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On relationships among intuitionistic fuzzy approximation operators, intuitionistic fuzzy topology and intuitionistic fuzzy automata*

S.P. Tiwari†
Department of Applied Mathematics
Indian School of Mines University
Dhanbad-826 004, India

Abstract

This paper is a study about the relationships among topologies and intuitionistic fuzzy topology induced, respectively, by approximation operators and an intuitionistic fuzzy approximation operator associated with an approximation space (X, R), when the relation R on X is precisely reflexive and transitive. In particular, we consider an intuitionistic fuzzy approximation operator on an approximation space X (i.e., a set X with a reflexive and transitive relation on it), which turns out to be an intuitionistic fuzzy closure operator. This intuitionistic fuzzy closure operator gives rise to two saturated fuzzy topologies on X and it turns out that all the level topologies of one of the fuzzy topology coincide and equal to the topology analogously induced on X by a crisp approximation operator. These observations are then applied to intuitionistic fuzzy automata.

Keywords:*Intuitionistic fuzzy set; Intuitionistic fuzzy approximation operator; Intuitionistic fuzzy topology; Fuzzy topology; Intuitionistic fuzzy automaton; Strong intuitionistic fuzzy subsystem.*

1 Introduction

In [7], the concepts of fuzzy subsystems and strong fuzzy subsystems of a fuzzy finite state machine (ffsm) were introduced and studied. In [4], a fuzzy topology on the state-set of a fuzzy automaton (a concept almost identical to that of a ffsm) was introduced and showed that fuzzy subsystems were precisely the closed fuzzy sets with respect to this fuzzy topology, while in [10], another fuzzy topology on the state-set of a fuzzy automaton was introduced and showed that strong fuzzy subsystems were precisely the closed fuzzy

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†sptiwarimaths@gmail.com

sets with respect to this fuzzy topology. Also in [10], the relationship among the 'level topologies' of this fuzzy topology and a (crisp) topology (introduced in [9]) on the state-set of a fuzzy automaton was discussed.
Jun [5], introduced and studied the concept of intuitionistic fuzzy finite state machine by using the notion of intuitionistic fuzzy sets (cf. [1]). In [11], two intuitionistic fuzzy topologies on the state-set of an intuitionistic fuzzy automaton were introduced and it was shown that the intuitionistic fuzzy subsystems and strong intuitionistic fuzzy subsystems of an intuitionistic fuzzy automaton can be characterize in terms of these intuitionistic fuzzy topologies (cf. [3]). In this paper, we introduce an intuitionistic fuzzy topology by using the concept of an intuitionistic fuzzy approximation operator (similar to as in [8, 13]), on the state-sets of intuitionistic fuzzy automata and establish its relationship with strong intuitionistic fuzzy subsystems. Interestingly, it turns out that all the *level topologies* of one of the fuzzy topologies, induced by the above mentioned intuitionistic fuzzy topology coincide with a (crisp) topology on the state-set of an intuitionistic fuzzy automaton.

2 Preliminaries

Throughout, $[0,1]^X$ and $1_A : X \to [0,1]$ shall, respectively, denote the set of all fuzzy sets in X and the characteristic function of a subset A of X.

In this section, we recall some basic definitions related to intuitionistic fuzzy sets and intuitionistic fuzzy topology from [1, 3, 11].

Definition 2.1 ([1]) *Let X be a nonempty set. An* **intuitionistic fuzzy set** *(IFS, in short) u is a pair (u_1, u_2) of fuzzy sets in X, i.e., functions $u_1, u_2 : X \to [0,1]$, such that $u_1(x) + u_2(x) \leq 1, \forall x \in X$.*

Remark 2.1 *An IFS $u = (u_1, u_2)$ in X will frequently be also viewed as a function $u : X \to [0,1] \times [0,1]$, given by $u(x) = (u_1(x), u_2(x)), x \in X$, such that $u_1(x) + u_2(x) \leq 1$. The IFSs $\tilde{0}$ and $\tilde{1}$ are given by $\tilde{0} = (\mathbf{0}, \mathbf{1})$ and $\tilde{1} = (\mathbf{1}, \mathbf{0})$, where $\mathbf{0}$ and $\mathbf{1}$ are respectively the 0-valued and the 1-valued constant fuzzy sets in X.*

We shall denote by $IFS(X)$, the family of all intuitionistic fuzzy subsets of X.

