Intuitionistic Fuzzy Metric over R_Δ

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Abstract

In this paper an approach to define a intuitionistic fuzzy metric is presented. There have been many suggestions for intuitionistic fuzzy metrics. In the sequel the author defines intuitionistic fuzzy metric as intuitionistic fuzzified distance mapping

Introduction

The interest in fuzzy sets has been constantly growing since L. Zadeh has first proposed the concept in 1965 [4]. A large amount of theoretical and applied results is achieved in the directions of algebra, logic, mathematical analysis, geometry, and topology. Even more applications have been developed: fuzzy sets are widely used in process control, linguistic analysis etc.

An important theoretical development is the way of defining a fuzzy real line as a generalisation of the set of real numbers taken with fuzzy degrees of membership (see, for example, [1] and [5]).

In 1983, K. Atanassov proposed a further generalisation of the notion of fuzzy set, known as *intuitionistic fuzzy set* (IFS) [2]. There, every element of a set has, in addition to its degree of membership μ_x, one more degree – of non-membership ν_x, and as a result – a degree of uncertainty $\pi_x = 1 - (\mu_x + \nu_x)$.

The present work is devoted to defining and studying the properties of the notion of intuitionistic fuzzy real number and the collection of these – the intuitionistic fuzzy real line.

The only definition of such a notion, as far as the author is informed, is in [6], where only the definition is given and some properties are studied concerning the relationship between the proposed notion and operators over IFS. The approach adopted there is principally different from the one we propose.

1 Intuitionistic fuzzy sets

An intuitionistic fuzzy sets A (IFS) (see [2]) is a set each element x of which is assigned three real numbers μ, ν, and π, called "degree of membership", "degree of non-membership", and "degree of uncertainty" of x to A, and $\mu + \nu + \pi = 1$.

Definition 1 *Let E be a fixed set. We will call the object*

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

IFS A^* in E, where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ specify the degrees of membership and non-membership of each element $x \in E$ to the set A, and

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

If

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x),$$

then $\pi_A(x)$ is the degree of uncertainty of the element $x \in E$ to the set A. With fuzzy sets, $\pi_A(x) = 0$ for all $x \in E$.

2 Intuitionistic fuzzy real line

The notion of intuitionistic fuzzy real line is a natural generalisation of the notion of fuzzy real line (presented by P. Eklund and W. Gähler in [1]), within the theory of IFS.

While fuzzy real line occupies a strip in R^2, intuitionistic fuzzy real line will, generally speaking, be within an infinite prism.

Consider the orthogonal triangle $\Delta \subset [0, 1] \times [0, 1]$:

$$\Delta = \{(\mu, \pi) | \mu > 0, \pi \ge 0, \mu + \pi \le 1\}$$

Since $\mu + \nu + \pi = 1$, for all $(\mu, \pi) \in \Delta$ the degree of non-membership ν is uniquely determined. For example, $\nu = 0$ iff (μ, π) belongs to the hypotenuse.

Definition 2 *The mappings $x : R \to \Delta$ are called* intuitionistic fuzzy numbers (IFN).

Everywhere below, where an angle range is specified, by π we will mean the constant $\pi = 3.14159\ldots$ and we will use π to denote the degree of uncertainty discussed above.

Let us have for $\varphi \in [0, \frac{\pi}{2})$

$$a_\varphi = \{(\mu, \pi) \in \Delta | \pi.\cos\varphi = \mu.\sin\varphi\}$$

$$\alpha_\varphi = \{(t, \mu, \pi) | t \in R, (\mu, \pi) \in a_\varphi\}$$

Definition 3 *An IF number $x : R \to \Delta$ is called* planar *iff there exists $\varphi \in [0, \frac{\pi}{2})$ such that for all $t \in R$ $(t, x(t)) \in \alpha_\varphi$. For the sake of convenience, such IF numbers will be denoted by x^φ.*

We define a non-strict total ordering over a_φ as follows:

$$(\mu_1, \pi_1) \leq (\mu_2, \pi_2) \Leftrightarrow \mu_1 \leq \mu_2$$

and a strict total ordering over a_φ by

$$(\mu_1, \pi_1) < (\mu_2, \pi_2) \Leftrightarrow \mu_1 < \mu_2.$$

The two relations above are introduced in [7], where all logically acceptable relations "$\leq$" over intuitionistic fuzzy values are studied.

