

Level operators over primary interval-valued intuitionistic fuzzy M group

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Abstract: The concept of interval-valued intuitionistic fuzzy M group is extended by introducing primary interval-valued intuitionistic fuzzy M group and primary interval-valued intuitionistic fuzzy anti M group using this concept primary interval-valued intuitionistic fuzzy M group and primary interval-valued intuitionistic fuzzy anti M group is defined and using level operators and their properties are established.

Keywords: Intuitionistic fuzzy set, Interval-valued intuitionistic fuzzy set, Primary interval-valued intuitionistic fuzzy M group, Primary interval-valued intuitionistic fuzzy anti M group.

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1 Introduction

The concept of fuzzy sets was initiated by L. A. Zadeh [8] then it has become a vigorous area of research in engineering, medical science, graph theory. Rosenfeld [7] gave the idea of fuzzy subgroup. H. J. Zimmermann [10] gave the idea of fuzzy set theory. The concept of IFS and IVIFS was introduced by K. T. Atanassov [1,2]. The author W. R. Zhang [9] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. K. Chakrabarty, R. Biwas and S. Nanda [4] investigated a note on union and intersection of intuitionistic fuzzy sets. G. Prasannavengeteswari, K. Gunasekaran and S. Nandakumar [5] introduced the definition of Primary Bipolar Intuitionistic M Fuzzy Group and anti M Fuzzy Group. A. Balasubramanian, K. L. Muruganantha Prasad, K. Arjunan [3] introduced the definition of Bipolar Interval Valued Fuzzy Subgroups of a Group. G. Prasannavengeteswari, K. Gunasekaran and S. Nandakumar [6] introduced the definition of primary interval-valued intuitionistic fuzzy M group and fuzzy anti M Group. In this study Level Operators over Primary interval-valued Intuitionistic Fuzzy M Group and Fuzzy anti M Group and some properties of the same are proved.

2 Preliminaries

Definition 1. An interval-valued intuitionistic fuzzy set (IVIFS) A over the set E is an object of the form $A = \{(x, M_A(x), N_A(x))|x \in E\}$, where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and $\sup M_A(x) + \sup N_A(x) \leq 1$, for every $x \in E$. Thus we can write IVIFS A as $A = \{[inf M_A(x), sup M_A(x)], [inf N_A(x), sup N_A(x)]|x \in E\}$. For simplicity, we write the intervals

$$[inf M_A(x), sup M_A(x)] = [\mu_A^-(x), \mu_A^+(x)]$$

and

$$[inf N_A(x), sup N_A(x)] = [\nu_A^-(x), \nu_A^+(x)],$$

where $\mu_A^+(x), \nu_A^+(x), \mu_A^-(x), \nu_A^-(x)$ are functions from E into $[0, 1]$ and $(\forall x \in E)$ ($\mu_A^-(x) \leq \mu_A^+(x), \nu_A^-(x) \leq \nu_A^+(x), \mu_A^+(x) + \nu_A^+(x) \leq 1$) are called the degree of positive membership, degree of negative membership, degree of positive non-membership, and the degree of negative non-membership, respectively. Note that we denote here $\mu_A^-(x) = inf M_A(x), \mu_A^+(x) = sup M_A(x), \nu_A^-(x) = inf N_A(x), \nu_A^+(x) = sup N_A(x)$.

Definition 2. Let G be an M group and A be an interval-valued intuitionistic fuzzy subgroup of G , then A is called a primary interval-valued intuitionistic fuzzy M group of G . If for all $x, y \in G$ and $m \in M$, then either $\mu_A^+(mxy) \leq \mu_A^+(x^p)$ and $\nu_A^+(mxy) \geq \nu_A^+(x^p)$, for some $p \in Z_+$ or else $\mu_A^+(mxy) \leq \mu_A^+(y^q)$ and $\nu_A^+(mxy) \geq \nu_A^+(y^q)$, for some $q \in Z_+$ and either $\mu_A^-(mxy) \geq \mu_A^-(x^p)$ and $\nu_A^-(mxy) \leq \nu_A^-(x^p)$, for some $p \in Z_+$ or else $\mu_A^-(mxy) \geq \mu_A^-(y^q)$ and $\nu_A^-(mxy) \leq \nu_A^-(y^q)$, for some $q \in Z_+$.

Example 1.

$$\mu_A^+(x) = \begin{cases} 0.7 & \text{if } x = 1 \\ 0.6 & \text{if } x = -1 \\ 0.4 & \text{if } x = i, -i \end{cases} \quad \nu_A^+(x) = \begin{cases} 0.2 & \text{if } x = 1 \\ 0.3 & \text{if } x = -1 \\ 0.5 & \text{if } x = i, -i \end{cases}$$

$$\mu_A^-(x) = \begin{cases} 0.6 & \text{if } x = 1 \\ 0.5 & \text{if } x = -1 \\ 0.3 & \text{if } x = i, -i \end{cases} \quad \nu_A^-(x) = \begin{cases} 0.1 & \text{if } x = 1 \\ 0.2 & \text{if } x = -1 \\ 0.5 & \text{if } x = i, -i \end{cases}$$

Definition 3. Let G be an M group and A be an interval-valued intuitionistic anti fuzzy subgroup of G , then A is called a primary interval-valued intuitionistic fuzzy anti M group of G . If for all $x, y \in G$ and $m \in M$, then either $\mu_A^+(mxy) \geq \mu_A^+(xp)$ and $\nu_A^+(mxy) \leq \nu_A^+(xp)$, for some $p \in Z_+$ or else $\mu_A^+(mxy) \geq \mu_A^+(yq)$ and $\nu_A^+(mxy) \leq \nu_A^+(yq)$, for some $q \in Z_+$ and either $\mu_A^-(mxy) \leq \mu_A^-(xp)$ and $\nu_A^-(mxy) \geq \nu(xp)$, for some $p \in Z_+$ or else $\mu_A^-(mxy) \leq \mu_A^-(yq)$ and $\nu_A^-(mxy) \geq \nu_A^-(yq)$, for some $q \in Z_+$.

