21st ICIFS, 22–23 May 2017, Burgas, Bulgaria
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 23, 2017, No. 2, 44–54

# Averaging of intuitionistic fuzzy differential equations

**A. El Allaoui, S. Melliani, Y. Allaoui and L. S. Chadli**

LMACS, Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University
PO Box 523, 23000 Beni Mellal, Morocco

**Received:** 28 February 2017 **Accepted:** 19 April 2017

**Abstract:** In this paper, we shall prove and discuss averaging of intuitionistic fuzzy differential equations. The main results generalize previous ones in fuzzy sets theory.
**Keywords:** Intuitionistic fuzzy sets, Intuitionistic fuzzy differential equations.
**AMS Classification:** 03E72, 34A07.

## 1 Introduction

In 1983, K. Atanassov laid the foundation for the development of the theory of intuitionistic fuzzy sets [1–3]. This concept is a generalization of fuzzy theory introduced by L. Zadeh in 1965 [12].

In [6], O. Kaleva gave the existence and uniqueness for a solution of the fuzzy differential equation

$$x'(t) = f(t, x(t)),$$

In [5], S. Melliani *et al.* discussed the existence and uniqueness for a solution of the intuitionistic fuzzy differential equation

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0.$$

Several works made in the study of the averaging of fuzzy differential equations [7, 8, 11].

In this paper, we establish averaging of intuitionistic fuzzy differential equations in order to generalize the results stated for fuzzy differential equations.

Consider the following problem with a small parameter $\varepsilon$:

$$\begin{cases} u'(t) = f\left(\dfrac{t}{\varepsilon}, u(t)\right) \\ u(0) = u_0 \in IF. \end{cases}, \tag{1}$$

where $f : \mathbb{R}^+ \times U \longrightarrow IF, U \subseteq IF$ is an open subset and $\varepsilon > 0$ is a small parameter.

# 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

**Definition 1.** *We denote*

$$IF = \left\{(u, v) : \mathbb{R} \to [0, 1]^2 \; |\forall x \in \mathbb{R} \,/ 0 \leq u(x) + v(x) \leq 1\right\}$$

*where*

1. $(u, v)$ *is normal i.e there exists* $x_0, x_1 \in \mathbb{R}$ *such that* $u(x_0) = 1$ *and* $v(x_1) = 1$.
2. $u$ *is fuzzy convex and* $v$ *is fuzzy concave.*
3. $u$ *is upper semicontinuous and* $v$ *is lower semicontinuous*
4. $supp(u, v) = cl(\{x \in \mathbb{R} : v(x) < 1\})$ *is bounded.*

*For* $\alpha \in [0, 1]$ *and* $(u, v) \in IF$*, we define*

$$[(u, v)]^\alpha = \{x \in \mathbb{R} \mid v(x) \leq 1 - \alpha\}$$

*and*

$$[(u, v)]_\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$

**Remark 1.** *We can consider* $[(u, v)]_\alpha$ *as* $[u]^\alpha$ *and* $[(u, v)]^\alpha$ *as* $[1 - v]^\alpha$ *in the fuzzy case.*

**Definition 2.** *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1, 0), & x = 0 \\ (0, 1), & x \neq 0 \end{cases}$$

**Definition 3.** *Let* $(u, v)$ *,*$(u', v') \in IF$ *and* $\lambda \in \mathbb{R}$*, we define the addition by :*

$$((u, v) \oplus (u', v'))(z) = \left(\sup_{z=x+y} \min(u(x), u'(y)); \inf_{z=x+y} \max(v(x), v'(y))\right)$$

$$\lambda(u, v) = \begin{cases} (\lambda u, \lambda v) & if\ \lambda \neq 0 \\ 0_{(0,1)} & if\ \lambda = 0 \end{cases}$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space $IF$ as follows:

$$[(u,v)\oplus(z,w)]^{\alpha}=[(u,v)]^{\alpha}+[(z,w)]^{\alpha}$$

$$[\lambda(u,v)]^{\alpha}=\lambda[(u,v)]^{\alpha}$$

$$[(u,v)\oplus(z,w)]_{\alpha}=[(u,v)]_{\alpha}+[(z,w)]_{\alpha}$$

$$[\lambda(u,v)]_{\alpha}=\lambda[(u,v)]_{\alpha}$$

where $(u,v),(z,w)\in IF$ and $\lambda\in\mathbb{R}$.
We denote

$$[(u,v)]_l^+(\alpha)=\inf\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$

$$[(u,v)]_r^+(\alpha)=\sup\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$

$$[(u,v)]_l^-(\alpha)=\inf\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}$$

$$[(u,v)]_r^-(\alpha)=\sup\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}$$

**Remark 2.**

$$\big[(u,v)\big]_{\alpha}=\big[[(u,v)]_l^+(\alpha),[(u,v)]_r^+(\alpha)\big]$$
$$\big[(u,v)\big]^{\alpha}=\big[[(u,v)]_l^-(\alpha),[(u,v)]_r^-(\alpha)\big]$$

**Theorem 1.** *( [10]) let $\mathcal{M}=\{M_\alpha,\ M^\alpha\ :\ \alpha\in[0,1]\}$ be a family of subsets in $\mathbb{R}$ satisfying Conditions $(i)-(iv)$*

**i)** $\alpha\leq\beta\Rightarrow M_\beta\subset M_\alpha$ *and* $M^\beta\subset M^\alpha$

**ii)** $M_\alpha$ *and* $M^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$ *for each* $\alpha\in[0,1]$.

**iii)** *for any nondecreasing sequence $\alpha_i\to\alpha$ on $[0,1]$, we have $M_\alpha=\bigcap_i M_{\alpha_i}$ and $M^\alpha=\bigcap_i M^{\alpha_i}$.*

**iv)** *For each $\alpha\in[0,1]$, $M_\alpha\subset M^\alpha$ and define $u$ and $v$, by*

$$u(x)=\begin{cases}0 & if\ x\notin M_0\\ \sup\{\alpha\in[0,1]\ :\ x\in M_\alpha\} & if x\in M_0\end{cases}$$

$$v(x)=\begin{cases}1 & if\ x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^\alpha\} & if x\in M^0\end{cases}$$

*Then $(u,v)\in IF$.*

The space $IF$ is metrizable by the distance of the following form:

$$d_\infty((u,v),(z,w))=\frac{1}{4}\sup_{0<\alpha\leq1}|[(u,v)]_r^+(\alpha)-[(z,w)]_r^+(\alpha)|$$
$$+\frac{1}{4}\sup_{0<\alpha\leq1}|[(u,v)]_l^+(\alpha)-[(z,w)]_l^+(\alpha)|$$

$$+\frac{1}{4}\sup_{0<\alpha\leq 1}|[(u,v)]_r^-(\alpha)-[(z,w)]_r^-(\alpha)|$$
$$+\frac{1}{4}\sup_{0<\alpha\leq 1}|[(u,v)]_l^-(\alpha)-[(z,w)]_l^-(\alpha)|$$

where $|.|$ denotes the usual Euclidean norm in $\mathbb{R}$.

**Theorem 2.** *( [10]) $(IF, d_\infty)$ is a complete metric space.*

On $IF$, we define the H-difference [9] as follows: $u \ominus v$ has sense if there exists $w \in IF$ such that

$$u \ominus v = w \Leftrightarrow u = v + w.$$

**Definition 4.** *A function $f : I \longrightarrow IF$ is continuous at a point $t_0 \in I$ if,*

$$\forall \varepsilon > 0,\ \exists \eta > 0, \quad t \in I \quad |t - t_0| < \eta \Rightarrow d_\infty(f(t), f(t_0)) < \varepsilon.$$

