

Intuitionistic fuzzy linear optimization

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Abstract: A new concept to the optimization problem in an intuitionistic fuzzy environment is introduced. A general model, of linear programming problems in which both the right hand side constants and the technological coefficients are intuitionistic fuzzy numbers, has been presented. An attempt has been made to find a general linear non-membership function and intuitionistic fuzzy index together with the membership function of the objective function and constraints in the solution of the given Intuitionistic Fuzzy Linear Programming Problem(IFLPP). Stepwise algorithm is given for solving an IFLPP and it is checked with a numerical example using intuitionistic fuzzy decisive set method.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy linear programming problem, Intuitionistic fuzzy optimization, Membership and non-membership functions, Intuitionistic index, Intuitionistic fuzzy decisive set method.

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1 Introduction

Linear programming is an optimization technique which is most frequently applied in real-world problems and therefore it is very important to introduce new tools in the approach that allow the model to fit into the real world as much as possible.

Many application-oriented mathematical models deal with real numbers. In real life, due to the inevitable measurement inaccuracy, the exact values of the measured quantities are not known and so the parameters of the problem are usually defined by the decision maker in an uncertain way. Therefore, it is desirable to consider the knowledge of experts about the parameters as fuzzy data. Fuzzy set theory has been extensively used to capture uncertainty and vagueness in decision making problems [8].

Bellman and Zadeh (1970) [4] proposed the concept of decision making in fuzzy environments. By adopting this concept to a mathematical programming problem, Tanaka, et al., [14] formulated the so-called fuzzy mathematical programming problem and showed that a compromise solution of decision maker could be obtained through an iterative use of linear programming technique. Zimmermann [15,16] proposed the first formulation of Fuzzy Linear Programming Problem (FLPP).

Shaocheng [13] considered the FLPP with fuzzy constraints and defuzzified it by first determining an upper bound for the objective function. Further he solved the so obtained crisp problem by using the fuzzy decisive set method introduced by Sakawa and Yana [12]. FLPP with linear membership functions is solved by Gasimov and Yenilmez [7].

Though fuzziness in decision making problems has been studied by various researchers who viewed the goals as fuzzy sets, they are very limited in scope and in many cases, do not represent the real problem very well. In practice, due to insufficiency of the information available, it is not easy to describe the fuzzy constraint conditions by ordinary fuzzy sets and the evaluation of membership and non-membership values upto decision maker's satisfaction is not always possible.

Consequently, there remains an indeterministic part of which hesitation survives. In such situations, the Intuitionistic Fuzzy Set (IFS) theory introduced by Atanassov [2] seems to be applicable to address this issue of uncertainty. In the case, when the degree of rejection is defined simultaneously with the degree of acceptance and when both these degrees are not complementary to each other, then IFS can be used as a more general and full tool for describing this fuzziness.

IFS is one of the extensions and evolutions of Zadeh Fuzzy Sets (ZFS). The membership, non-membership functions and intuitionistic index of an IFS can be used to express three states - support, opponent and neutral, with more delicate depiction and expression of fuzzy essence of objective world. Further more, it appears more agile and applied when it comes to deal with uncertain problems.

Atanassov and Kreinovich [3] implemented fuzzy interpretation of interval data. Angelov [1] implemented the optimization in an IF environment. Burillo., et al, [5] proposed the definition of Intuitionistic Fuzzy Number (IFN) and studied perturbations of IFN. B. S. Mahapatra and G. S. Mahapatra [10] have presented intuitionistic fuzzy fault tree analysis using trapezoidal intuitionistic fuzzy number. Fuzzy linear programs with symmetric trapezoidal fuzzy numbers have been proposed by K. Ganesan and P. Veeramani [6]. Lei Yang, et al., [9] proposed a normalization technique for ascertaining non-membership functions of IFS.

This paper introduces a solution procedure for IFLPP in which the right hand side constants and the technological coefficients are intuitionistic fuzzy numbers.