Definition 2.2 *(1) For two IFSs $u = (u_1, u_2), v = (v_1, v_2)$ in X, we write $u \leq v$ if $u_1(x) \leq u_2(x)$ and $v_1(x) \geq v_2(x)$.*

(2) The **supremum** *and* **infimum** *of IFSs $u_j = (u_{j1}, u_{j2})$ in $X, j \in J$, are respectively defined as*

(i) $\vee_j u_j = (\vee_j u_{j1}, \wedge_j u_{j2})$ and

(ii) $\wedge_j u_j = (\wedge_j u_{j1}, \vee_j u_{j2})$.

(3) The **complement** *u^c of an IFS $u = (u_1, u_2)$ is defined as $u^c = (u_2, u_1)$.*

Definition 2.3 ([3]) *An* **intuitionistic fuzzy topology** *(IFT, in short) on a nonempty set X is a family τ of IFSs in X, such that* *(i)* $\tilde{0}, \tilde{1} \in \tau$,

(ii) $u_j \in \tau, j \in J \Rightarrow \vee_{j \in J} u_j \in \tau$,

(iii) $u, v \in \tau \Rightarrow u \wedge v \in \tau$.

The pair (X, τ) *is called an* **intuitionistic fuzzy topological space** *(IFTS, in short) and the IFSs in* τ *are called* **intuitionistic fuzzy open** *(ifo, in short). The complement* u^c *of an ifo set* u *in an IFTS* (X, τ) *is called* **intuitionistic fuzzy closed** *in* X.

Let (X, τ) be an IFTS. Clearly, τ induces the following two topologies on X (in the sense of Chang [2]):

$$\tau_1 = \{u_1 : u \in \tau\} \quad \text{and} \quad \tau_2 = \{1 - u_2 : u \in \tau\}.$$

Definition 2.4 ([11]) *Let* X *be a nonempty set. An* **intuitionistic fuzzy closure operator** *on* X *is a map* $c : IFS(X) \to IFS(X)$*, such that* $\forall u, v \in IFS(X)$*, the following conditions hold:*

(i) $c(\tilde{0}) = \tilde{0}$,

(ii) $u \leq c(u)$,

(iii) $c(u \vee v) = c(u) \vee c(v)$,

(iv) $c(c(u)) = c(u)$.

It is easy to check that an intuitionistic fuzzy closure operator c on X, as defined above, induces an IFT on X viz., $\{(u_1, u_2) : c(u_2, u_1) = (u_2, u_1)\}$.

Remark 2.1 *In [11], it is also discussed that an intuitionistic fuzzy closure operator* c *can be seen to lead to a pair of maps* $c_1, c_2 : [0,1]^X \to [0,1]^X$*, such that* $c_1(u_1)(x) + c_2(u_2)(x) \leq 1, \forall u = (u_1, u_2) \in IFS(X)$*, which are respectively a fuzzy closure operator and a fuzzy interior operator on* X *and vice-versa. Accordingly, an intuitionistic fuzzy closure operator* c *can be denoted as* (c_1, c_2).

3 Apprximation operators and the associated topologies

In this section, we indicate that study of an intuitionistic fuzzy topology can be carried out much on the same lines as done in [10].

We shall begin with the following definition introduced in [10].

Definition 3.1 *(i) A pair* (X, R) *is called an* **approximation space** *if* X *is a set and* R *is a binary relation on* X.