*$f$ continuous on $I$ if it is continuous at every point $t_0 \in I$.*

**Definition 5.** *A function $f : I \times IF \longrightarrow IF$ is continuous at a point $(t_0, u_0) \in I \times IF$ if,*

$$\forall \varepsilon > 0,\ \exists \eta > 0, \quad (t,u) \in I \times IF \quad |t - t_0| < \eta \text{ and } d_\infty(u, u_0) < \eta \Rightarrow d_\infty(f(t,u), f(t_0, u_0)) < \varepsilon.$$

*$f$ continuous on $I \times IF$ if it is continuous at every point $(t_0, u_0) \in I$.*

**Definition 6.** *A function $f : I \times IF \longrightarrow IF$ is continuous in $u_0 \in IF$ uniformly with respect to $t \in I$ if, for any $u \in IF$*

$$\forall \varepsilon > 0,\ \exists \eta > 0, \quad u \in IF,\ d_\infty(u, u_0) < \eta \Rightarrow d_\infty(f(t,u), f(t_0, u_0)) < \varepsilon, \qquad \forall t \in I.$$

**Definition 7.** *A mapping $f : [a,b] \longrightarrow IF$ is said to be differentiable at $t_0$ if there exist $f'(t_0) \in IF$ such that the following limits:*

$$\lim_{h\to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} \text{ and } \lim_{h\to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h}$$

*exist and they are equal to $f'(t_0)$.*

**Theorem 3.** *( [5]) Let $f : I \longrightarrow IF$ be differentiable and $f'$ is integrable over $I$. Let $a \in I$, then, for each $t \in I$, we have*

$$f(t) = f(a) + \int_a^t f'(s)ds.$$

# 3 Main results

**Definition 8.** *A mapping $u : [0,a) \longrightarrow U$, $0 < a \leq \infty$, is called a solution of problem (1) if it is continuous, for all $t \in [0,a)$ and satisfies the integral equation*

$$u(t) = u_0 + \int_0^t f\left(\frac{s}{\varepsilon}, u(s)\right) ds.$$

**Definition 9.** *A mapping $u$ is called a maximal solution of problem (1) if there exists a maximal positive interval of definition $I$ of $u$, such that $u$ is a solution of (1) on $I$.*

We associate Eq.(1) with the averaging equation

$$\begin{cases} v'(t) = \bar{f}(v(t)) \\ v(0) = u_0. \end{cases} \tag{2}$$

Where the function $\bar{f} : U \longrightarrow IF$, is such that,

$$\bar{f}(u) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(s,u)ds, \qquad \forall u \in U,$$

with the metric $d_\infty$.

To establish our results, we introduce the following assumptions:

(i) the function $f : \mathbb{R}^+ \times U \longrightarrow IF$ is continuous;

(ii) the function $f$ is continuous in $u \in U$ uniformly with respect to $t \in \mathbb{R}^+$;

(iii) there exists a locally integrable function $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $M > 0$ such that

$$d_\infty\left(f(t,u), 0_{(1,0)}\right) \leq \varphi(t), \qquad \forall t \in \mathbb{R}^+, \quad \forall u \in U,$$

and

$$\int_{t_1}^{t_2} \varphi(t)dt \leq M(t_2 - t_1), \qquad \forall t_1, t_2 \in \mathbb{R}^+;$$

(iv) the limit

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(s,u)ds = \bar{f}(u),$$

exists for all $u \in U$;

(v) there exists a constant $N > 0$ such that, for all continuous fuctions $u, v : \mathbb{R}^+ \longrightarrow U$ and $t \geq 0$,

$$d_\infty\left(\int_0^t \bar{f}(u(s))ds, \int_0^t \bar{f}(v(s))ds\right) \leq N\int_0^t d_\infty(u(s), v(s))ds.$$

To establish our main result we will prove the following lemmas:

**Lemma 1.** *Let assumptions $(ii)$, $(iii)$ and $(iv)$ be satisfied. Then the function $\bar{f}$ is continuous and*

$$d_\infty(\bar{f}\left(u\right), 0_{(1,0)}) \leq M, \qquad \forall u \in U.$$

*Proof.* Let $u_1 \in U$, From the assumption $(ii)$, we get, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, $\forall u \in U$

$$d_\infty(u, u_1) < \delta \Rightarrow d_\infty(f(s,u), f(s,u_1)) < \frac{\varepsilon}{2}, \forall s \in \mathbb{R}^+.$$