This paper is organized as follows: Section 2 briefly describes the basic definitions and notations of IFS, IFN. A general model of IFLPP with intuitionistic fuzzy right hand side constants and the technological coefficients is presented in Section 3. The Intuitionistic fuzzy decisive set method is presented and the application of this method is illustrated in Section 4. Finally, the paper is concluded in Section 5.

2 Preliminaries

Definition 2.1. [2] An Intuitionistic Fuzzy Set (IFS) A in X is defined as an object of the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}$ where the functions $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively, and for every $x \in X$ in A , $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ holds.

Note: Here after, in this paper, μ represents membership values and ν represents non-membership values.

Definition 2.2. [2] For every common fuzzy subset A on X , Intuitionistic Fuzzy Index of x in A is defined as $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. It is also known as degree of hesitancy or degree of uncertainty of the element x in A .

Obviously, for every $x \in X$, $0 \leq \pi_A(x) \leq 1$.

Definition 2.3. [10] An Intuitionistic Fuzzy Number (IFN) \tilde{A}^I is

i) an intuitionistic fuzzy subset of the real line,

ii) normal, that is, there is some $x_0 \in R$ such that $\mu_{\tilde{A}^I}(x_0) = 1$, $\nu_{\tilde{A}^I}(x_0) = 0$,

iii) convex for the membership function $\mu_{\tilde{A}^I}(x)$, that is,

$\mu_{\tilde{A}^I}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2))$, for every $x_1, x_2 \in R$, $\lambda \in [0, 1]$,

iv) concave for the non-membership function $\nu_{\tilde{A}^I}(x)$, that is,

$\nu_{\tilde{A}^I}(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2))$ for every $x_1, x_2 \in R$, $\lambda \in [0, 1]$.

3 Basic framework of an IFLPP

It is noted that in the basic framework of an IFLPP, there are no additional assumptions about the nature of objective functions and constraints. According to different hypothesis being considered, distinct IFLP problems shall be obtained.

Consider the case in which a decision maker tolerates violations in the accomplishment of the constraints, that is, he permits the technological co-efficients and the right hand side constants in the constraints to be satisfied 'as well as possible'. (In this paper a_{ij} 's and b_j 's because the costs C_j 's and x_j 's are supposed to be fixed throughout). Obviously, all of these imprecisions about the coefficients and constants can be modeled by means of intuitionistic fuzzy numbers and here the general form of such an IFLPP is considered and stepwise algorithm is given for solving an IFLPP and it is checked with a numerical example.

Definition 3.1. An IFLPP with intuitionistic fuzzy technological coefficients and right hand side constants is defined as

$$\begin{aligned}
 & \text{Maximize } z = \sum_{j=1}^n c_j x_j \\
 & \text{subject to } \sum_{j=1}^n \tilde{a}_{ij}^I x_j \leq \tilde{b}_i^I, \quad 1 \leq i \leq m \\
 & \quad \quad \quad x_j \geq 0, \quad 1 \leq j \leq n
 \end{aligned} \tag{1}$$

where atleast one $x_j > 0$ and \tilde{a}_{ij}^I and \tilde{b}_i^I are IF numbers.

Definition 3.2. Any vector $x \in R^n$ which satisfies the constraints and non negative restrictions of (1) is said to be an Intuitionistic Fuzzy Feasible Solution.

Definition 3.3. Let S be the set of all intuitionistic fuzzy feasible solutions of (1). Any vector $x_0 \in S$ is said to be an Intuitionistic Fuzzy Optimum Solution to (1) if $Cx_0 \geq Cx$ for all $x \in S$ where $C = (c_1, c_2, \dots, c_n)$ and $Cx = c_1x_1 + c_2x_2 + \dots + c_nx_n$.

Definition 3.4. The IF set of optimal values G which is a subset of R^n , is defined as

$$\mu_G(z) = \begin{cases} 0 & \text{if } z < z_l \\ \frac{z-z_l}{z_u-z_l} & \text{if } z_l \leq z < z_u \\ 1 & \text{if } z \geq z_u \end{cases}$$

and (2)

$$\nu_G(z) = \begin{cases} 1 & \text{if } z < z_l \\ 1 - c - \mu_G(z) & \text{if } z_l \leq z < z_u \\ 0 & \text{if } z \geq z_u \end{cases}$$

Here 'c' is called the intuitionistic fuzzy index and the value of 'c' is chosen such that $0 < c < \frac{z_u - z}{z_u - z_l} < 1$.