(ii) For an approximation space (X, R),

(a) $c : [0,1]^X \to [0,1]^X$*, defined as,* $c(\lambda)(x) = \vee\{\lambda(y) : y \in R(x)\}, \lambda \in [0,1]^X, x \in X$*, is called the* **fuzzy approximation operator** *on* X *(induced by* R*).*

(b) $\bar{s} : 2^X \to 2^X$*, defined as* $\bar{s}(A) = (c(1_A))^{-1}(\mathbf{1}), A \in 2^X$*, is called the* **approximation operator** *on* X*.*

(Here $R(x) = \{y \in X : xRy\}$.) We note that $\bar{s}(A)$ can be alternatively expressed as $\bar{s}(A) = \{x \in X : R(x) \cap A \neq \phi\}$. This is easy to see from Definition 3.1 (ii) (b) and the fact that $(c(1_A))^{-1}(\mathbf{1}) = \{x \in X : c(1_A)(x) = 1\} = \{x \in X : \vee\{1_A(y) : y \in R(x)\} = 1\} = \{x \in X : \text{for some } y \in R(x) \text{ with } y \in A\} = \{x \in X : R(x) \cap A \neq \phi\}$.

Remark 3.1 *Note that, what we have named above as a (fuzzy) approximation operator on* X*, has been named in ([8, 12]) as an upper (fuzzy) approximation operator, since in [8], a lower (fuzzy) approximation operator is also defined. However, since both the upper and lower approximation operators induce the same topology on* X*, the name approximation operator rather than an upper approximation operator has choosen. We will follow the same nomenclature in the case of an intuitionistic fuzzy upper approximation operator defined below.*

Definition 3.2 ([14]) *For an approximation space* $(X, R), \bar{c} : IFS(X) \to IFS(X)$*, defined as,* $\bar{c}(u)(x) = \vee\{u(y) : y \in R(x)\}, u \in IFS(X), x \in X$*, is called the* **intuitionistic fuzzy approximation operator** *on* X *(induced by* R*).*

Remark 3.2 *If the intuitionistic fuzzy approximation operator* $\bar{c} : IFS(X) \to IFS(X)$*, is expressed as* $\bar{c}(u) = (\bar{c}_1(u_1), \bar{c}_2(u_2)), \forall u = (u_1, u_2) \in IFS(X)$*, then we are clearly led to two maps* $\bar{c}_1, \bar{c}_2 : [0,1]^X \to [0,1]^X$*, which,* $\forall u = (u_1, u_2) \in IFS(X), x \in X$*, satisfy* $\bar{c}_1(u_1)(x) = \vee\{u_1(y) : y \in R(x)\}, \bar{c}_2(u_2)(x) = \wedge\{u_2(y) : y \in R(x)\}$*. Thus an intuitionistic fuzzy approximation operator* $\bar{c}$ *can be seen to lead to a pair of maps* $\bar{c}_1, \bar{c}_2 : [0,1]^X \to [0,1]^X$*, such that* $\bar{c}_1(u_1)(x) + \bar{c}_2(u_2)(x) \leq 1, \forall u = (u_1, u_2) \in IFS(X)$ *and vice-versa. Accordingly, we shall denote an intuitionistic fuzzy approximation operator* $\bar{c}$ *as* $(\bar{c}_1, \bar{c}_2)$*.*

Proposition 3.1 ([12]) *A relation* R *on a set* X *is reflexive and transitive if and only if the (associated) approximation operator* $\bar{s}$ *is a Kuratowski saturated*[1] *closure operator on* X*.*

As a consequence of the above proposition, the approximation operator $\bar{s}$ on X associated with an approximation space (X, R), induces a saturated[2] topology on X, which we shall denote as $\bar{T}(X)$.

Remark 3.3 *Let* R^* *be another relation on* X *such that* $y \in R^*(x)$ *if and only if* $x \in R(y)$*. Then it is obvious to see that* R^* *is also reflexive and transitive if* R *is reflexive and transitive. Thus, by Proposition 3.1,* R^* *also induces a saturated topology on* X*, which we shall denote by* $\bar{T}^*(X)$*, and the corresponding approximation operator by* $\bar{s}^*$*.*

The relationship between the topologies $\bar{T}(X)$ and $\bar{T}^*(X)$ is given by the following proposition.

