_k(f)$ and $(\tau^*_s)_k(f) \leq \tau^{j*}_k(f) \vee \tau^{j*}_k(0_X) = \tau^{j*}_k(f)$.

If $(\mathcal{U}, \mathcal{U}^*)$ is the intuitionistic supra fuzzy soft topology on X which is finer than (τ^1, τ^{1*}) and (τ^2, τ^{2*}) for every family $\{f_1, f_2\}$ such that $f = f_1 \sqcup f_2$, we have

$$\tau^1_k(f_1) \wedge \tau^2_k(f_2) \leq \mathcal{U}_k(f_1) \wedge \mathcal{U}_k(f_2) \leq \mathcal{U}_k(f_1 \sqcup f_2),$$

$$\tau^{1*}_k(f_1) \vee \tau^{2*}_k(f_2) \geq \mathcal{U}^*_k(f_1) \vee \mathcal{U}^*_k(f_2) \geq \mathcal{U}^*_k(f_1 \sqcup f_2).$$

By the definition of (τ_s, τ^*_s), we get $(\tau_s)_k(f) \leq \mathcal{U}_k(f)$ and $(\tau^*_s)_k(f) \geq \mathcal{U}^*_k(f)$ for all $f \in (L^X)^E$.

(2) Suppose there exist $e \in E, x \in X, r \in M_0$ and $f \in (L^X)^E$ such that $\mathcal{C}_{12}(k, f, r)_e(x) \not\geq \mathcal{C}_{\tau_s,\tau^*_s}(k, f, r)_e(x)$. By the definition of $\mathcal{C}_{12}$, for $i \in \{1, 2\}$, there exist $h_i \in (L^X)^E$ with $h_i \sqsupseteq f, \tau^i_k(h'_i) \geq r$ and $\tau^{i*}_k(h'_i) \leq r'$ such that $\mathcal{C}_{12}(k, f, r)_e(x) \leq (h_1 \sqcap h_2)_e(x) < \mathcal{C}_{\tau_s,\tau^*_s}(k, f, r)_e(x)$. Since $f \sqsubseteq h_1 \sqcap h_2$ and $(\tau_s)_k((h_1 \sqcap h_2)') = (\tau_s)_k(h'_1 \sqcup h'^1_2 \geq \tau_k(h'_1) \wedge \tau_k(h'_2) \geq r$,

$(\tau_s^*)_k((h_1 \sqcap h_2)') = (\tau_s^*)_k(h_1' \sqcup h_2'^1 \geq \tau_k^*(h_1'^*) \wedge \tau_k(h_2'^*) \geq r$, by the definition of $\mathcal{C}_{\tau_s,\tau_s^*}$, it is clear that $\mathcal{C}_{\tau_s,\tau_s^*}(k,f,r) \sqsubseteq h_1 \sqcap h_2$. It is a contradiction. Hence, $\mathcal{C}_{12} \leq \mathcal{C}_{\tau_s,\tau_s^*}$.

Suppose there exist $e \in E, x_1 \in X, k \in K, r_1 \in M_0$ and $w \in (L^X)^E$ such that

$$\mathcal{C}_{12}(k,w,r_1)_e(x_1) \not\leq \mathcal{C}_{\tau_s,\tau_s^*}(k,w,r_1)_e(x_1).$$

By the definition of $\mathcal{C}_{\tau_s,\tau_s^*}$ there exists $h \in (L^X)^E$ with $h \sqsupseteq w$, $(\tau_s)_k(h') \geq r_1$ and $(\tau_s^*)_k(h') \leq r_1'$ such that $\mathcal{C}_{12}(k,w,r_1)_e(x_1) > h_e(x_1) \geq \mathcal{C}_{\tau_s,\tau_s^*}(k,w,r_1)_e(x_1)$. On the other hand, since $(\tau_s)_k(h') \geq r_1$ and $(\tau_s^*)_k(h') \leq r_1'$, by the definition of (τ_s,τ_s^*), for all $s \not\geq r_1$, there exist $g_1, g_2 \in (L^X)^E$ with $h' = g_1 \sqcup g_2$ such that $(\tau_s)_k(h_k'^1(g_1) \wedge \tau_k^2(g_2) > s$ and $(\tau_s^*)_k(h_k'^{1*}(g_1) \vee \tau_k^{2*}(g_2) < s'$.

It follows that for $i \in \{1,2\}, \mathcal{C}_{\tau^i,\tau^{i*}}(k,g_i',s) = g_i'$. Since $h \sqsubseteq g_i'$, for $i \in \{1,2\}$, $\mathcal{C}_{\tau^i,\tau^{i*}}(k,h,s)$ $\sqsubseteq \mathcal{C}_{\tau^i,\tau^{i*}}(k,g_i',s)$.

So for all $s \not\geq r_i$,

$$\begin{aligned}\mathcal{C}_{12}(k,h,s) &= \mathcal{C}_{\tau^1,\tau^{1*}}(k,h,s) \sqcap \mathcal{C}_{\tau^2,\tau^{2*}}(k,h,s)\\ &\sqsubseteq \mathcal{C}_{\tau^1,\tau^{1*}}(k,g_1',s) \sqcap \mathcal{C}_{\tau^2,\tau^{2*}}(k,g_2',s) = g_1' \sqcap g_2' = h.\end{aligned}$$

Thus, $\mathcal{C}_{12}(k,h,s) = h$ from (C2) of Definition 2.2. Hence by Lemma 3.6,

$$h = \bigsqcup_{s \not\geq r_1} \mathcal{C}_{12}(k,h,s) = \mathcal{C}_{12}(k,h,r_1).$$

Since $w \sqsubseteq h$, we have $\mathcal{C}_{12}(k,w,r_1) \sqsubseteq \mathcal{C}_{12}(k,h,r_1) = h$. It is a contradiction. Hence, $\mathcal{C}_{12} \geq \mathcal{C}_{\tau_s,\tau_s^*}$. □

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