Intuitionistic Fuzzy Sets - An Alternative Look

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Abstract

This paper concerns the theory of intuitionistic fuzzy sets according to Atanassov. If triangular norms, especially nonstrict Archimedean ones, are used, we propose a revision and a flexibilizing generalization of some fundamental notions and constructions of that theory. Its application to group decision making is outlined.

Keywords: Intuitionistic fuzzy set, Triangular norm, Scalar cardinality, Group decision making.

1 Introduction

The term "intuitionistic fuzzy sets" does function in two contexts in the literature devoted to fuzziness. In the first one, we then mean intuitionistic logic-based fuzzy sets developed in 1984 by Takeuti and Titani (see [11, 12]). At almost the same time and disregarding that the adjective "intuitionistic" is reserved for contructions related to intuitionistic logic, Atanassov proposed a generalization of fuzzy sets and called his constructions intuitionistic fuzzy sets, too (see e.g. [1, 2]). This a bit unfortunate and colliding name was and is a source of some reservations about his concept. On the other hand, one must emphasize that Atanassov's intuitionistic fuzzy sets do offer some interesting new possibilities of applications, e.g. in problems of decision making (see [8], [9]). More precisely, in comparison with fuzzy sets, they seem to be better suited for expressing a degree of hesitation of a decision maker. This paper is devoted to intuitionistic fuzzy sets from [1] equipped with triangular norm-based operations (see [3]). We will show in Section 3 that some fundamental notions of that theory do require a revision when nonstrict Archimedean triangular norms are used. That modified look will be applied in Section 4 to group decision making. Section 2 recollects the original concept of intuitionistic fuzzy sets with triangular norms using a bit modified notation and terminology.

2 Intuitionistic Fuzzy Sets

A fuzzy set is a nebular collection of elements from a universe $\boldsymbol{M}$ described by and identified with a (membership) function $A\colon \boldsymbol{M} \to [0, 1]$. An *intuitionistic fuzzy set* is instead a nebular collection of elements from $\boldsymbol{M}$ identified with a pair $\mathscr{E} = (A, A^d)$, where $A, A^d\colon \boldsymbol{M} \to [0, 1]$ and

$$\forall x \in \boldsymbol{M}\colon A(x) + A^d(x) \le 1. \tag{2.1}$$

Again, one interprets A as a membership function: $A(x)$ is a degree of membership of x in the intuitionistic fuzzy set $\mathscr{E}$, whereas A^d, a function *dual* to A, is understood as a *nonmembership function*, i.e. $A^d(x)$ does express a *degree of nonmembership* of x in that intuitionistic fuzzy set. A double optics is thus exploited in intuitionistic fuzzy set theory by applying in A a many-valued form of the classical membership predicate $\in$ and, on the other hand, by using a many-valued form of the dual classical predicate $\notin$ to construct A^d. The number

$$\chi_{\mathscr{E}}(x) = 1 - A(x) - A^d(x) \tag{2.2}$$

is called and understood as a *degree of hesitation* whether or not x is in $\mathscr{E}$. It is obvious that each fuzzy set A can be seen as an intuitionistic fuzzy set, namely as (A, A'), with the standard complementation $A'(x) = 1 - A(x)$. Trivially,

$$\forall x \in \boldsymbol{M}\colon A(x) + A^d(x) + \chi_{\mathscr{E}}(x) = 1. \tag{2.3}$$

Basic relations for and operations on intuitionistic fuzzy sets $\mathscr{E} = (A, A^d)$ and $\mathscr{F} = (B, B^d)$ with triangular norms are defined in the following way ([3]):

$$\mathscr{E} = \mathscr{F} \iff A = B \;\&\; A^d = B^d, \qquad (\textit{equality})$$

$$\mathscr{E} \subset \mathscr{F} \iff A \subset B \;\&\; B^d \subset A^d, \qquad (\textit{inclusion})$$

$$\mathscr{E} \cup_{\boldsymbol{t},\boldsymbol{s}} \mathscr{F} = (A \cup_{\boldsymbol{s}} B, A^d \cap_{\boldsymbol{t}} B^d),$$
(*sum induced by* ***t*** *and* ***s***)

$$\mathscr{E} \cap_{\boldsymbol{t},\boldsymbol{s}} \mathscr{F} = (A \cap_{\boldsymbol{t}} B, A^d \cup_{\boldsymbol{s}} B^d),$$
(*intersection induced by* ***t*** *and* ***s***)