Definition 4 *Let x^φ be a planar IF number. The set*

$$x_{(\mu,\pi)}^\varphi = \{t \in R | x^\varphi(t) \geq (\mu, \pi)\}$$

for $(\mu, \pi) \in a_\varphi \backslash \{(0,0)\}$, is called (μ, π)-level set of x^φ.

Let

$$b_{\varphi,\nu} = \{(t, \mu, \pi) \in \alpha_\varphi | \mu + \pi = 1 - \nu\}$$

Definition 5 *An IF number x^φ is called* normal *if there exists $t \in R$ such that $(t, x^\varphi(t)) \in b_{\varphi,0}$.*

Definition 6 *An IF number x^φ is called* convex *if for all $r, s, t \in R$, $r \leq s \leq t$ implies $\min\{x^\varphi(r), x^\varphi(t)\} \leq x^\varphi(s)$.*

Theorem 2.1 *An IF number x^φ is convex iff every one of its level sets $x_{(\mu,\pi)}^\varphi$ is a convex set in R.*

Proof:
$\Rightarrow$) Let $r, s \in x_{(\mu,\pi)}^\varphi$ and let $p \in [r, s]$. From the convexity of x^φ it follows that if $\min\{x^\varphi(r), x^\varphi(s)\} \leq x^\varphi(p)$, then $(\mu, \pi) \leq x^\varphi(p)$, i. e., $p \in x_{(\mu,\pi)}^\varphi$, whence $x_{(\mu,\pi)}^\varphi$ is a convex set.

$\Leftarrow$) Let $x_{(\mu,\pi)}^\varphi$ be convex set for all $(\mu, \pi) \in a_\varphi \backslash \{(0,0)\}$. Then, there exists no p such that if $p \in [r, s]$ for $r, s \in x_{(\mu,\pi)}^\varphi$, then $p \notin x_{(\mu,\pi)}^\varphi$. Therefore $\min\{x^\varphi(r), x^\varphi(s)\} \leq x^\varphi(p)$. $\Diamond$

Definition 7 *An IF number x^φ is called* upper semicontinuous *if for all $t \in R$ and $(\mu, \pi) \in a_\varphi$ and $x^\varphi(t) < (\mu, \pi)$ there exists $\delta > 0$ such that if $|s - t| \leq \delta$, then $x^\varphi(s) < (\mu, \pi)$.*

Definition 8 *An IF number x^φ is called* IF regular, *if it is planar, convex, normal, upper semicontinuous and all of its level sets are bounded.*

Definition 9 *The set of all IF regular numbers is called* IF real line *and is denoted by R_Δ.*

Theorem 2.2 *The level sets of any IF regular number x^φ are closed finite intervals:*

$$x_{(\mu,\pi)}^\varphi = [x_{(\mu,\pi)1}^\varphi, x_{(\mu,\pi)2}^\varphi] \subset R.$$

Proof: From theorem 2.1 we have that x^φ is convex. Therefore, so are its level sets $x^\varphi_{(\mu,\pi)}$ for $(\mu,\pi) \in a_\varphi \backslash \{(0,0)\}$.

Then, they are intervals of the real line.

From the definition of $x^\varphi_{(\mu,\pi)}$ and the fact that they are bounded it follows that they are closed finite intervals. $\diamond$

The IF real line R_Δ can be regarded as an infinite prism, whose basis is the rectangular triangle Δ. The fuzzy real line R_L [1], in the case when $L = [0,1]$, is a subset α_0 of R_Δ. Therefore the IF real line is a correct generalisation of the fuzzy case.

Definition 10 *Every real number* $t \in R$ *generates for all* $\varphi \geq 0$ *IF regular numbers* $(t^\varphi)^\sim$, *determined by the formula*

$$(t^\varphi)^\sim(s) = \begin{cases} (\frac{1}{1+\ tg\,\varphi}, \frac{tg\,\varphi}{1+\ tg\,\varphi}), & \text{if } s = t \\ (0,0), & \text{if } s \neq t, \end{cases}$$

where $s \in R$.

Numbers of the kind $(t^\varphi)^\sim$ *will be called* non-fuzzy IF numbers.