Example 2.

$$\mu_A^+(x) = \begin{cases} 0.4 & \text{if } x = 1 \\ 0.6 & \text{if } x = -1 \\ 0.7 & \text{if } x = i, -i \end{cases} \quad \nu_A^+(x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.3 & \text{if } x = -1 \\ 0.2 & \text{if } x = i, -i \end{cases}$$

$$\mu_A^-(x) = \begin{cases} 0.3 & \text{if } x = 1 \\ 0.5 & \text{if } x = -1 \\ 0.4 & \text{if } x = i, -i \end{cases} \quad \nu_A^-(x) = \begin{cases} 0.4 & \text{if } x = 1 \\ 0.2 & \text{if } x = -1 \\ 0.1 & \text{if } x = i, -i \end{cases}$$

Definition 4. Let A be an IVIFS over a set E , then the level operator $!A$ and $?A$ are defined as

$$!A = \left\{ \langle x, \left[\min \left\{ \frac{1}{2}, \mu_A^+(x) \right\}, \max \left\{ \frac{1}{2}, \nu_A^+(x) \right\} \right], \left[\max \left\{ \frac{1}{2}, \mu_A^-(x) \right\}, \min \left\{ \frac{1}{2}, \nu_A^-(x) \right\} \right] \rangle \mid x \in E \right\}$$

$$?A = \left\{ \langle x, \left[\max \left\{ \frac{1}{2}, \mu_A^+(x) \right\}, \min \left\{ \frac{1}{2}, \nu_A^+(x) \right\} \right], \left[\min \left\{ \frac{1}{2}, \mu_A^-(x) \right\}, \max \left\{ \frac{1}{2}, \nu_A^-(x) \right\} \right] \rangle \mid x \in E \right\}$$

3 Some operations on primary interval-valued intuitionistic fuzzy M group and primary interval-valued intuitionistic fuzzy anti M group

Theorem 1. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $!A$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned} \text{Consider } \mu_{!A}^+(mxy) &= \min \left(\frac{1}{2}, \mu_A^+(mxy) \right) = \min \left(\frac{1}{2}, \sup M_A(mxy) \right) \\ &\leq \min \left(\frac{1}{2}, \sup M_A(xp) \right) = \min \left(\frac{1}{2}, \mu_A^+(xp) \right) \\ &= \mu_{!A}^+(xp) \end{aligned}$$

Therefore, $\mu_{!A}^+(mxy) \leq \mu_{!A}^+(xp)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } v_{!A}^+(mxy) &= \max\left(\frac{1}{2}, v_A^+(mxy)\right) = \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ &\geq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) = \max\left(\frac{1}{2}, v_A^+(x^p)\right) \\ &= v_{!A}^+(x^p) \end{aligned}$$

Therefore, $v_{!A}^+(mxy) \geq v_{!A}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \mu_{!A}^-(mxy) &= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) = \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\ &\geq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) = \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ &= \mu_{!A}^-(x^p) \end{aligned}$$

Therefore, $\mu_{!A}^-(mxy) \geq \mu_{!A}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } v_{!A}^-(mxy) &= \min\left(\frac{1}{2}, v_A^-(mxy)\right) = \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ &\leq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) = \min\left(\frac{1}{2}, v_A^-(x^p)\right) \\ &= v_{!A}^-(x^p) \end{aligned}$$

Therefore, $v_{!A}^-(mxy) \leq v_{!A}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!A$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 2. If A and B are primary interval-valued intuitionistic fuzzy M groups of G , then $!(A \cap B) = !A \cap !B$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $x, y \in B$ and $m \in M$.

$$\begin{aligned} \text{Consider } \mu_{!(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) = \min\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\ &= \min\left(\frac{1}{2}, \min(\sup M_A(mxy), \sup M_B(mxy))\right) \\ &\leq \min\left(\frac{1}{2}, \min(\sup M_A(x^p), \sup M_B(x^p))\right) \\ &= \min\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\ &= \min\left(\min\left(\frac{1}{2}, \mu_A^+(x^p)\right), \min\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\ &= \min(\mu_{!A}^+(x^p), \mu_{!B}^+(x^p)) \\ &= \mu_{!A \cap !B}^+(x^p) \end{aligned}$$

Therefore, $\mu_{!(A \cap B)}^+(mxy) \leq \mu_{!A \cap !B}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } v_{!(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, v_{A \cap B}^+(mxy)\right) = \max\left(\frac{1}{2}, \max(v_A^+(mxy), v_B^+(mxy))\right) \\ &= \max\left(\frac{1}{2}, \max(\sup N_A(mxy), \sup N_B(mxy))\right) \\ &\geq \max\left(\frac{1}{2}, \max(\sup N_A(x^p), \sup N_B(x^p))\right) \\ &= \max\left(\frac{1}{2}, \max(v_A^+(x^p), v_B^+(x^p))\right) \\ &= \max\left(\max\left(\frac{1}{2}, v_A^+(x^p)\right), \max\left(\frac{1}{2}, v_B^+(x^p)\right)\right) \\ &= \max(v_{!A}^+(x^p), v_{!B}^+(x^p)) \\ &= v_{!A \cap !B}^+(x^p) \end{aligned}$$