And, by assumption $(iv)$, we have, for all $\eta > 0$, there exists $T_0 > 0$ such that

$$\forall T \geq T_0, \qquad d_\infty\left(\frac{1}{T}\int_0^T f(s,u)ds, \bar{f}(u)\right) < \eta, \ \forall u \in U.$$

Hence,

$$\begin{aligned}
&d_\infty(\bar{f}(u), \bar{f}(u_1)) \\
&\leq d_\infty\left(\bar{f}(u), \frac{1}{T}\int_0^T f(s,u)ds\right) + d_\infty\left(\frac{1}{T}\int_0^T f(s,u)ds, \frac{1}{T}\int_0^T f(s,u_1)ds\right) \\
&+ d_\infty\left(\frac{1}{T}\int_0^T f(s,u_1)ds, \bar{f}(u_1)\right) \leq 2\eta + \frac{1}{T}\int_0^T d_\infty(f(s,u), f(s,u_1))ds \\
&\leq 2\eta + \frac{\varepsilon}{2}
\end{aligned}$$

For $\eta = \dfrac{\varepsilon}{4}$, we get

$$d_\infty(\bar{f}(u), \bar{f}(u_1)) \leq \varepsilon.$$

Then, $\bar{f}$ is continuous at $u_1$.
From the assumption $(iv)$, we have for all $\eta > 0$, there exists $T_0 > 0$ such that $\forall T \geq T_0$

$$d_\infty\left(\bar{f}(u), \frac{1}{T}\int_0^T f(s,u)ds\right) < \eta, \qquad \forall u \in U.$$

Therefore,

$$\begin{aligned}
d_\infty(\bar{f}(u), 0_{(1,0)}) &\leq d_\infty\left(\bar{f}(u), \frac{1}{T}\int_0^T f(s,u)ds\right) + d_\infty\left(\frac{1}{T}\int_0^T f(s,u)ds, 0_{(1,0)}\right) \\
&\leq \eta + \frac{1}{T}\int_0^T d\infty(f(s,u)ds, 0_{(1,0)}) \\
&\leq \eta + M.
\end{aligned}$$

Since $\eta$ is arbitrary, hence the result is proved. □

**Lemma 2.** *Let assumption $(iv)$ be satisfied. Then for all $b > 0$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,b]} d_\infty\left(\frac{\varepsilon}{\alpha}\int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) = 0, \qquad \forall u \in U.$$

*Proof.* Let $u \in U$, $b > 0$ and $\alpha > 0$. It is easy to note that from $(iv)$, if $t = 0$, we have

$$\lim_{\varepsilon \to 0} d_\infty\left(\frac{\varepsilon}{\alpha}\int_0^{\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) = 0, \qquad \forall u \in U.$$

Now, for $t \in (0, b]$, we have that

$$\frac{\varepsilon}{\alpha}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds = \frac{\varepsilon}{\alpha}\int_0^{\frac{t}{\varepsilon}} f(s,u)ds + \frac{\varepsilon}{\alpha}\int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds,$$

since

$$\frac{\varepsilon}{\alpha} = \frac{1}{\frac{\alpha}{\varepsilon}} = \frac{\frac{t}{\alpha}+1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}},$$

Thus,

$$\begin{aligned}
&\frac{t}{\alpha}\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds + \frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds \\
&= \frac{t}{\alpha}\frac{1}{\frac{t}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}} f(s,u)ds + \frac{\varepsilon}{\alpha}\int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds,
\end{aligned} \tag{3}$$