As 'z' approaches its maximum value, the value of 'c' approaches 'zero'.

Definition 3.5. The intuitionistic fuzzy set of the i^{th} constraint, C_i , which is a subset of R^n , is defined as

$$\mu_{C_i}(x) = \begin{cases} 0 & \text{if } b_i < \sum_{j=1}^n a_{ij}x_j \\ \frac{b_i - \sum_{j=1}^n a_{ij}x_j}{\sum_{j=1}^n d_{ij}x_j + p_i} & \text{if } \sum_{j=1}^n a_{ij}x_j \leq b_i < \sum_{j=1}^n (a_{ij} + d_{ij})x_j + p_i \\ 1 & \text{if } b_i \geq \sum_{j=1}^n (a_{ij} + d_{ij})x_j + p_i \end{cases}$$

and (3)

$$\nu_{C_i}(x) = \begin{cases} 1 & \text{if } b_i < \sum_{j=1}^n a_{ij}x_j \\ 1 - c - \mu_{C_i}(x) & \text{if } \sum_{j=1}^n a_{ij}x_j \leq b_i < \sum_{j=1}^n (a_{ij} + d_{ij})x_j + p_i \\ 0 & \text{if } b_i \geq \sum_{j=1}^n (a_{ij} + d_{ij})x_j + p_i \end{cases}$$

Definition 3.6. The linear membership and non-membership functions of the IFN \tilde{a}_{ij}^I are defined as

$$\mu_{\tilde{a}_{ij}^I}(x) = \begin{cases} 1 & \text{if } x < a_{ij} \\ \frac{a_{ij}+d_{ij}-x}{d_{ij}} & \text{if } a_{ij} \leq x < a_{ij} + d_{ij} \\ 0 & \text{if } x \geq a_{ij} + d_{ij} \end{cases}$$

and (4)

$$\nu_{\tilde{a}_{ij}^I}(x) = \begin{cases} 0 & \text{if } x < a_{ij} \\ 1 - c - \mu_{\tilde{a}_{ij}^I}(x) & \text{if } a_{ij} \leq x < a_{ij} + d_{ij} \\ 1 & \text{if } x \geq a_{ij} + d_{ij} \end{cases}$$

Definition 3.7. The linear membership and non-membership functions of the IFN \tilde{b}_i^I are defined as

$$\mu_{\tilde{b}_i^I}(x) = \begin{cases} 1 & \text{if } x < b_i \\ \frac{b_i+p_i-x}{p_i} & \text{if } b_i \leq x < b_i + p_i \\ 0 & \text{if } x \geq b_i + p_i \end{cases}$$

and

$$\nu_{\tilde{b}_i^I}(x) = \begin{cases} 0 & \text{if } x < b_i \\ 1 - c - \mu_{\tilde{b}_i^I}(x) & \text{if } b_i \leq x < b_i + p_i \\ 1 & \text{if } x \geq b_i + p_i \end{cases}$$

where $x \in R$ and $d_{ij}, p_i > 0$ for all $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$ and also $0 < c < \frac{z_u - z_l}{z_u - z_l} < 1$.