Therefore, $\nu_{!(A \cap B)}^+(mxy) \geq \nu_{!A \cap !B}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \mu_{!(A \cap B)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^-(mxy)\right) = \max\left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy))\right) \\
&= \max\left(\frac{1}{2}, \max(\inf M_A(mxy), \inf M_B(mxy))\right) \\
&\geq \max\left(\frac{1}{2}, \max(\inf M_A(x^p), \inf M_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p))\right) \\
&= \max\left(\max\left(\frac{1}{2}, \mu_A^-(x^p)\right), \max\left(\frac{1}{2}, \mu_B^-(x^p)\right)\right) \\
&= \max(\mu_{!A}^-(x^p), \mu_{!B}^-(x^p)) \\
&= \mu_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{!(A \cap B)}^-(mxy) \geq \mu_{!A \cap !B}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \nu_{!(A \cap B)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{A \cap B}^-(mxy)\right) = \min\left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy))\right) \\
&= \min\left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy))\right) \\
&\leq \min\left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \min(\nu_A^-(x^p), \nu_B^-(x^p))\right) \\
&= \min\left(\min\left(\frac{1}{2}, \nu_A^-(x^p)\right), \min\left(\frac{1}{2}, \nu_B^-(x^p)\right)\right) \\
&= \min(\nu_{!A}^-(x^p), \nu_{!B}^-(x^p)) \\
&= \nu_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{!(A \cap B)}^-(mxy) \leq \nu_{!A \cap !B}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!(A \cap B) = !A \cap !B$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 3. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $?A$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned}
\text{Consider } \mu_{?A}^+(mxy) &= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) = \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \sup M_A(x^p)\right) = \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?A}^+(mxy) \leq \mu_{?A}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \nu_{?A}^+(mxy) &= \min\left(\frac{1}{2}, \nu_A^+(mxy)\right) = \min\left(\frac{1}{2}, \sup N_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{?A}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{?A}^+(mxy) \geq \nu_{?A}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{?A}^-(mxy) &= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?A}^-(mxy) \geq \mu_{?A}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{?A}^-(mxy) &= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{?A}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{?A}^-(mxy) \leq \nu_{?A}^-(x^p)$, for some $p \in Z_+$.

Therefore, $?A$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 4. If A and B are primary interval-valued intuitionistic fuzzy M groups of G , then $?A \cap B = ?A \cap ?B$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $x, y \in B$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{?(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\
&= \max\left(\frac{1}{2}, \min(sup M_A(mxy), sup M_B(mxy))\right) \\
&\leq \max\left(\frac{1}{2}, \min(sup M_A(x^p), sup M_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\
&= \min\left(\max\left(\frac{1}{2}, \mu_A^+(x^p)\right), \max\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\
&= \min(\mu_{?A}^+(x^p), \mu_{?B}^+(x^p)) \\
&= \mu_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?(A \cap B)}^+(mxy) \leq \mu_{?A \cap ?B}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{?(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{A \cap B}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max(\nu_A^+(mxy), \nu_B^+(mxy))\right) \\
&= \min\left(\frac{1}{2}, \max(sup N_A(mxy), sup N_B(mxy))\right) \\
&\geq \min\left(\frac{1}{2}, \max(sup N_A(x^p), sup N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \max(\nu_A^+(x^p), \nu_B^+(x^p))\right)
\end{aligned}$$

$$\begin{aligned}
&= \max \left(\min \left(\frac{1}{2}, \nu_A^+(x^p) \right), \min \left(\frac{1}{2}, \nu_B^+(x^p) \right) \right) \\
&= \max(\nu_{?A}^+(x^p), \nu_{?B}^+(x^p)) \\
&= \nu_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{?(A \cap B)}^+(mxy) \geq \nu_{?A \cap ?B}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \mu_{?(A \cap B)}^-(mxy) &= \min \left(\frac{1}{2}, \mu_{A \cap B}^-(mxy) \right) \\
&= \min \left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy)) \right) \\
&= \min \left(\frac{1}{2}, \max(\inf M_A(mxy), \inf M_B(mxy)) \right) \\
&\geq \min \left(\frac{1}{2}, \max(\inf M_A(x^p), \inf M_B(x^p)) \right) \\
&= \min \left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p)) \right) \\
&= \max \left(\min \left(\frac{1}{2}, \mu_A^-(x^p) \right), \min \left(\frac{1}{2}, \mu_B^-(x^p) \right) \right) \\
&= \max(\mu_{?A}^-(x^p), \mu_{?B}^-(x^p)) \\
&= \mu_{?A \cap ?B}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?(A \cap B)}^-(mxy) \geq \mu_{?A \cap ?B}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \nu_{?(A \cap B)}^-(mxy) &= \max \left(\frac{1}{2}, \nu_{A \cap B}^-(mxy) \right) \\
&= \max \left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy)) \right) \\
&= \max \left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy)) \right) \\
&\leq \max \left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p)) \right) \\
&= \max \left(\frac{1}{2}, \min(\nu_A^-(x^p), \nu_B^-(x^p)) \right) \\
&= \min \left(\max \left(\frac{1}{2}, \nu_A^-(x^p) \right), \max \left(\frac{1}{2}, \nu_B^-(x^p) \right) \right) \\
&= \min(\nu_{?A}^-(x^p), \nu_{?B}^-(x^p)) \\
&= \nu_{?A \cap ?B}^-(x^p).
\end{aligned}$$

Therefore, $\nu_{?(A \cap B)}^-(mxy) \leq \nu_{?A \cap ?B}^-(x^p)$, for some $p \in Z_+$. Therefore, $?A \cap B = ?A \cap ?B$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 5. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $\overline{?A} = !A$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned}
\text{Consider } \mu_{?\overline{A}}^+(mxy) &= \nu_{?A}^+(mxy) \\
&= \min \left(\frac{1}{2}, \nu_A^+(mxy) \right) \\
&= \min \left(\frac{1}{2}, \mu_A^+(mxy) \right) \\
&= \min \left(\frac{1}{2}, \sup M_A(mxy) \right) \\
&\leq \min \left(\frac{1}{2}, \sup M_A(x^p) \right)
\end{aligned}$$

$$= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\ = \mu_{!A}^+(x^p)$$

Therefore, $\mu_{?A}^+(mxy) \leq \mu_{!A}^+(x^p)$, for some $p \in Z_+$.

