Thirteenth Int. Conf. on IFSs, Sofia, 9-10 May 2009
NIFS Vol. 15 (2009), 1, 26-31

Theorem for equivalence of the two most general intuitionistic fuzzy modal operators

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In a series of research the two types of modal operators, defined over the Intuitionistic Fuzzy Sets (IFSs, [3] were generalized to two operators. Here, we will prove that they coincide.

Over IFSs there have been defined not only operations and relations similar to the ordinary fuzzy set ones, but also operators that cannot be defined in the case of ordinary fuzzy sets.

Let a set E be fixed. An IFS A over E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let for every $x \in E$:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

For every two IFSs A and B a lot of relations and operations are defined (see, e.g. [3]), the most important of these are:

$$A \subset B \quad \text{iff} \quad (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x));$$
$$A = B \quad \text{iff} \quad (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x));$$

The standard modal operators (see, e.g. [10]) obtained IFS-form in the first author's paper [1]:

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

After this, they were object of a sequence of extensions:

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\};$$
$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1;$$
$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\}.$$
$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$
$$H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$
$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$
$$J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\}.$$

This sequence finishes with operator

$$X_{a,b,c,d,e,f}(A) = \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)),$$
$$d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}, \quad (1)$$

where $a, b, c, d, e, f \in [0, 1]$ and:

$$a + e - e.f \leq 1, \quad (2)$$
$$b + d - b.c \leq 1. \quad (3)$$

This operator can represent all previous ones (see, e.g., [3]).

On the other hand, a modification of the modal operator were introduced (see, [2, 3, 4, 5, 9]):

$$\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle | x \in E\},$$
$$\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle | x \in E\}.$$

and they also were extended sequentially to:

$$\boxplus_\alpha A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + 1 - \alpha\rangle | x \in E\},$$
$$\boxtimes_\alpha A = \{\langle x, \alpha.\mu_A(x) + 1 - \alpha, \alpha.\nu_A(x)\rangle | x \in E\},$$

for $\alpha \in [0, 1]$;

$$\boxplus_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + \beta\rangle | x \in E\},$$
$$\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x) + \beta, \alpha.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$ (this extension is introduced in [9] by Katerina Dencheva),

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x) + \gamma\rangle | x \in E\},$$
$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$,

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha.\mu_A(x) + 1 - \alpha), \alpha(\beta.\nu_A(x) + 1 - \beta)\rangle | x \in E\},$$

where $\alpha, \beta \in [0,1]$ (this extension is introduced in [8] by Gökhan Cuvalcioĝlu),

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0,1]$ and

$$\max(\alpha, \beta) + \gamma + \delta \leq 1$$

introduced in [5, 6].

The most general form of these operators is operator:

$$\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A = \{\langle x, \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma, \beta.\nu_A(x) - \zeta.\mu_A(x) + \delta\rangle | x \in E\}, \quad (4)$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$ and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1, \quad (5)$$

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0 \quad (6)$$

introduced in [7].

Here, we shall prove the following

Theorem. Operators $X_{a,b,c,d,e,f}$ and $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A$ are equivalent.

Proof Let $a, b, c, d, e, f \in [0,1]$ and satisfy (2) and (3) . Let us put

$$\alpha = a - b,$$

$$\beta = d - e,$$

$$\gamma = b,$$

$$\delta = e,$$

$$\varepsilon = bc,$$

$$\zeta = ef.$$

Let

$$X \equiv \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma = (a - b).\mu_A(x) - b.c.\nu_A(x) + b,$$

$$Y \equiv \beta.\nu_A(x) - \zeta.\mu_A(x) + \delta = (d - e).\nu_A(x) - e.f.\mu_A(x) + e.$$

Then

$$X \geq (a - b).0 - b.c.1 + b = b.(1 - c) \geq 0,$$

$$X \leq (a - b).1 - b.c.0 + b = a \leq 1,$$

$$Y \geq (d - e).0 - e.f.1 + e = e.(1 - f) \geq 0,$$

$$Y \leq (d - e).1 - e.f.0 + e = d \leq 1$$

and

$$X + Y = (a - b).\mu_A(x) - b.c.\nu_A(x) + b + (d - e).\nu_A(x) - e.f.\mu_A(x) + e$$

$$= (a - b - e.f).\mu_A(x) + (d - e - b.c).\nu_A(x) + b + e$$
$$\leq (a - b - e.f).\mu_A(x) + (d - e - b.c).(1 - \mu_A(x)) + b + e$$
$$= d - e - b.c + b + e + (a - b - e.f - d + e + b.c).\mu_A(x))$$
$$\leq d - b.c + b + (a - b - e.f - d + e + b.c)$$
$$= a - e.f + e \leq 1$$

(from (2)).

Then we obtain that

$$\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A = \{\langle x, \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma, \beta.\nu_A(x) - \zeta.\mu_A(x) + \delta\rangle | x \in E\}$$
$$= \{\langle x, (a - b).\mu_A(x) - b.c.\nu_A(x) + b, (d - e).\nu_A(x) - e.f.\mu_A(x) + e\rangle | x \in E\}$$
$$= \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)), d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$$
$$= X_{a,b,c,d,e,f}(A).$$

In the opposite case, let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ and satisfy (5) and (6). From (6) it follows that for $\alpha = \beta = \delta = \zeta = 0$: $\varepsilon \leq \gamma$, while for $\alpha = \beta = \gamma = \varepsilon = 0$: $\zeta \leq \delta$; from (5) it follows that for $\beta = \delta = \varepsilon = \zeta = 0$: $\alpha + \gamma \leq 1$, while for $\alpha = \gamma = \varepsilon = \zeta = 0$: $\beta + \delta \leq 1$. Then, let us put

$$a = \alpha + \gamma \ (\leq 1),$$
$$b = \gamma,$$
$$c = \frac{\varepsilon}{\gamma} \ (\leq 1),$$
$$d = \beta + \delta \ (\leq 1),$$
$$e = \delta,$$
$$f = \frac{\zeta}{\delta} \ (\leq 1).$$