Therefore, from (3), we have

$$\begin{aligned}
&d_\infty\left(\frac{\varepsilon}{\alpha}\int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) \\
&= d_\infty\Bigg(\frac{t}{\alpha}\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds \\
&+ \frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds \ominus \frac{t}{\alpha}\frac{1}{\frac{t}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}} f(s,u)ds, \bar{f}(u) + \frac{t}{\alpha}\bar{f}(u) \ominus \frac{t}{\alpha}\bar{f}(u)\Bigg) \\
&\le \frac{t}{\alpha}d_\infty\left(\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) + d_\infty\left(\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) \\
&+ \frac{t}{\alpha}d_\infty\left(\frac{1}{\frac{t}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{t\in(0,b]} d_\infty\left(\frac{\varepsilon}{\alpha}\int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) &\le \frac{b}{\alpha}\sup_{t\in(0,b]} d_\infty\left(\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) \\
&+ \sup_{t\in(0,b]} d_\infty\left(\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) \\
&+ \frac{b}{\alpha}\sup_{t\in(0,b]} d_\infty\left(\frac{1}{\frac{t}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right).
\end{aligned}$$

Now, from $(iv)$, we get that

$$\lim_{\varepsilon\to 0}\sup_{t\in(0,b]} d_\infty\left(\frac{1}{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}+\frac{\alpha}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) = 0,$$

and

$$\lim_{\varepsilon\to 0}\sup_{t\in(0,b]} d_\infty\left(\frac{1}{\frac{t}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}} f(s,u)ds, \bar{f}(u)\right) = 0.$$

Then, the result is proved. □

**Corollary 1.** *Let assumptions $(i)$, $(iii)$ and $(iv)$ be satisfied. Let $u_\varepsilon$ be a maximal solution of (1) on $[0, a_\varepsilon)$, $0 < a_\varepsilon \leq \infty$. Then for all $b \in [0, a_\varepsilon)$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,b]} d_\infty \left( \frac{\varepsilon}{\alpha} \int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon} + \frac{\alpha}{\varepsilon}} f(s, u_\varepsilon) ds, \bar{f}(u_\varepsilon) \right) = 0.$$

*Proof.* It is easy to prove that from $(i)$ and $(iii)$, $u_\varepsilon$ is well defined. Then the result follows directly from Lemma 2. $\square$

**Lemma 3.** *Let assumptions $(i) - (iv)$ be satisfied. Let $u_\varepsilon$ be a maximal solution of (1) on $[0, a_\varepsilon)$, $0 < a_\varepsilon \leq \infty$. Then for all $b \in [0, a_\varepsilon)$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,b]} d_\infty \left( \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^t \bar{f}(u_\varepsilon(s)) ds \right) = 0.$$

*Proof.* Let $b \in [0, a_\varepsilon)$, We divide the segment $[0, b]$ into $n$ equal parts by the points $t_i$,

$$t_0 = 0 < t_1 < \cdots < t_n = b, \qquad n \in \mathbb{N},$$

let $e_\varepsilon = t_{i+1} - t_i$, $i = 0, 1, \cdots, n-1$ with $\lim\limits_{\varepsilon \to 0} e_\varepsilon = 0$.

For $t \in [t_p, t_{p+1}]$, $p \in \{0, 1, \cdots, n-1\}$, we have

$$\begin{aligned}
& d_\infty \left( \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^t \bar{f}(u_\varepsilon(s) ds) \right) \\
&= d_\infty \left( \int_0^{t_p} f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds + \int_{t_p}^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds \; , \int_0^{t_p} \bar{f}(u_\varepsilon(s)) ds + \int_{t_p}^t \bar{f}(u_\varepsilon(s)) ds \right) \\
&\leq d_\infty \left( \int_0^{t_p} f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^{t_p} \bar{f}(u_\varepsilon(s)) ds \right) \\
&+ d_\infty \left( \int_{t_p}^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_{t_p}^t \bar{f}(u_\varepsilon(s)) ds \right) \\
&\leq \sum_{i=0}^{p-1} d_\infty \left( \int_{t_i}^{t_{i+1}} f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_{t_i}^{t_{i+1}} \bar{f}(u_\varepsilon(s)) ds \right) \\
&+ d_\infty \left( \int_{t_p}^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_{t_p}^t \bar{f}(u_\varepsilon(s)) ds \right).
\end{aligned} \tag{4}$$