Consider $\nu_{?A}^+(mxy) = \mu_{?A}^+(mxy)$

$$= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ \geq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ = \nu_{!A}^+(x^p)$$

Therefore, $\nu_{?A}^+(mxy) \geq \nu_{!A}^+(x^p)$, for some $p \in Z_+$.

Consider $\mu_{?A}^-(mxy) = \nu_{?A}^-(mxy)$

$$= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ = \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\ = \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\ \geq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\ = \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ = \mu_{!A}^-(x^p)$$

Therefore, $\mu_{?A}^-(mxy) \geq \mu_{!A}^-(x^p)$, for some $p \in Z_+$

Consider $\nu_{?A}^-(mxy) = \mu_{?A}^-(mxy)$

$$= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\ = \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ = \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ \leq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\ = \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\ = \nu_{!A}^-(x^p)$$

Therefore, $\nu_{?A}^-(mxy) \leq \nu_{!A}^-(x^p)$, for some $p \in Z_+$. Therefore, $?A = !A$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 6. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $!(?A) = ?(!A)$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider $\mu_{!(?A)}^+(mxy) = \min\left(\frac{1}{2}, \mu_{?A}^+(mxy)\right)$

$$= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(mxy)\right)\right)$$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(mxy)\right)\right) \\
&\leq \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \mu_{!A}^+(x^p)\right) \\
&= \mu_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{!(!A)}^+(mxy) \leq \mu_{?(!A)}^+(x^p)$, for some $p \in Z_+$.

Consider $\nu_{!(!A)}^+(mxy) = \max\left(\frac{1}{2}, \nu_{?A}^+(mxy)\right)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \nu_A^+(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(mxy)\right)\right) \\
&\geq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \nu_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \nu_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \nu_{!A}^+(x^p)\right) \\
&= \nu_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{!(!A)}^+(mxy) \geq \nu_{?(!A)}^+(x^p)$, for some $p \in Z_+$.

Consider $\mu_{!(!A)}^-(mxy) = \max\left(\frac{1}{2}, \mu_{?A}^-(mxy)\right)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(mxy)\right)\right) \\
&\geq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \mu_{!A}^-(x^p)\right) \\
&= \mu_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{!(!A)}^-(mxy) \geq \mu_{?(!A)}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
v_{! (?A)}^-(mxy) &= \min \left(\frac{1}{2}, v_{?A}^-(mxy) \right) \\
&= \min \left(\frac{1}{2}, \max \left(\frac{1}{2}, v_A^-(mxy) \right) \right) \\
&= \min \left(\frac{1}{2}, \max \left(\frac{1}{2}, \inf N_A(mxy) \right) \right) \\
&\leq \min \left(\frac{1}{2}, \max \left(\frac{1}{2}, \inf N_A(x^p) \right) \right) \\
&= \min \left(\frac{1}{2}, \max \left(\frac{1}{2}, v_A^-(x^p) \right) \right) \\
&= \max \left(\frac{1}{2}, \min \left(\frac{1}{2}, v_A^-(x^p) \right) \right) \\
&= \max \left(\frac{1}{2}, v_{!A}^-(x^p) \right) \\
&= v_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore, $v_{! (?A)}^-(mxy) \leq v_{?(!A)}^-(x^p)$, for some $p \in Z_+$. Therefore, $! (?A) = ? (!A)$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 7. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $! (\square A) = \square (!A)$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{! (\square A)}^+(mxy) &= \min \left(\frac{1}{2}, \mu_{\square A}^+(mxy) \right) \\
&= \min \left(\frac{1}{2}, \mu_A^+(mxy) \right) \\
&= \min \left(\frac{1}{2}, \sup M_A(mxy) \right) \\
&\leq \min \left(\frac{1}{2}, \sup M_A(x^p) \right) \\
&= \min \left(\frac{1}{2}, \mu_A^+(x^p) \right) \\
&= \mu_{!A}^+(x^p) \\
&= \mu_{\square (!A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{! (\square A)}^+(mxy) \leq \mu_{\square (!A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
v_{! (\square A)}^+(mxy) &= \max \left(\frac{1}{2}, v_{\square A}^+(mxy) \right) \\
&= \max \left(\frac{1}{2}, 1 - \mu_A^+(mxy) \right) \\
&= \max \left(\frac{1}{2}, 1 - \sup M_A(mxy) \right) \\
&\geq \max \left(\frac{1}{2}, 1 - \sup M_A(x^p) \right) \\
&= \max \left(\frac{1}{2}, 1 - \mu_A^+(x^p) \right) \\
&= \max \left(\frac{1}{2}, v_A^+(x^p) \right) \\
&= v_{!A}^+(x^p)
\end{aligned}$$