$$\mathscr{E}' = (A^d, A) \qquad (\textit{complement})$$

with a t-norm $\boldsymbol{t}$, a t-conorm $\boldsymbol{s}$, and $(A \cap_{\boldsymbol{t}} B)(x) = A(x)\,\boldsymbol{t}\,B(x)$, $(A \cup_{\boldsymbol{s}} B)(x) = A(x)\,\boldsymbol{s}\,B(x)$ (see [4, 6]). However, the choice of $\boldsymbol{t}$ and $\boldsymbol{s}$ cannot be quite arbitrary. If we like $\mathscr{E} \cup_{\boldsymbol{t},\boldsymbol{s}} \mathscr{F}$ and $\mathscr{E} \cap_{\boldsymbol{t},\boldsymbol{s}} \mathscr{F}$ to be intuitionistic fuzzy sets, one has to assume that $\boldsymbol{t} \le \boldsymbol{s}^*$ (equivalently: $\boldsymbol{s} \le \boldsymbol{t}^*$), where $\boldsymbol{s}^*$ denotes the t-norm *associated* with $\boldsymbol{s}$, i.e. $a\,\boldsymbol{s}^*\,b = 1 - (1-a)\,\boldsymbol{s}\,(1-b)$.

3 An Alternative Look at Intuitionistic Fuzzy Sets

Whichever t-norm $\boldsymbol{t}$ and t-conorm $\boldsymbol{s}$ are used to generate sums and intersections of intuitionistic fuzzy sets the condition (2.1), being fundamental in the Atanassov's theory, offers the same way of aggregating $A(x)$ and $A^d(x)$. This seems to be a bit inconsistent and causes some discomfort. Let us try to find a more flexible method of that aggregation.

It is easy to notice that (1.1) can be rewritten as $A^d \subset A'$ or, more generally, as

$$A^d \subset A^{\nu} \tag{3.1}$$

with a negation ν, where $A^{\nu}(x) = \nu(A(x))$; a slightly modified form of this inclusion can be found in [13] in which one shows that intuitionistic fuzzy sets are L-fuzzy sets. The dual function A^d is thus something smaller than the complement A^{ν} of A. If $\boldsymbol{t}$ is (at least) left continuous and ν is the *negation induced by* $\boldsymbol{t}$, i.e. $\nu = \nu_{\boldsymbol{t}}$ with

$$\nu_{\boldsymbol{t}}(a) = \bigvee\{c \in [0,1]\colon a\,\boldsymbol{t}\,c = 0\},$$

then (3.1) is equivalent to the equality

$$A \cap_{\boldsymbol{t}} A^d = 1_{\varnothing}. \tag{3.2}$$

Since *strict t-norms*, i.e. t-norms being continuous and strictly increasing on $(0,1)\times(0,1)$, as well as the minimum t-norm $\wedge$ do not have zero divisors, the condition (3.2) is not interesting for that class of t-norms and, then, (2.1) can be maintained.

By an *Archimedean t-norm* we mean a continuous t-norm $\boldsymbol{t}$ such that $a\,\boldsymbol{t}\,a < a$ for $a \in (0,1)$ (see [4, 6, 15] for more details). It is easy to see that strict t-norms are Archimedean. A typical example of a nonstrict Archimedean t-norm is the *Łukasiewicz t-norm* $\boldsymbol{t}_{Ł}$ with $a\,\boldsymbol{t}_{Ł}\,b = 0 \vee (a + b - 1)$. More generally, each *Schweizer t-norm* $\boldsymbol{t}_{S,p}$ with

$$a\,\boldsymbol{t}_{S,p}\,b = [0 \vee (a^p + b^p - 1)]^{1/p},$$

$p > 0$, is nonstrict Archimedean and, then,

$$\nu_{\boldsymbol{t}}(a) = (1 - a^p)^{1/p}.$$

Nonstrict Archimedean t-norms do have zero divisors, i.e. $a\,\boldsymbol{t}\,b = 0$ is possible for $a, b > 0$. That a t-norm does have zero divisors can be a useful and desired feature in many applications because that t-norm then shows some *inertia* in attaining positive values: a positive argument < 1 is still treated as zero if the other, though positive, is not sufficiently "large". For instance, $0.3\,\boldsymbol{t}_{Ł}\,0.6 = 0$, whereas $0.3\,\boldsymbol{t}_{Ł}\,0.8 > 0$.