Theorem 2.3 *Let* $x^\varphi_{(\mu_1,\pi_1)i}$ *be real numbers and* (μ_1, π_1) *ranges within* $a_\varphi \backslash \{(0,0)\}$,
for $\varphi \in [0, \frac{\pi}{2})$ *and* $i = 1, 2$.
The intervals $\left[x^\varphi_{(\mu_1,\pi_1)1}, x^\varphi_{(\mu_1,\pi_1)2} \right]$ *are level sets of a regular IF number* x^φ *iff*
1)$x^\varphi_{(\mu,\pi)1} \leq x^\varphi_{(\mu_1,\pi_1)1}$ *and* $x^\varphi_{(\mu_1,\pi_1)2} \leq x^\varphi_{(\mu,\pi)2}$
for all (μ,π) *and* $(\mu_1, \pi_1) \in a_\varphi \backslash \{(0,0)\}$, *for which* $(\mu,\pi) < (\mu_1, \pi_1)$.
2) for all $(\mu_1, \pi_1) \in a_\varphi \backslash \{(0,0)\}, i = 1, 2$

$$\lim_{(\mu,\pi)\uparrow(\mu_1,\pi_1)} x^\varphi_{(\mu,\pi)i} = x^\varphi_{(\mu_1,\pi_1)i}$$

(i. e. for all $\varepsilon > 0$ *there exists a* $(\mu_2, \pi_2) \in a_\varphi \backslash \{(0,0)\}$ *such that if* $(\mu_2, \pi_2) < (\mu,\pi) \leq (\mu_1, \pi_1)$, *then* $|x^\varphi_{(\mu,\pi)i} - x^\varphi_{(\mu_1,\pi_1)i}| < \varepsilon$.)

Definition 11 *If* x^{φ_1} *and* $x^{\varphi_2} \in R_\Delta$, *then*

$$(x^{\varphi_1} + y^{\varphi_2})(t) = \sup_{r+s=t} \min\{x^\varphi(r), y^\varphi(s)\}$$

$$(x^{\varphi_1}.y^{\varphi_2})(t) = \sup_{r.s=t} \min\{x^\varphi(r), y^\varphi(s)\},$$

where $\varphi = \varphi_1 + \varphi_2 - \frac{2\varphi_1.\varphi_2}{\pi}$

Comment: It is easy to see that $\varphi \geq \varphi_1$ and $\varphi \geq \varphi_2$, therefore the result of the addition or multiplication of IF numbers is an IF number with higher degree of uncertainty and with lower degree of membership. The φ defined in this way ensures that $+$ and $.$ have unique identities $(0^0)^\sim$ and $(1^0)^\sim$.

We define a unary "$-$" operation using "$+$" and "$.$".

Definition 12 $-y^\varphi = (-1^0)^\sim.y^\varphi$.

By "+" and ".", subtraction is uniquely determined.

$$x^{\varphi_1} - y^{\varphi_2} = x^{\varphi_1} + (-1^0)^{\sim}.y^{\varphi_2}$$

Definition 13 *For $\varphi \in [0, \frac{\pi}{2})$ and for all $(\mu, \pi) \in a_\varphi \backslash \{(0,0)\}$, $i \in \{1, 2\}$,*

$$x^{\varphi_1} \leq y^{\varphi_2} \Leftrightarrow x^\varphi_{(\mu,\pi)i} \leq y^\varphi_{(\mu,\pi)i}$$

Definition 14
$$x^{\varphi_1} < y^{\varphi_2},$$

if $x^{\varphi_1} + (\delta^\varphi)^{\sim} \leq y^{\varphi_2}$ for some $\delta > 0$ and for some $\varphi \in [0, \frac{\pi}{2})$.

Some properties of arithmetic operations and order relations see in [8].

3 Intuitionistic fuzzy metric over R_Δ

Definition 15 *By IF metric over the set X we will mean any mapping $d : X \times X \to R_\Delta$ with the following properties:*
1)$d(x, y) = 0^{\sim} \Leftrightarrow x = y$
2)$d(x, y) = d(y, x)$
3)$d(x, z) \leq d(x, y) + d(y, z)$

Let

$$\delta(x^{\varphi_1}, y^{\varphi_2})(\mu, \pi) = |x^{|\varphi_1 - \varphi_2|}_{(\mu,\pi)1} - y^{|\varphi_1 - \varphi_2|}_{(\mu,\pi}}1| + |x^{|\varphi_1 - \varphi_2|}_{(\mu,\pi)2} - y^{|\varphi_1 - \varphi_2|}_{(\mu,\pi}}2|$$

for all $x^{\varphi_1}, y^{\varphi_2} \in R_\Delta$ and $(\mu, \pi) \in a_{|\varphi_1 - \varphi_2|} \backslash \{(0,0)\}$.