$$\begin{aligned}
&= 1 - \mu_{!A}^+(x^p) \\
&= 1 - \mu_{\square(!A)}^+(x^p) \\
&= \nu_{\square(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{!(\square A)}^+(mxy) \geq \nu_{\square(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \mu_{!(\square A)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{\square A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p) \\
&= \mu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{!(\square A)}^-(mxy) \geq \mu_{\square(!A)}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \nu_{!(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= 1 - \mu_{!A}^-(x^p) \\
&= 1 - \mu_{\square(!A)}^-(x^p) \\
&= \nu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{!(\square A)}^-(mxy) \leq \nu_{\square(!A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!(\square A) = \square(!A)$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 8. If A is a primary interval-valued intuitionistic fuzzy M group, then $?(\square A) = \square(?A)$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned}
\text{Consider } \mu_{?(\square A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\square A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \sup M_A(x^p)\right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= \mu_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{\square(A)}^+(mxy) \leq \mu_{\square(?A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{?(\square A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \sup M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, 1 - \sup M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{?A}^+(x^p) \\
&= 1 - \mu_{?A}^+(x^p) \\
&= 1 - \mu_{\square(?A)}^+(x^p) \\
&= \nu_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{?(\square A)}^+(mxy) \geq \nu_{\square(?A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{?(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p) \\
&= \mu_{\square(?A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?(\square A)}^-(mxy) \geq \mu_{\square(?A)}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{?(\square A)}^-(mxy) &= \max\left(\frac{1}{2}, \nu_{\square A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{?A}^-(x^p) \\
&= 1 - \mu_{?A}^-(x^p)
\end{aligned}$$

$$= 1 - \mu_{\square(?A)}^-(x^p) \\ = \nu_{\square(?A)}^-(x^p)$$

Therefore, $\nu_{\square(?A)}^-(mxy) \leq \nu_{\square(?A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $?(\square A) = \square(?A)$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 9. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $!(\Diamond A) = \Diamond(!A)$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\text{Consider } \mu_{!(\Diamond A)}^+(mxy) = \min\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\ = \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\ = \min\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\ = \min\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\ \leq \min\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\ = \min\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\ = \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\ = \mu_{!A}^+(x^p) \\ = 1 - \nu_{!A}^+(x^p) \\ = 1 - \nu_{\Diamond(!A)}^+(x^p) \\ = \mu_{\Diamond(!A)}^+(x^p)$$

Therefore, $\mu_{!(\Diamond A)}^+(mxy) \leq \mu_{\Diamond(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\text{Consider } \nu_{!(\Diamond A)}^+(mxy) = \max\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ \geq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ = \nu_{!A}^+(x^p) \\ = \nu_{\Diamond(!A)}^+(x^p),$$

Therefore, $\nu_{!(\Diamond A)}^+(mxy) \geq \nu_{\Diamond(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\text{Consider } \mu_{!(\Diamond A)}^-(mxy) = \max\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\ = \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\ = \max\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right)$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p) \\
&= 1 - \nu_{!A}^-(x^p) \\
&= 1 - \nu_{\Diamond(!A)}^-(x^p) \\
&= \mu_{\Diamond(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{\Diamond(!A)}^-(mxy) \geq \mu_{\Diamond(!A)}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{\Diamond(!A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&\leq \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= \nu_{\Diamond(!A)}^-(x^p).
\end{aligned}$$

Therefore, $\nu_{\Diamond(!A)}^-(mxy) \leq \nu_{\Diamond(!A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!(\Diamond A) = \Diamond(!A)$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 10. If A is a primary interval-valued intuitionistic fuzzy M group of G , then $?(\Diamond A) = \Diamond(?A)$ is a primary interval-valued intuitionistic fuzzy M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{?(\Diamond A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= 1 - \nu_{?A}^+(x^p) \\
&= 1 - \nu_{\Diamond(?A)}^+(x^p)
\end{aligned}$$

$$= \mu_{\Diamond(\Diamond A)}^+(x^p)$$

Therefore, $\mu_{\Diamond(\Diamond A)}^+(mxy) \leq \mu_{\Diamond(\Diamond A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \nu_{\Diamond(\Diamond A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\ &= \min\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ &= \min\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ &= \nu_{\Diamond A}^+(x^p) \\ &= \nu_{\Diamond(\Diamond A)}^+(x^p) \end{aligned}$$

Therefore, $\nu_{\Diamond(\Diamond A)}^+(mxy) \geq \nu_{\Diamond(\Diamond A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \mu_{\Diamond(\Diamond A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\ &= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ &= \mu_{\Diamond A}^-(x^p) \\ &= 1 - \nu_{\Diamond A}^-(x^p) \\ &= 1 - \nu_{\Diamond(\Diamond A)}^-(x^p) \\ &= \mu_{\Diamond(\Diamond A)}^-(x^p) \end{aligned}$$

Therefore, $\mu_{\Diamond(\Diamond A)}^-(mxy) \geq \mu_{\Diamond(\Diamond A)}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \nu_{\Diamond(\Diamond A)}^-(mxy) &= \max\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\ &= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ &= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ &\leq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\ &= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\ &= \nu_{\Diamond A}^-(x^p) \\ &= \nu_{\Diamond(\Diamond A)}^-(x^p) \end{aligned}$$

Therefore, $\nu_{\Diamond(\Diamond A)}^-(mxy) \leq \nu_{\Diamond(\Diamond A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $\Diamond(\Diamond A) = \Diamond(\Diamond A)$ is a primary interval-valued intuitionistic fuzzy M group of G . \square

Theorem 11. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $!A$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned} \text{Consider } \mu_{!A}^+(mxy) &= \min\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\ &= \min\left(\frac{1}{2}, \sup M_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, \sup M_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\ &= \mu_{!A}^+(x^p) \end{aligned}$$

Therefore, $\mu_{!A}^+(mxy) \geq \mu_{!A}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \nu_{!A}^+(mxy) &= \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ &= \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ &\leq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ &= \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ &= \nu_{!A}^+(x^p) \end{aligned}$$

Therefore, $\nu_{!A}^+(mxy) \leq \nu_{!A}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \mu_{!A}^-(mxy) &= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\ &= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\ &\leq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\ &= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ &= \mu_{!A}^-(x^p) \end{aligned}$$