From $(iii)$ and Lemma 1, we have

$$\begin{aligned}
& d_\infty \left( \int_{t_p}^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_{t_p}^t \bar{f}(u_\varepsilon(s)) ds \right) \\
&\leq d_\infty \left( \int_{t_p}^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, 0_{(1,0)} \right) + d_\infty \left( \int_{t_p}^t \bar{f}(u_\varepsilon(s)) ds, 0_{(1,0)} \right) \\
&\leq \int_{t_p}^t d_\infty \left( f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, 0_{(1,0)} \right) + \int_{t_p}^t d_\infty \left( \bar{f}(u_\varepsilon(s)) ds, 0_{(1,0)} \right) \\
&\leq 2M(t - t_p) \\
&\leq 2M(t_{p+1} - t_p) \leq 2M e_\varepsilon.
\end{aligned} \tag{5}$$

From $i = 0, 1, \cdots, n$ and $s \in [t_i, t_{i+1}]$ and from $(iii)$, we have

$$\begin{aligned}
d_\infty\left(u_\varepsilon(s), u_\varepsilon(t_i)\right) &= d_\infty\left(u_0 + \int_0^s f(\tau, u_\varepsilon(\tau))d\tau, u_0 + \int_0^{t_i} f(\tau, u_\varepsilon(\tau))d\tau\right)\\
&\le d_\infty\left(\int_0^{t_i} f(\tau, u_\varepsilon(\tau))d\tau + \int_{t_i}^{s} f(\tau, u_\varepsilon(\tau))d\tau, \int_0^{t_i} f(\tau, u_\varepsilon(\tau))d\tau\right)\\
&\le d_\infty\left(\int_{t_i}^{s} f(\tau, u_\varepsilon(\tau))d\tau, 0_{(1,0)}\right)\\
&\le \int_{t_i}^{s} d_\infty\left(f(\tau, u_\varepsilon(\tau)), 0_{(1,0)}\right)d\tau\\
&\le M(s - t_i) \le M e_\varepsilon.
\end{aligned}$$

Hence, by $(ii)$, we get

$$d_\infty\left(f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right), f\left(\frac{s}{\varepsilon}, u_\varepsilon(t_i)\right)\right) \le \beta_\varepsilon^i, \qquad \text{with} \quad \lim_{\varepsilon\to 0}\beta_\varepsilon^i = 0, \tag{6}$$

and from Lemma 1,

$$d_\infty\left(\bar{f}\left(u_\varepsilon(s)\right), \bar{f}\left(u_\varepsilon(t_i)\right)\right) \le \gamma_\varepsilon^i, \qquad \text{with} \quad \lim_{\varepsilon\to 0}\gamma_\varepsilon^i = 0. \tag{7}$$