The definition of IF metric *over R_Δ is given in the following.*

Theorem 3.1 *Let $x^{\varphi_1}, y^{\varphi_2} \in R_\Delta$. Then*

$$|x^{\varphi_1}, y^{\varphi_2}|_{(\mu,\pi)} = [0, \sup_{(\mu',\pi') \geq \mu,\pi} \delta(x^{\varphi_1}, y^{\varphi_2})(\mu', \pi')],$$

$(\mu', \pi') \in a_{|\varphi_1 - \varphi_2|} \backslash \{(0,0)\}$, are the level sets of a regular IF number $|x^{\varphi_1}, y^{\varphi_2}|$. The mapping $|., .| : (x^{\varphi_1}, y^{\varphi_2}) \to |x^{\varphi_1}, y^{\varphi_2}|$, $x^{\varphi_1}, y^{\varphi_2} \in R_\Delta$, is a IF metric over R_Δ.

Proof: Let $x^{\varphi_1}, y^{\varphi_2} \in R_\Delta$ and $(\mu, \pi) \in a_{|\varphi_1 - \varphi_2|} \backslash \{(0,0)\}$. Since

$$|x^{|\varphi_1 - \varphi_2|}_{(\mu',\pi')i} - y^{|\varphi_1 - \varphi_2|}_{(\mu',\pi'}}i| \leq |x^{|\varphi_1 - \varphi_2|}_{(\mu,\pi)i} - x^{|\varphi_1 - \varphi_2|}_{(\mu+\pi=1)}i| + |x^{|\varphi_1 - \varphi_2|}_{(\mu,\pi)i} - y^{|\varphi_1 - \varphi_2|}_{(\mu,\pi}}i| + |y^{|\varphi_1 - \varphi_2|}_{(\mu,\pi)i} - y^{|\varphi_1 - \varphi_2|}_{(\mu+\pi=1)}i|$$

for all $(\mu', \pi') \geq \mu, \pi \in a_{|\varphi_1 - \varphi_2|} \backslash \{(0,0)\}$ and $i = 1, 2$, $\sup_{(\mu',\pi') \geq \mu,\pi} \delta(x^{\varphi_1}, y^{\varphi_2})(\mu', \pi')$ is finite. That $|x^{\varphi_1}, y^{\varphi_2}|$ is a regular IF number follows from 2.3. We have to show that

$$\lim_{(\mu',\pi') \uparrow (\mu,\pi)} \sup_{(\mu'',\pi'') \geq \mu',\pi'} \delta(x^{\varphi_1}, y^{\varphi_2})(\mu'', \pi'') = \sup_{(\mu'',\pi'') \geq \mu,\pi} \delta(x^{\varphi_1}, y^{\varphi_2})(\mu'', \pi'').$$

But this is a consequence of

$$| \sup_{(\mu'',\pi'')\geq\mu',\pi'} \delta(x^{\varphi_1}, y^{\varphi_2})(\mu'', \pi'') - \sup_{(\mu'',\pi'')\geq\mu,\pi} \delta(x^{\varphi_1}, y^{\varphi_2})(\mu'', \pi'')| \leq$$

$$\sup_{(\mu'',\pi'')\geq\mu',\pi',(\mu'',\pi'')>\mu,\pi} |\delta(x^{\varphi_1}, y^{\varphi_2})(\mu'', \pi'') - \delta(x^{\varphi_1}, y^{\varphi_2})(\mu, \pi)|$$

and

$$lim_{(\mu',\pi')\uparrow(\mu,\pi)}\delta(x^{\varphi_1}, y^{\varphi_2})(\mu', \pi') = \delta(x^{\varphi_1}, y^{\varphi_2})(\mu, \pi).$$

Obviously, $|.,.|$ has the properties 1) - 3) of IF metric. $\diamondsuit$

Proposition 1. Let $x^{\varphi_1}, y^{\varphi_2}, z^{\varphi_3} \in R_\Delta$. Then

$$|x^{\varphi_1} + y^{\varphi_2}, x^{\varphi_1} + z^{\varphi_3}| = |y^\varphi, z^0|,$$

where $\varphi = (\varphi_2 - \varphi_3)(1 - \frac{2\varphi_1}{\pi})$.

Proof: Obvious. $\diamondsuit$

4 Conclusion

The goal of this paper is to define a new concept of Intuitionistic Fuzzy metric over IF real line. The results present various perspectives for further research. A natural continuation is the possibility of developing and investigating intuitionistic fuzzy topology.

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