Therefore, $\mu_{!A}^-(mxy) \leq \mu_{!A}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned} \text{Consider } \nu_{!A}^-(mxy) &= \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\ &= \nu_{!A}^-(x^p) \end{aligned}$$

Therefore, $\nu_{!A}^-(mxy) \geq \nu_{!A}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!A$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 12. If A and B are primary interval-valued intuitionistic fuzzy anti M group of G , then $!(A \cap B) = !A \cap !B$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $x, y \in B$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{!(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\
&= \min\left(\frac{1}{2}, \min(supM_A(mxy), supM_B(mxy))\right) \\
&\geq \min\left(\frac{1}{2}, \min(supM_A(x^p), supM_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\
&= \min\left(\min\left(\frac{1}{2}, \mu_A^+(x^p)\right), \min\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\
&= \min(\mu_{!A}^+(x^p), \mu_{!B}^+(x^p)) \\
&= \mu_{!A \cap !B}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{!(A \cap B)}^+(mxy) \geq \mu_{!A \cap !B}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{!(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, \nu_{A \cap B}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \max(\nu_A^+(mxy), \nu_B^+(mxy))\right) \\
&= \max\left(\frac{1}{2}, \max(supN_A(mxy), supN_B(mxy))\right) \\
&\leq \max\left(\frac{1}{2}, \max(supN_A(x^p), supN_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \max(\nu_A^+(x^p), \nu_B^+(x^p))\right) \\
&= \max\left(\max\left(\frac{1}{2}, \nu_A^+(x^p)\right), \max\left(\frac{1}{2}, \nu_B^+(x^p)\right)\right) \\
&= \max(\nu_{!A}^+(x^p), \nu_{!B}^+(x^p)) \\
&= \nu_{!A \cap !B}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{!(A \cap B)}^+(mxy) \leq \nu_{!A \cap !B}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{!(A \cap B)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy))\right) \\
&= \max\left(\frac{1}{2}, \max(infM_A(mxy), infM_B(mxy))\right) \\
&\leq \max\left(\frac{1}{2}, \max(infM_A(x^p), infM_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p))\right) \\
&= \max\left(\max\left(\frac{1}{2}, \mu_A^-(x^p)\right), \max\left(\frac{1}{2}, \mu_B^-(x^p)\right)\right) \\
&= \max(\mu_{!A}^-(x^p), \mu_{!B}^-(x^p)) \\
&= \mu_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{!(A \cap B)}^-(mxy) \leq \mu_{!A \cap !B}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{!(A \cap B)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{A \cap B}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy))\right)
\end{aligned}$$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy))\right) \\
&\geq \min\left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \min(v_A^-(x^p), v_B^-(x^p))\right) \\
&= \min\left(\min\left(\frac{1}{2}, v_A^-(x^p)\right), \min\left(\frac{1}{2}, v_B^-(x^p)\right)\right) \\
&= \min(v_{!A}^-(x^p), v_{!B}^-(x^p)) \\
&= v_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore, $v_{!A \cap !B}^-(mxy) \geq v_{!A \cap !B}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!(A \cap B) = !A \cap !B$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 13. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $?A$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{?A}^+(mxy) &= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \sup M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?A}^+(mxy) \geq \mu_{?A}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
v_{?A}^+(mxy) &= \min\left(\frac{1}{2}, v_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \sup N_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, v_A^+(x^p)\right) \\
&= v_{?A}^+(x^p)
\end{aligned}$$

Therefore, $v_{?A}^+(mxy) \leq v_{?A}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{?A}^-(mxy) &= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?A}^-(mxy) \leq \mu_{?A}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
v_{?A}^-(mxy) &= \max\left(\frac{1}{2}, v_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, v_A^-(x^p)\right) \\
&= v_{?A}^-(x^p)
\end{aligned}$$

Therefore, $v_{?A}^-(mxy) \geq v_{?A}^-(x^p)$, for some $p \in Z_+$.

Therefore, $?A$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 14. If A and B are primary interval-valued intuitionistic fuzzy anti M group of G , then $?A \cap B = ?A \cap ?B$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $x, y \in B$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{?(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\
&= \max\left(\frac{1}{2}, \min(\sup M_A(mxy), \sup M_B(mxy))\right) \\
&\geq \max\left(\frac{1}{2}, \min(\sup M_A(x^p), \sup M_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\
&= \min\left(\max\left(\frac{1}{2}, \mu_A^+(x^p)\right), \max\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\
&= \min(\mu_{?A}^+(x^p), \mu_{?B}^+(x^p)) \\
&= \mu_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?(A \cap B)}^+(mxy) \geq \mu_{?A \cap ?B}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
v_{?(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, v_{A \cap B}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max(v_A^+(mxy), v_B^+(mxy))\right) \\
&= \min\left(\frac{1}{2}, \max(\sup N_A(mxy), \sup N_B(mxy))\right) \\
&\leq \min\left(\frac{1}{2}, \max(\sup N_A(x^p), \sup N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \max(v_A^+(x^p), v_B^+(x^p))\right) \\
&= \max\left(\min\left(\frac{1}{2}, v_A^+(x^p)\right), \min\left(\frac{1}{2}, v_B^+(x^p)\right)\right) \\
&= \max(v_{?A}^+(x^p), v_{?B}^+(x^p)) \\
&= v_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore, $v_{?(A \cap B)}^+(mxy) \leq v_{?A \cap ?B}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{?(A \cap B)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{A \cap B}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy))\right) \\
&= \min\left(\frac{1}{2}, \max(\inf M_A(mxy), \inf M_B(mxy))\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \min\left(\frac{1}{2}, \max(\inf M_A(x^p), \inf M_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p))\right) \\
&= \max\left(\min\left(\frac{1}{2}, \mu_A^-(x^p)\right), \min\left(\frac{1}{2}, \mu_B^-(x^p)\right)\right) \\
&= \max(\mu_{?A}^-(x^p), \mu_{?B}^-(x^p)) \\
&= \mu_{?A \cap ?B}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?(A \cap B)}^-(mxy) \leq \mu_{?A \cap ?B}^-(x^p)$, for some $p \in Z_+$.