Then, from (4), (5), (6) and (7), it follows that

$$\begin{aligned}
&d_\infty\left(\int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right)ds, \int_0^t \bar{f}(u_\varepsilon(s)ds)\right)\\
&\le \sum_{i=0}^{p-1} d_\infty\left(\int_{t_i}^{t_{i+1}} f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right)ds, \int_{t_i}^{t_{i+1}} f\left(\frac{s}{\varepsilon}, u_\varepsilon(t_i))ds\right)\right)\\
&+ \sum_{i=0}^{p-1} d_\infty\left(\int_{t_i}^{t_{i+1}} f\left(\frac{s}{\varepsilon}, u_\varepsilon(t_i)\right)ds, \int_{t_i}^{t_{i+1}} \bar{f}(u_\varepsilon(t_i))ds\right)\\
&+ \sum_{i=0}^{p-1} d_\infty\left(\int_{t_i}^{t_{i+1}} \bar{f}(u_\varepsilon(t_i))ds, \int_{t_i}^{t_{i+1}} \bar{f}(u_\varepsilon(s))ds\right)\\
&+ d_\infty\left(\int_{t_p}^{t} f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right)ds, \int_{t_p}^{t} \bar{f}(u_\varepsilon(s))ds\right)\\
&\le \sum_{i=0}^{p-1} e_\varepsilon d_\infty\left(\frac{\varepsilon}{e_\varepsilon}\int_{\frac{t_i}{\varepsilon}}^{\frac{t_i}{\varepsilon}+\frac{e_\varepsilon}{\varepsilon}} f\left(s, u_\varepsilon(t_i)\right)ds, \bar{f}(u_\varepsilon(t_i))\right)\\
&+ \sum_{i=0}^{p-1}\int_{t_i}^{t_{i+1}}\left(\beta_\varepsilon^i + \gamma_\varepsilon^i\right)ds + 2Me_\varepsilon \qquad (8)\\
&\le \sup_{t\in[0,b]} d_\infty\left(\frac{\varepsilon}{e_\varepsilon}\int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{e_\varepsilon}{\varepsilon}} f\left(s, u_\varepsilon(t)\right)ds, \bar{f}(u_\varepsilon(t))\right)\sum_{i=0}^{p-1} e_\varepsilon\\
&+ \max_{i\in\{0,1,\cdots,p-1\}}\left(\beta_\varepsilon^i + \gamma_\varepsilon^i\right)\sum_{i=0}^{p-1}\int_{t_i}^{t_{i+1}} ds + 2Me_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in[0,b]} d_\infty \left( \frac{\varepsilon}{e_\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{e_\varepsilon}{\varepsilon}} f\left(s, u_\varepsilon(t)\right) ds, \bar{f}(u_\varepsilon(t)) \right) \sum_{i=0}^{p-1} e_\varepsilon \\
&+ \max_{i\in\{0,1,\cdots,p-1\}} \left(\beta_\varepsilon^i + \gamma_\varepsilon^i\right) \sum_{i=0}^{p-1} \int_{t_i}^{t_{i+1}} ds + 2Me_\varepsilon \\
&\leq \sup_{t\in[0,b]} d_\infty \left( \frac{\varepsilon}{e_\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{e_\varepsilon}{\varepsilon}} f\left(s, u_\varepsilon(t)\right) ds, \bar{f}(u_\varepsilon(t)) \right) \sum_{i=0}^{p-1} (t_{i+1}-t_i) \\
&+ \max_{i\in\{0,1,\cdots,p-1\}} \left(\beta_\varepsilon^i + \gamma_\varepsilon^i\right) \sum_{i=0}^{p-1} (t_{i+1}-t_i) + 2Me_\varepsilon \\
&\leq b \sup_{t\in[0,b]} d_\infty \left( \frac{\varepsilon}{e_\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}+\frac{e_\varepsilon}{\varepsilon}} f\left(s, u_\varepsilon(t)\right) ds, \bar{f}(u_\varepsilon(t)) \right) \\
&+ b \max_{i\in\{0,1,\cdots,p-1\}} \left(\beta_\varepsilon^i + \gamma_\varepsilon^i\right) + 2Me_\varepsilon.
\end{aligned}
$$

Consequently, according to Corollary 1, (6), (7) and (8), the result is obtained. □

Now, we are in the position to establish our result.

**Theorem 4.** *Let assumptions $(iii)-(v)$ be satisfied. Let $u_0 \in U$, $u_\varepsilon$ be a maximal solution of (1) on $[0, a_\varepsilon)$, $0 < a_\varepsilon \leq \infty$ and $v$ be the maximal solution of (2) on $[0, a)$, $0 < a \leq \infty$. Then for all $b \in (0, a_\varepsilon) \cap (0, a)$ and $\xi > 0$, there exists $\kappa_b^\xi > 0$ such that*

$$
d_\infty\left(u_\varepsilon(t), v(t)\right) < \xi, \qquad \forall t \in (0, \kappa_b^\xi], \quad t \in [0, b].
$$