[1]A Kuratowski closure operator $k : 2^X \to 2^X$ on X is being called here *saturated* if the (usual) requirement $k(A \cup B) = k(A) \cup k(B)$ is replaced by $k(\cup A_j) = \cup k(A_j)$, where $A, B, A_j \in 2^X, j \in J$.

[2]in the sense that arbitrary intersection of open sets is also open.

Proposition 3.2 *The topologies $\bar{T}(X)$ and $\bar{T}^*(X)$ on X are dual, i.e., $A \subseteq X$ is $\bar{T}(X)$-open iff A is $\bar{T}^*(X)$-closed.*

Proof: Let A be $\bar{T}(X)$-open. Then $\bar{s}(X-A) = X-A$. Now $x \in \bar{s}^*(A) \Rightarrow R^*(x) \cap A \neq \phi \Rightarrow \exists y \in X$ such that $y \in R^*(x)$ and $y \in A$, or that $x \in R(y)$ and $y \in A$. But $y \in A \Rightarrow y \notin X-A \Rightarrow y \notin \bar{c}(X-A) \Rightarrow R(y) \cap (X-A) = \phi \Rightarrow R(y) \subseteq A \Rightarrow x \in A$. Thus $\bar{c}^*(A) \subseteq A$, whereby $\bar{c}^*(A) = A$, implying that A is $\bar{T}^*(X)-$closed. Conversly, let $A \subseteq X$ be $\bar{T}^*(X)-$closed. Then $\bar{c}^*(A) = A$. Now $x \in \bar{c}(X-A) \Rightarrow R(x) \cap (X-A) \neq \phi \Rightarrow \exists y \in X$ such that $y \in R(x)$ and $y \notin A$, or that $x \in R^*(y)$ and $y \notin A$. But $y \notin A \Rightarrow y \notin c^*(A) \Rightarrow R^*(y) \cap A = \phi \Rightarrow R^*(y) \subseteq X-A \Rightarrow x \in X-A$. Thus $\bar{c}(X-A) \subseteq X-A$ and so $\bar{c}(X-A) = X-A$, implying that A is $\bar{T}(X)$-open.

Proposition 3.3 ([10]) *A relation R on a set X is reflexive and transitive if and only if the (associated) fuzzy approximation operator is a Kuratowski saturated fuzzy closure operator on X.*

We now observe that an analogue of above proposition for *intuitionistic fuzzy* approximation operator exists and is the following.

Proposition 3.4 *A relation R on a set X is reflexive and transitive if and only if the (associated) intuitionistic fuzzy approximation operator $\bar{c}$ is a Kuratowski saturated intuitionistic fuzzy closure operator on X.*

Proof: Let R be a reflexive and transitive relation on X. Then, first we need to show that $\bar{c}_1$ is a fuzzy closure operator, i.e., $\forall u = (u_1, u_2), u_j = (u_{1j}, u_{2j}) \in IFS(X),\ j \in J, \bar{c}_1$ satisfies

(i) $\bar{c}_1(\mathbf{0}) = \mathbf{0}$,

(ii) $u_1 \leq \bar{c}_1(u_1)$,

(iii) $\bar{c_1}(\bar{c_1}(u_1)) = \bar{c_1}(u_1)$,

(iv) $\bar{c_1}(\vee\{u_{1j} : j \in J\}) = \vee\{\bar{c_1}(u_{1j}) : j \in J\}$.

(i) is obvious. (ii) and (iii) follow by using the reflexivity and the transitivity of R respectively. Finally, given $x \in X$ and $u_j = (u_{1j}, u_{2j}) \in IFS(X),\ j \in J,\ \bar{c_1}(\vee\{u_{1j} : j \in J\})(x) = \vee\{\vee\{u_{1j} : j \in J\}(y) : y \in R(x)\} = \vee\{\vee\{u_{1j}(y) : j \in J\} : y \in R(x)\} = \vee\{\vee\{u_{1j}(y) : y \in R(x)\} : j \in J\} = \vee\{\bar{c_1}(u_{1j}) : j \in J\}(x)$. Thus $\bar{c_1}(\vee\{u_{1j} : j \in J\}) = \vee\{\bar{c_1}(u_{1j}) : j \in J\}$.
Further, to show that $\bar{c}_2$ is a fuzzy interior operator on X, we need to verify the following conditions, $u = (u_1, u_2), u_j = (u_{1j}, u_{2j}) \in IFS(X),\ j \in J$:

(i) $\bar{c}_2(\mathbf{0}) = \mathbf{0}$,

(ii) $u_2 \geq \bar{c}_2(u_2)$,

(iii) $\bar{c_2}(\bar{c_2}(u_2)) = \bar{c_2}(u_2)$,

(iv) $\bar{c_2}(\wedge\{u_{2j} : j \in J\}) = \wedge\{\bar{c_2}(u_{2j}) : j \in J\}$.

Again, (i) is obvious. (ii) and (iii) follow by using the reflexivity and the transitivity of R respectively. Finally, given $x \in X$ and $u_j = (u_{1j}, u_{2j}) \in IFS(X)$, $j \in J$, $\bar{c_2}(\wedge\{u_{2j} : j \in J\})(x) = \wedge\{\wedge\{u_{2j} : j \in J\}(y) : y \in R(x)\} = \wedge\{\wedge\{u_{2j}(y) : j \in J\} : y \in R(x)\} = \wedge\{\wedge\{u_{2j}(y) : y \in R(x)\} : j \in J\} = \wedge\{\bar{c_2}(u_{2j}) : j \in J\}(x)$. Thus $\bar{c_2}(\vee\{u_{2j} : j \in J\}) = \wedge\{\bar{c_2}(u_{2j}) : j \in J\}$.
Lastly, we need to show that $\bar{c}_1(u_1)(x)+\bar{c}_2(u_2)(x) \leq 1, \forall u = (u_1, u_2) \in IFS(X)$, $\forall x \in X$, which is satisfied as shown follows.
$\bar{c}_1(u_1)(x) + \bar{c}_2(u_2)(x)$
$= \vee\{u_1(y) : y \in R(x)\} + \wedge\{u_2(y) : y \in R(x)\}$
$\leq \vee\{u_1(y) : y \in R(x)\} + \wedge\{1 - u_1(y) : y \in R(x)\}$, (as $u_1(y) + u_2(y) \leq 1, \forall y \in X$)
$= \vee\{u_1(y) : y \in R(x)\} + 1 - \vee\{u_1(y) : y \in R(x)\}$
$= 1$.
Thus $\bar{c}_1(u_1)(x) + \bar{c}_2(u_2)(x) \leq 1, \forall u = (u_1, u_2) \in IFS(X), \forall x \in X$.
Hence $\bar{c} = (\bar{c}_1, \bar{c}_2)$ is an intuitionistic fuzzy closure operator on X.
Conversely, let $\bar{c}$ be an intuitionistic fuzzy closure operator and $x \in X$. Then $\tilde{1} = (\mathbf{1}, \mathbf{0}) = (1_{\{x\}}(x), 1_{X-\{x\}}(x)) \leq \bar{c}(1_{\{x\}}(x), 1_{X-\{x\}}(x))$. Thus $\bar{c}(1_{\{x\}}(x), 1_{X-\{x\}}(x)) = \tilde{1}$, whence $\vee\{1_{\{x\}}(y) : y \in R(x)\} = 1$ and $\wedge\{1_{X-\{x\}}(y) : y \in R(x)\} = 0 \Rightarrow x \in R(x)$. Hence R is reflexive. Also, let $y \in R(x)$ and $z \in R(y)$. Then $y \in R(x) \Rightarrow \bar{c}(1_{\{z\}}, 1_{X-\{z\}})(y) \leq \bar{c}(\bar{c}(1_{\{z\}}, 1_{X-\{z\}})(x) = \bar{c}(1_{\{z\}}, 1_{X-\{z\}})(x)$, while $z \in R(y) \Rightarrow \tilde{1} = (1_{\{z\}}, 1_{X-\{z\}})(z) \leq \bar{c}(1_{\{z\}}, 1_{X-\{z\}})(y)$. Hence $\tilde{1} \leq \bar{c}(1_{\{z\}}, 1_{X-\{z\}})(x)$, whereby $\bar{c}(1_{\{z\}}, 1_{X-\{z\}})(x) = \tilde{1}$, or that $\bar{c}_1(1_{\{z\}})(x) = 1$ and $\bar{c}_2(1_{X-\{z\}})(x) = 0$. Thus $\exists w \in R(x)$ with $(1_{\{z\}})(w) = 1$ and $(1_{X-\{z\}})(w) = 0$, implying that $z = w$ and so $z \in R(x)$. Hence R is transitive also.