Consider $\nu_{?(A \cap B)}^-(mxy) = \max\left(\frac{1}{2}, \nu_{A \cap B}^-(mxy)\right)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy))\right) \\
&= \max\left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy))\right) \\
&\geq \max\left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \min(\nu_A^-(x^p), \nu_B^-(x^p))\right) \\
&= \min\left(\max\left(\frac{1}{2}, \nu_A^-(x^p)\right), \max\left(\frac{1}{2}, \nu_B^-(x^p)\right)\right) \\
&= \min(\nu_{?A}^-(x^p), \nu_{?B}^-(x^p)) \\
&= \nu_{?A \cap ?B}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{?(A \cap B)}^-(mxy) \geq \nu_{?A \cap ?B}^-(x^p)$, for some $p \in Z_+$.

Therefore, $?A \cap B = ?A \cap ?B$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 15. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $\overline{?A} = !A$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider $\mu_{?\overline{A}}^+(mxy) = \nu_{?\overline{A}}^+(mxy)$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \nu_{\overline{A}}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \sup M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{!A}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?\overline{A}}^+(mxy) \geq \mu_{!A}^+(x^p)$, for some $p \in Z_+$.

Consider $\nu_{?\overline{A}}^+(mxy) = \mu_{?\overline{A}}^+(mxy)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \mu_{\overline{A}}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup N_A(mxy)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, v_A^+(x^p)\right) \\
&= v_{!A}^+(x^p)
\end{aligned}$$

Therefore, $v_{?A}^+(mxy) \leq v_{!A}^+(x^p)$, for some $p \in Z_+$.

Consider $\mu_{?A}^-(mxy) = v_{?A}^-(mxy)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, v_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?A}^-(mxy) \leq \mu_{!A}^-(x^p)$, for some $p \in Z_+$.

Consider $v_{?A}^-(mxy) = \mu_{?A}^-(mxy)$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, v_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, v_A^-(x^p)\right) \\
&= v_{!A}^-(x^p)
\end{aligned}$$

Therefore, $v_{?A}^-(mxy) \geq v_{!A}^-(x^p)$, for some $p \in Z_+$.

Therefore, $?A = !A$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 16. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $!(?A) = ?(!A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned}
\text{Consider } \mu_{!(?A)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{?A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(mxy)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(mxy)\right)\right) \\
&\geq \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \mu_{!A}^+(x^p)\right) \\
&= \mu_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{!(!A)}^+(mxy) \geq \mu_{?(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } v_{!(!A)}^+(mxy) &= \max\left(\frac{1}{2}, v_{?A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, v_A^+(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(mxy)\right)\right) \\
&\leq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, v_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, v_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, v_{!A}^+(x^p)\right) \\
&= v_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore, $v_{!(!A)}^+(mxy) \leq v_{?(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \mu_{!(!A)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{?A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(mxy)\right)\right) \\
&\leq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \mu_{!A}^-(x^p)\right) \\
&= \mu_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{!(!A)}^-(mxy) \leq \mu_{?(!A)}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } v_{!(!A)}^-(mxy) &= \min\left(\frac{1}{2}, v_{?A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, v_A^-(mxy)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \inf N_A(mxy)\right)\right) \\
&\geq \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \inf N_A(x^p)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \min \left(\frac{1}{2}, \max \left(\frac{1}{2}, \nu_A^-(x^p) \right) \right) \\
&= \max \left(\frac{1}{2}, \min \left(\frac{1}{2}, \nu_A^-(x^p) \right) \right) \\
&= \max \left(\frac{1}{2}, \nu_{!A}^-(x^p) \right) \\
&= \nu_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{!(!A)}^-(mxy) \geq \nu_{?(!A)}^-(x^p)$, for some $p \in Z_+$. Therefore, $!(!A) = ?(!A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 17. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $!(\square A) = \square(!A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{!(\square A)}^+(mxy) &= \min \left(\frac{1}{2}, \mu_{\square A}^+(mxy) \right) \\
&= \min \left(\frac{1}{2}, \mu_A^+(mxy) \right) \\
&= \min \left(\frac{1}{2}, \sup M_A(mxy) \right) \\
&\geq \min \left(\frac{1}{2}, \sup M_A(x^p) \right) \\
&= \min \left(\frac{1}{2}, \mu_A^+(x^p) \right) \\
&= \mu_{!A}^+(x^p) \\
&= \mu_{\square(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{!(\square A)}^+(mxy) \geq \mu_{\square(!A)}^+(x^p)$, for some $p \in Z_+$

Consider

$$\begin{aligned}
\nu_{!(\square A)}^+(mxy) &= \max \left(\frac{1}{2}, \nu_{\square A}^+(mxy) \right) \\
&= \max \left(\frac{1}{2}, 1 - \mu_A^+(mxy) \right) \\
&= \max \left(\frac{1}{2}, 1 - \sup M_A(mxy) \right) \\
&\leq \max \left(\frac{1}{2}, 1 - \sup M_A(x^p) \right) \\
&= \max \left(\frac{1}{2}, 1 - \mu_A^+(x^p) \right) \\
&= \max \left(\frac{1}{2}, \nu_A^+(x^p) \right) \\
&= \nu_{!A}^+(x^p) \\
&= 1 - \mu_{!A}^+(x^p) \\
&= 1 - \mu_{\square(!A)}^+(x^p) \\
&= \nu_{\square(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{!(\square A)}^+(mxy) \leq \nu_{\square(!A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{!(\square A)}^-(mxy) &= \max \left(\frac{1}{2}, \mu_{\square A}^-(mxy) \right) \\
&= \max \left(\frac{1}{2}, \mu_A^-(mxy) \right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p) \\
&= \mu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{!(\square A)}^-(mxy) \leq \mu_{\square(!A)}^-(x^p)$, for some $p \in Z_+$