*Proof.* For $t \in [0, b]$ and from $(v)$, we have

$$
\begin{aligned}
d_\infty\left(u_\varepsilon(t), v(t)\right) &= d_\infty \left( u_0 + \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, u_0 + \int_0^t \bar{f}(v(s)) ds \right) \\
&\leq d_\infty \left( \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^t \bar{f}(v(s)) ds \right) \\
&\leq d_\infty \left( \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^t \bar{f}\left(u_\varepsilon(s)\right) ds \right) \\
&+ d_\infty \left( \int_0^t \bar{f}\left(u_\varepsilon(s)\right) ds, \int_0^t \bar{f}(v(s)) ds \right) \\
&\leq \sup_{t\in[0,b]} d_\infty \left( \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^t \bar{f}\left(u_\varepsilon(s)\right) ds \right) \\
&+ N \int_0^t d_\infty\left(u_\varepsilon(s), v(s)\right) ds.
\end{aligned}
$$

Denote

$$
\theta_\varepsilon = \sup_{t\in[0,b]} d_\infty \left( \int_0^t f\left(\frac{s}{\varepsilon}, u_\varepsilon(s)\right) ds, \int_0^t \bar{f}\left(u_\varepsilon(s)\right) ds \right),
$$

From Lemma 3, we have $\lim_{\varepsilon \to 0} \theta_\varepsilon = 0$. By Gronwall Lemma, we get

$$d_\infty \left(u_\varepsilon(t), v(t)\right) \leq \theta_\varepsilon e^{Nt} \leq \theta_\varepsilon e^{Nb}.$$

Finally, we obtain

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,b]} d_\infty \left(u_\varepsilon(t), v(t)\right) = 0.$$

This completes the proof. □

# References


[1] Atanassov K. T. (1983). Intuitionistic fuzzy sets, *VII ITKR Session*, Sofia, 20–23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: *Int. J. Bioautomation*, 2016, 20(S1), S1–S6.

[2] Atanassov, K. (1986). Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 87–96.

[3] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer Physica-Verlag, Heidelberg.

[4] Atanassov, K. T., Vassilev, P. M., & Tsvetkov, R. T. (2013). *Intuitionistic Fuzzy Sets, Measures and Integrals*, Bulgarian Academic Monographs Series, Vol. 12, "Professor Marin Drinov" Academic Publishing House, Sofia.

[5] Ettoussi, R., Melliani, S., Elomari, M., & Chadli, L. S. (2015). Solution of intuitionistic fuzzy differential equations by successive approximations method, *Notes on Intuitionistic Fuzzy Sets*, 21(2), 51–62.

[6] Kaleva, O. (1990). The Cauchy problem for fuzzy differential equations, *Fuzzy Sets and Systems*, 35(3), 389–396.

[7] Kichmarenko, O. D., & Skripnik, N. V. (2008). Averaging of fuzzy differential equations with delay, *Nonlinear Oscillations*, 11(3), 331–344.

[8] Kichmarenko, O., & Skripnik, N. V. (2011). One scheme of averaging of fuzzy differential equations with maxima, *J. Adv. Resear. Appl. Math*, 3(1), 94–103.

[9] Melliani, S., Elomari, M., Chadli, L. S., & Ettoussi, R. . Extension of Hukuhara difference in intuitionistic fuzzy set theory, *Notes on Intuitionistic Fuzzy Sets*, 21(4), 34–47.

[10] Melliani, S., Elomari, M., Ettoussi, R., & Chadli, L. S. (2015). Intuitionistic fuzzy metric space, *Notes on Intuitionistic Fuzzy Sets*, 21(1), 43–53.

[11] Plotnikov, A. V., & Komleva, T. A. (2012). Averaging of the fuzzy differential equations, *J. Uncertain Syst*. 6(1), 30–37.

[12] Zadeh, L. A. (1965). Fuzzy sets, *Information and Control*, 8, 338–353.