We shall denote the IFT induced by $\bar{c} = (\bar{c}_1, \bar{c}_2)$ on X as $\bar{\tau}(X)$ and the two fuzzy topologies induced on X by $\bar{c}_1$ and $\bar{c}_2$, as $\bar{\tau}_1(X)$ and $\bar{\tau}_2(X)$.

4 Relationship between topologies $\bar{\tau}_1(X)$ and $\bar{T}^*(X)$ as well as $\bar{\tau}_2(X)$ and $\bar{T}(X)$

From [6], recall that for a given fuzzy topology Δ on a set X, and any $\alpha \in [0, 1)$, $\iota_\alpha(\Delta) = \{\lambda^{-1}(\alpha, 1] : \lambda \in \Delta\}$ is well-known to be a topology on X, referred to as the *α-level topology* of Δ. It is customary to denote by $\iota(\Delta)$, the supremum of the topologies $\iota_\alpha(\Delta), \alpha \in [0, 1)$, and call it the *topological modification* of the fuzzy topology Δ.

In the remaining part of this section, X is a set with a reflexive and transitive relation R, $\bar{\tau}_1(X), \bar{\tau}_2(X)$ are the fuzzy topologies, and $\bar{T}(X), \bar{T}^*(X)$ are the topologies on X, as defined in the previous section.

Proposition 4.1 $\iota_\alpha(\bar{\tau}_1(X)) = \bar{T}^*(X)$ *and* $\iota_\alpha(\bar{\tau}_2(X)) \subseteq \bar{T}(X), \forall \alpha \in [0, 1)$.

Proof: Fix any $\alpha \in [0, 1)$. Then $\iota_\alpha(\bar{\tau}_1(X)) = \{u_1^{-1}(\alpha, 1] : u_1 \in [0, 1]^X, \bar{c_1}(1 - u_1) = 1-u_1\} = \{(1-u_1)^{-1}[0, 1-\alpha) : u_1 \in [0, 1]^X, \bar{c_1}(1-u_1) = 1-u_1\}$. Also, $\bar{T}^*(X) = \{A \in 2^X : \bar{s}^*(X - A) = X - A\}$. Let $A = (1-u_1)^{-1}[0, 1-\alpha) \in \iota_\alpha(\bar{\tau}_1(X))$. Then $\bar{c_1}(1-u_1) = 1-u_1$. To show that $\bar{s}^*(X - A) = X - A$, it is enough to show that $\bar{s}^*(X - A) \subseteq X - A$. Let $x \in \bar{s}^*(X - A)$. Then $R^*(x) \cap X - A \neq \phi$. So, let $y \in R^*(x) \cap (X - A)$. Also, $y \in X - A \Rightarrow y \notin A \Rightarrow (1 - u_1)(y) \geq 1 - \alpha$. Now, $(1 - u_1)(x) = \bar{c_1}(1 - u_1)(x) =$

$\vee\{(1-u_1)(y) : y \in R^*(x)\} \geq 1-\alpha \Rightarrow x \notin (1-u_1)^{-1}[0,1-\alpha) = A \Rightarrow x \in X-A$. Hence $\bar{s}^*(X-A) \subseteq X-A$. Thus $\iota_\alpha(\bar{\tau}_1(X)) \subseteq \bar{T}^*(X)$.
Conversely, let $A \in \bar{T}^*(X)$, so that $\bar{s}^*(X-A) = X-A$. To prove that $A \in \iota_\alpha(\bar{\tau}_1(X))$, it is enough to show that

(i) $A = 1_{X-A}^{-1}[0,1-\alpha)$ and

(ii) $\bar{c_1}(1_{X-A}) = 1_{X-A}$.