Let

$$X \equiv a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)) = (\alpha + \gamma).\mu_A(x) + \gamma.(1 - \mu_A(x) - \frac{\varepsilon}{\gamma}.\nu_A(x)),$$

$$Y \equiv d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x)) = (\beta + \delta).\nu_A(x) + \delta.(1 - \frac{\zeta}{\delta}.\mu_A(x) - \nu_A(x)).$$

Then from above we obtain:

$$0 \leq \gamma - \varepsilon \leq X = \alpha.\mu_A(x) + \gamma - \varepsilon.\nu_A(x) \leq \alpha + \gamma \leq 1,$$

$$0 \leq \delta - \zeta \leq Y = \beta.\nu_A(x) + \delta - \zeta.\mu_A(x) \leq \beta + \delta \leq 1,$$

$$X + Y = \alpha.\mu_A(x) + \gamma - \varepsilon.\nu_A(x) + \beta.\nu_A(x) + \delta - \zeta.\mu_A(x)$$
$$= (\alpha - \zeta).\mu_A(x) - (\beta - \varepsilon).\nu_A(x) + \gamma + \delta$$

$$\leq (\alpha - \zeta).\mu_A(x) - (\beta - \varepsilon).(1 - \mu_A(x)) + ga + \delta$$

$$= (\alpha - \zeta + \beta - \varepsilon).\mu_A(x) - \beta + \gamma + \delta + \varepsilon$$

$$\leq \alpha - \zeta + \beta - \varepsilon - \beta + \gamma + \delta + \varepsilon$$

$$= \alpha - \zeta + \gamma + \delta$$

$$\leq \max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1$$

(from (5)).

Then we obtain that

$$X_{a,b,c,d,e,f}(A)$$

$$= \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)), d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$$

$$= \{\langle x, (\alpha+\gamma).\mu_A(x)+\gamma.(1-\mu_A(x)-\frac{\varepsilon}{\gamma}.\nu_A(x)), (\beta+\delta).\nu_A(x)+\delta.(1-\frac{\zeta}{\delta}.\mu_A(x)-\nu_A(x))\rangle | x \in E\}$$

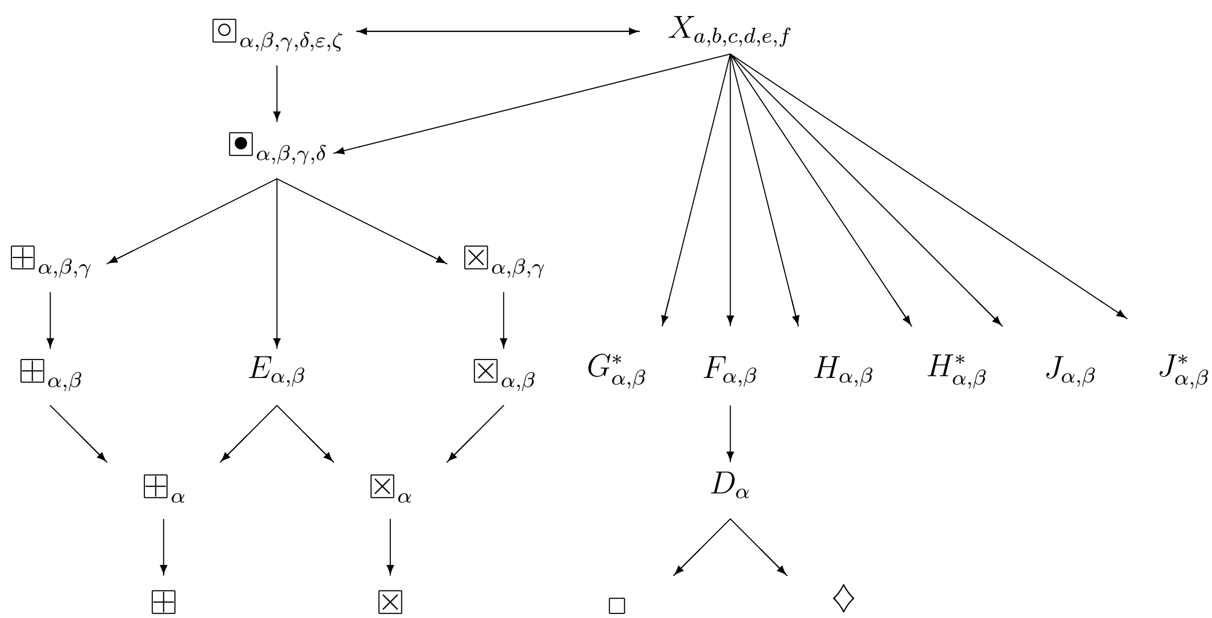
$$= \{\langle x, (\alpha+\gamma).\mu_A(x)+\gamma-\gamma.\mu_A(x)-\varepsilon.\nu_A(x), (\beta+\delta).\nu_A(x)+\delta-\zeta.\mu_A(x)-\delta.\nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma, \beta.\nu_A(x) - \zeta.\mu_A(x) + \delta\rangle | x \in E\}$$

$$= \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A.$$

Therefore, the two operators are equivalent.

Finally, we shall construct the following diagram in which "$X \to Y$" denotes that operator X represents operator Y, while the opposite is not valid.



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