A general model for IFLPP

When the degree of rejection(non-membership) is defined simultaneously with degree of acceptance(membership) of the objective function and constraints and when both of these degrees are not complementary to each other, then IF sets can be used as a more general tool for describing uncertainty. In [1], maximization of the degree of acceptance and minimization of the degree of rejection of the IF objective function and constraints are defined as follows:

$$\begin{aligned} & \max \mu(x), \min \nu(x) \\ & \text{subject to} \\ & \mu(x) \geq \nu(x) \\ & 0 \leq \mu(x) + \nu(x) \leq 1 \\ & \mu(x), \nu(x) \geq 0, x \geq 0 \end{aligned}$$

where $\mu(x)$ and $\nu(x)$ denote the degrees of acceptance and rejection of x respectively. It is an extension of fuzzy optimization in which the degrees of rejection of objectives and constraints

are considered together with the degrees of satisfaction. The above problem is equivalent to the following:

$$\begin{aligned}
& \max \alpha, \min \beta \\
& \text{subject to} \\
& \alpha \leq \mu(x) \\
& \beta \geq \nu(x) \\
& \alpha \geq \beta \quad \text{and} \quad 0 \leq \alpha + \beta \leq 1; \quad \alpha, \beta \geq 0; \quad x \geq 0
\end{aligned}$$

where α denotes the minimal acceptable degree and β denotes the maximal degree of rejection. The IFO model can be changed into the following certainty(non-fuzzy) optimization model [1] as follows:

$$\begin{aligned}
& \max (\alpha - \beta) \\
& \text{subject to} \\
& \alpha \leq \mu(x) \\
& \beta \geq \nu(x) \\
& \mu(x) \geq \nu(x) \\
& \alpha \geq \beta \quad \text{and} \quad 0 \leq \alpha + \beta \leq 1; \quad \alpha, \beta \geq 0; \quad x \geq 0.
\end{aligned} \tag{5}$$

It should be emphasized here that α and β are treated as decisive variables rather than constants and so the problem is a non-linear programming problem which can not be solved by using usual simplex methods. So, Intuitionistic Fuzzy Decisive Set Method[11] is used to solve LPP in an IF environment.

4 The intuitionistic fuzzy decisive set method

In this method, a combination of the bisection method and phase one of the simplex method of linear programming problem is used to obtain a feasible solution.

For fixed values of α and β , the problem (5) is a linear programming problem. Obtaining the optimal solution (α^*, β^*) to (5) is equivalent to determining the maximum value of α and the minimum value of β so that the feasible set is nonempty. The algorithm of this method for (5) is presented below.

Algorithm for solving an IFLPP

Step 1 : Set α, β in the interval $(0,1)$ such that $\beta = 1 - c - \alpha$ where $c \in (0, 1)$ and the difference between α and β should not approach the value zero and test whether a feasible set satisfying the constraints of the problem (5) exists or not using phase one of the simplex method.

If a feasible set exists,

$$\text{set } \alpha^L = \alpha; \beta^L = \beta \text{ and } \alpha^R = \beta; \beta^R = \alpha$$

Otherwise,

$$\text{set } \alpha^L = \beta; \beta^L = \alpha \text{ and } \alpha^R = \alpha; \beta^R = \beta$$

and go to the next step.

Step 2 : For the value of $\alpha = (\alpha^L + \alpha^R)/2$ and $\beta = (\beta^L + \beta^R)/2$, update the values of $\alpha^L, \alpha^R, \beta^L$ and β^R , using the bisection method as follows :

(i) If feasible set is nonempty for α and β , set $\alpha^L = \alpha$ and α^R as it's value in the preceding step. $\beta^L = \beta$ and β^R as it's value in the preceding step.

(ii) If feasible set is empty for α and β , set $\alpha^R = \alpha$ and α^L as it's value in the preceding step. $\beta^R = \beta$ and β^L as it's value in the preceding step.

Consequently, for each α and β , test whether a feasible set of (5) exists or not, using phase one of the simplex method and determine the maximum value α^* and the minimum value β^* satisfying the constraints of (5).

When the intuitionistic fuzzy decisive set method is applied to a given IFLPP, the values of α and β in (0,1) remain constant for a particular value of 'c' in (0,1) which is the optimum solution.

Conversely when the solution is optimum, the value of 'c' remains constant in (0,1).