(i) is obviously true. To show (ii), let $x \in X$. If $\bar{c_1}(1_{X-A})(x) = 0$, (ii) follows obviously. If $\bar{c_1}(1_{X-A})(x) > 0$, then $\exists y \in R^*(x)$ such that $1_{X-A}(y) > 0$. But then $1_{X-A}(y) = 1$, whereby $y \in X-A$. Thus $y \in R^*(x) \cap (X-A)$ and so $R^*(x) \cap (X-A) \neq \phi$. Hence $x \in \bar{s}^*(X-A) = X-A$, whereby $1_{X-A}(x) = 1$, implying that $\bar{c_1}(1_{X-A})(x) \leq 1_{X-A}(x)$. Hence $\bar{T}^*(X) \subseteq \iota_\alpha(\bar{\tau}_1(X))$. Thus $\iota_\alpha(\bar{\tau}_1(X)) = \bar{T}^*(X), \forall \alpha \in [0,1)$.
We now show that $\iota_\alpha(\bar{\tau}_2(X)) \subseteq \bar{T}(X), \forall \alpha \in [0,1)$. Fix any $\alpha \in [0,1)$. By definition, $\iota_\alpha(\bar{\tau}_2(X)) = \{u_2^{-1}(\alpha,1] : u_2 \in [0,1]^X, \bar{c_2}(u_2) = u_2\}$ and $\bar{T}(X) = \{A \in 2^X : \bar{s}^*(A) = A\}$. Let $A = u_2^{-1}(\alpha,1] \in \iota_\alpha(\bar{\tau}_2(X))$. Then $\bar{c_2}(u_2) = u_2$. We show that $\bar{s}^*(A) = A$, for which it suffices to show that $\bar{s}^*(A) \subseteq A$. Let $x \in \bar{s}^*(A)$. Then $R^*(x) \cap A \neq \phi$. So, let $y \in R^*(x) \cap A$. Then $y \in R^*(x)$ and $y \in A$. Also, $u_2(y) > \alpha$, as $y \in A$. Thus $u_2(x) = \bar{c_2}(u_2)(x) = \wedge\{u_2(y) : y \in R^*(x)\} > \alpha$. Hence $x \in u_2^{-1}(\alpha,1]$, or that $x \in A$, whereby $\bar{s}^*(A) \subseteq A$. Hence $\iota_\alpha(\bar{\tau}_2(X)) \subseteq \bar{T}(X), \forall \alpha \in [0,1)$.

From the above proposition, following is obvious.

Proposition 4.2 *$\bar{T}(X)$ is the topological modification of $\bar{\tau}(X)$.*

5 An Intuitionistic fuzzy topology for intuitionistic fuzzy automata

In this section, we introduce two (crisp) topologies and an intuitionistic fuzzy topology on the state-sets of intuitionistic fuzzy automata (by using the concept of approximation operators), which are precisely the same, what have introduced in [11]. Lastly, we indicate the relationship between intuitionistic fuzzy automata and intuitionistic fuzzy topology, as done in [11].

Definition 5.1 ([11])*An* **intuitionistic fuzzy automaton** *(IFA, in short) is a triple $M = (Q, X, \delta)$, where Q is a set (of* **states** *of M), X is a monoid (the* **input monoid** *of M with identity e), and δ is an IFS in $Q \times X \times Q$, such that $\forall q, p \in Q, \forall x, y \in X$,*

$$\delta_1(q,e,p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p, \end{cases} \quad \delta_2(q,e,p) = \begin{cases} 1 & \text{if } q \neq p \\ 0 & \text{if } q = p, \end{cases}$$

$$\delta_1(q,xy,p) = \vee\{\delta_1(q,x,r) \wedge \delta_1(r,y,p) : r \in Q\}, \text{ and}$$
$$\delta_2(q,xy,p) = \wedge\{\delta_2(q,x,r) \vee \delta_2(r,y,p) : r \in Q\}.$$

In what follows, Q will *throughout* denote the state-set of an intuitionistic fuzzy automaton (Q, X, δ).