Consider

$$\begin{aligned}
\nu_{!(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= 1 - \mu_{!A}^-(x^p) \\
&= 1 - \mu_{\square(!A)}^-(x^p) \\
&= \nu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{!(\square A)}^-(mxy) \geq \nu_{\square(!A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!(\square A) = \square(!A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 18. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $?(\square A) = \square(?A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{?(\square A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\square A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \sup M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= \mu_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?(\square A)}^+(mxy) \geq \mu_{\square(?A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{?(\square A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(mxy)\right)
\end{aligned}$$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, 1 - \sup M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, 1 - \sup M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(x^p)\right) \\
&= \min\left(\frac{1}{2}, v_A^+(x^p)\right) \\
&= v_{?A}^+(x^p) \\
&= 1 - \mu_{?A}^+(x^p) \\
&= 1 - \mu_{\square(?A)}^+(x^p) \\
&= v_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore, $v_{?(\square A)}^+(mxy) \leq v_{\square(?A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\mu_{?(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p) \\
&= \mu_{\square(?A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{?(\square A)}^-(mxy) \leq \mu_{\square(?A)}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
v_{?(\square A)}^-(mxy) &= \max\left(\frac{1}{2}, v_{\square A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, v_A^-(x^p)\right) \\
&= v_{?A}^-(x^p) \\
&= 1 - \mu_{?A}^-(x^p) \\
&= 1 - \mu_{\square(?A)}^-(x^p) \\
&= v_{\square(?A)}^-(x^p).
\end{aligned}$$

Therefore, $v_{?(\square A)}^-(mxy) \geq v_{\square(?A)}^-(x^p)$, for some $p \in Z_+$. Therefore, $?(\square A) = \square(?A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 19. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $!(\diamond A) = \diamond !(A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

$$\begin{aligned}
\text{Consider } \mu_{!(\Diamond A)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{!A}^+(x^p) \\
&= 1 - \nu_{!A}^+(x^p) \\
&= 1 - \nu_{\Diamond(!A)}^+(x^p) \\
&= \mu_{\Diamond(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{!(\Diamond A)}^+(mxy) \geq \mu_{\Diamond(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \nu_{!(\Diamond A)}^+(mxy) &= \max\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{!A}^+(x^p) \\
&= \nu_{\Diamond(!A)}^+(x^p)
\end{aligned}$$

Therefore, $\nu_{!(\Diamond A)}^+(mxy) \leq \nu_{\Diamond(!A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \mu_{!(\Diamond A)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p)
\end{aligned}$$

$$\begin{aligned}
&= 1 - \nu_{!A}^-(x^p) \\
&= 1 - \nu_{\Diamond(!A)}^-(x^p) \\
&= \mu_{\Diamond(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{\Diamond(!A)}^-(mxy) \leq \mu_{\Diamond(!A)}^-(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{!(\Diamond A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&\geq \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= \nu_{\Diamond(!A)}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{!(\Diamond A)}^-(mxy) \geq \nu_{\Diamond(!A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $!(\Diamond A) = \Diamond(!A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

Theorem 20. If A is a primary interval-valued intuitionistic fuzzy anti M group of G , then $?(\Diamond A) = \Diamond(?A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G .

Proof. Consider $x, y \in A$ and $m \in M$.

Consider

$$\begin{aligned}
\mu_{?(\Diamond A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= 1 - \nu_{?A}^+(x^p) \\
&= 1 - \nu_{\Diamond(?A)}^+(x^p) \\
&= \mu_{\Diamond(?A)}^+(x^p)
\end{aligned}$$

Therefore, $\mu_{?(\Diamond A)}^+(mxy) \geq \mu_{\Diamond(?A)}^+(x^p)$, for some $p \in Z_+$.

Consider

$$\begin{aligned}
\nu_{?(\Diamond A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \sup N_A(mxy)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{?A}^+(x^p) \\
&= \nu_{\Diamond(?A)}^+(x^p).
\end{aligned}$$

Therefore, $\nu_{\Diamond(?A)}^+(mxy) \leq \nu_{\Diamond(?A)}^+(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \mu_{\Diamond(?A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p) \\
&= 1 - \nu_{?A}^-(x^p) \\
&= 1 - \nu_{\Diamond(?A)}^-(x^p) \\
&= \mu_{\Diamond(?A)}^-(x^p)
\end{aligned}$$

Therefore, $\mu_{\Diamond(?A)}^-(mxy) \leq \mu_{\Diamond(?A)}^-(x^p)$, for some $p \in Z_+$.

$$\begin{aligned}
\text{Consider } \nu_{\Diamond(?A)}^-(mxy) &= \max\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{?A}^-(x^p) \\
&= \nu_{\Diamond(?A)}^-(x^p)
\end{aligned}$$

Therefore, $\nu_{\Diamond(?A)}^-(mxy) \geq \nu_{\Diamond(?A)}^-(x^p)$, for some $p \in Z_+$.

Therefore, $?(\Diamond A) = \Diamond(?A)$ is a primary interval-valued intuitionistic fuzzy anti M group of G . \square

4 Conclusion

In this paper the main idea of primary interval-valued intuitionistic fuzzy M group and primary interval-valued intuitionistic fuzzy anti M group are a new algebraic structures of fuzzy algebra and it is used through the level operators. We believe that our ideas can also applied for other algebraic system.

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