A numerical example: Production allocation problem

A manufacturer produces two types of models M_1 and M_2 . It is noted that the time required to manufacture each model can vary from day to day due to breakdown of machines, overtime work, etc. At the same time, each M_1 model requires *some what close* to 1 hour of grinding and 2 hours of polishing; whereas each M_2 model requires *some what close to* 3 hours of grinding and 5 hours of polishing with hesitancy degree .3 each. The manufacturer has 2 grinders and 3 polishers. Finally, the time availability of each grinder and polisher can vary from day to day due to breakdown, overtime work, power failure, etc. However, it is expected by the manufacturer that the working hours of each grinder and polisher are somewhat close to 10 hours and 20 hours a week respectively. Profit on an M_1 model is Rs.3 and on an M_2 model is Rs.4. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week?

Here is a real situation from an industry. The manufacturer can if he so chooses, produce only model M_2 , but then his grinders would be idle and his profit may not be maximum. This is only a guess. To be definite, it must be found how many M_1 and M_2 models should be produced per week, in order that his profit may be maximum.

Since the time required to manufacture each model and the available time of each grinder and polisher are uncertain with hesitancy degree 0.3, the given production allocation problem can be modeled as an IFLPP with intuitionistic fuzzy technological coefficients and right-hand side numbers. This calls for introducing variables x_1 and x_2 to stand for the desired number of M_1 and M_2 models respectively. Once it is done, the problem can be put in the language of Algebra as follows:

$$\text{Maximize } 3x_1 + 4x_2$$

subject to

$$\tilde{1}^I x_1 + \tilde{2}^I x_2 \leq \tilde{20}^I \quad (6)$$

$$\begin{aligned} \tilde{3}^I x_1 + \tilde{5}^I x_2 &\leq \tilde{60}^I \\ x_1, x_2 &\geq 0 \end{aligned}$$

which take intuitionistic fuzzy parameters as $\tilde{1}^I = L(1, 1)$, $\tilde{2}^I = L(2, 3)$, $\tilde{3}^I = L(3, 2)$ and $\tilde{5}^I = L(5, 3)$, $\tilde{20}^I = L(20, 10)$ and $\tilde{60}^I = L(60, 50)$ as used in [7] and [13].

By using the IF optimization model, problem(6) can be reduced to the following equivalent non-linear programming problem:

$$\text{maximize } (\alpha - \beta)$$

subject to

$$3x_1 + 4x_2 \geq 30 + 60\alpha ; 3x_1 + 4x_2 \geq 90 - 6c - 60\beta$$

$$(1 + \alpha)x_1 + (2 + 3\alpha)x_2 \leq 20 - 10\alpha ; (2 - c - \beta)x_1 + (5 - 3c - 3\beta)x_2 \leq 10 + 10c + 10\beta \quad (7)$$

$$\begin{aligned} (2 + 3\alpha)x_1 + (5 + 3\alpha)x_2 &\leq 60 - 50\alpha; (5 - 2c - 2\beta)x_1 + (8 - 3c - 3\beta)x_2 \leq 10 + 50c + 50\beta \\ 6x_1 + 8x_2 &\geq 120 - 60c \\ (3 - c)x_1 + (7 - 3c)x_2 &\leq 30 + 10c \\ (8 - 2c)x_1 + (13 - 3c)x_2 &\leq 70 + 50c \\ 0 &\leq \alpha, \beta \leq 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

By using the Intuitionistic Fuzzy Decisive Set Method, the optimal solution to the problem (7) is obtained as $\alpha^* = 0.59951$, $\beta^* = 0.10048$, $x_1^* = 7.15034$ and $x_2^* = 0$.

This means that the vector (x_1^*, x_2^*) is a solution to the problem (6) which has the best membership grade α^* and the least non-membership grade β^* with intuitionistic index $c = 0.3$.

5 Conclusion

A General model for studying LPP with IF goals has been presented in this paper. In this approach, the degree of acceptance and rejection of the objective function and the constraints are introduced together. These values can not be simply considered as a complement of each other and the sum of their value is less than or equal to one. Intuitionistic fuzzification of objective function and constraints has been modified as an extension of the authors' previous work. The solution of IF goal in IFS environment is obtained by solving an IFLPP and the proposed method is illustrated by a numerical example. This model is very useful in regression analysis to minimize the spread among the data.

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