Definition 5.2 ([5]) *Let $A \subseteq Q$. The* **intuitionistic source** *and the* **intuitionistic successor** *of A, are respectively the following sets:*

$$\sigma_Q(A) = \{q \in Q : \delta_1(q,x,p) > 0 \text{ and } \delta_2(q,x,p) < 1, \text{ for some } (x,p) \in X \times A\}$$
$$s_Q(A) = \{p \in Q : \delta_1(q,x,p) > 0 \text{ and } \delta_2(q,x,p) < 1, \text{ for some } (x,q) \in X \times A\}.$$

Note that the intuitionistic source operator σ (respectively the intuitionistic successor operator s), defined in Definition 5.2, induces a reflexive and transitive relation R (respectively R^*) on Q, given by pRq (respectively pR^*q) if $p \in \sigma(q)$ (respectively $p \in s(q)$), which must give rise to a topology (and also, its dual topology) on Q (cf. Proposition 3.1, 3.2). We shall denote the topologies so given by R and R^* by $IT(Q)$ and $IT^*(Q)$ respectively (which are precisely the same as introduced in [11]).

Definition 5.3 ([11])*Let $M = (Q, X, \delta)$ be an IFA and u be an IFS in Q. Then u is called a* **strong intuitionistic fuzzy subsystem** *of M if $\forall p, q \in Q$,*

$$p \in \sigma(q) \Rightarrow u_1(p) \leq u_1(q) \text{ and } u_2(p) \geq u_2(q).$$

We next show that there is an intuitionistic fuzzy topology on Q such that the strong intuitionistic fuzzy subsystems of an IFA (Q, X, δ) turn out to be precisely the $\check{\tau}(Q)$-closed IFSs of Q. Let (Q, X, δ) be an IFA. Consider the reflexive and transitive relation R on Q, defined as

$$pRq \text{ iff } p \in \sigma(q), \forall p, q \in Q$$

Obviously, $\sigma(q) = R(q), \forall q \in Q$. So as in Definition 3.2, there is an intuitionistic fuzzy approximation operator on Q given by

$$\check{c}(u)(q) = \vee\{u(p) : p \in \sigma(q)\}, \forall u \in IFS(Q), \forall q \in Q.$$

This operator $\check{c}$ must be a saturated intuitionistic fuzzy closure operator on Q (Proposition 3.4). Thus $\check{c}$ induces an intuitionistic fuzzy topology (which is precisely the same as introduced in [11]) on Q. We shall denote the IFT induced by $\check{c} = (\check{c}_1, \check{c}_2)$ on Q as $\check{\tau}(Q)$ and the two fuzzy topologies induced on Q by $\check{c}_1$ and $\check{c}_2$, as $\check{\tau}_1(Q)$ and $\check{\tau}_2(Q)$.

Proposition 5.1 *$u \in IFS(Q)$ is a strong intuitionistic fuzzy subsystem of an IFA (Q, X, δ) if and only if u is intuitionistic fuzzy $\check{\tau}(Q)$-closed.*

Proof: Similar to as done in [11].

The relationship between the fuzzy topologies $\check{\tau}_1(Q)$ and $\check{\tau}_2(Q)$, induced by the intuitionistic fuzzy topology $\check{\tau}(Q)$, and the topologies $IT(Q)$ and $IT^*(Q)$ is given by the following Proposition, which is evident from Proposition 4.1.

Proposition 5.2 *$\iota_\alpha(\tau_1(Q)) = IT^*(Q)$ and $\iota_\alpha(\tau_2(Q)) \subseteq IT(Q), \forall \alpha \in [0, 1)$.*

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