Our further discussion in this section will be limited to nonstrict Archimedean t-norms. For intuitionistic fuzzy sets with a t-norm $\boldsymbol{t}$ from that class, (3.2) or equivalently (3.1) with $\nu = \nu_{\boldsymbol{t}}$ become a worth considering alternative to (2.1). Each fuzzy set F can then be represented in the language of intuitionistic fuzzy sets as (F, F^{ν}). Clearly, if $\boldsymbol{t} = \boldsymbol{t}_{Ł}$, then $\nu_{\boldsymbol{t}}$ is the *Łukasiewicz negation* $\nu_{Ł}$ with $\nu_{Ł}(a) = 1 - a$, and both (3.1) and (3.2) collapse to (2.1). The question one must pose in this place is that about a counterpart of the hesitation formula (2.2) when (3.2) is used. Intuitively speaking, our hesitation as to the status of x does mean that we cannot agree that x belongs to an intuitionistic fuzzy set and, simultaneously, we cannot agree that x does not belong to it. Following that intuition, it seems that a suitable candidate is

$$\chi_{\mathscr{E}}(x) = \nu_{\boldsymbol{t}}(A(x))\,\boldsymbol{t}\,\nu_{\boldsymbol{t}}(A^d(x)) = \nu_{\boldsymbol{t}}(A(x)\,\boldsymbol{t}^\circ A^d(x)), \tag{3.3}$$

where $\boldsymbol{t}^\circ$ denotes the t-conorm *complementary* to $\boldsymbol{t}$, i.e.

$$a\,\boldsymbol{t}^\circ b = \nu_{\boldsymbol{t}}(\nu_{\boldsymbol{t}}(a)\,\boldsymbol{t}\,\nu_{\boldsymbol{t}}(b)).$$

Since $a\,\boldsymbol{t}^\circ \nu_{\boldsymbol{t}}(a) = 1$, we have

$$\forall x \in \boldsymbol{M}\colon A(x)\,\boldsymbol{t}^\circ A^d(x)\,\boldsymbol{t}^\circ \chi_{\mathscr{E}}(x) = 1, \tag{3.4}$$

which forms a counterpart and a generalization of the relationship (2.3). It collapses to (2.3) for $\boldsymbol{t} = \boldsymbol{t}_Ł$; $\boldsymbol{t}^\circ$ is then the *Łukasiewicz t-conorm* $\boldsymbol{s}_Ł$ with $a\,\boldsymbol{s}_Ł\,b = 1 \wedge (a+b)$. We easily notice that

$$\chi_{\mathscr{E}}(x) = 0 \;\Leftrightarrow\; A^d(x) = A^\nu(x)$$

and

$$\chi_{\mathscr{E}}(x) = 1 \;\Leftrightarrow\; A(x) = A^d(x) = 0. \tag{3.5}$$

The geometrical interpretation of a hesitation degree $\chi_{\mathscr{E}}(x)$ in (2.3) via a triangle in the unit 3-dimensional cube analysed in [10] is in (3.4) replaced by an interpretation involving more complex surfaces in that cube (see [7]). In the orthogonal projection of the triangle connected with (2.3), segments parallel to the diagonal of the unit square do represents pairs $(A(x), A^d(x))$ to which the same fixed hesitation degree is related. If (3.4) is used and an analogous projection is made, pairs $(A(x), A^d(x))$ with the same hesitation degree do generally form some curves. For instance, if $\boldsymbol{t} = \boldsymbol{t}_{S,p}$, one gets (cf. (2.2))

$$\chi_{\mathscr{E}}(x) = [1 - (A(x))^p - (A^d(x))^p]^{1/p}. \tag{3.6}$$

Thus, say, for $p = 2$, the pairs $(A(x), A^d(x))$ with $\chi_{\mathscr{E}}(x) = 0.5$ do form in the unit square a quarter of the circle with radius ≈ 0.87 and center at (0, 0).

Theorem 3.1. *If* $\mathscr{E}$ *and* $\mathscr{F}$ *are intuitionistic fuzzy sets with respect to the condition* (3.1) *with* $\nu = \nu_{\boldsymbol{t}}$ *and a nonstrict Archimedean t-norm* $\boldsymbol{t}$, *then so are* $\mathscr{E} \cup_{\boldsymbol{t},\boldsymbol{s}} \mathscr{F}$ *and* $\mathscr{E} \cap_{\boldsymbol{t},\boldsymbol{s}} \mathscr{F}$ *provided that* $\boldsymbol{s} \le \boldsymbol{t}^\circ$.

Generally, $\boldsymbol{s} \le \boldsymbol{t}^\circ$ cannot be replaced by $\boldsymbol{t} \le \boldsymbol{s}^\circ$.

4 Applications to Group Decision Making

Since the method we are going to present does require a familiarity with axiomatic theory of scalar cardinalities of fuzzy sets, we like to begin our discussion by recalling some basic notions and facts from that theory (see [14-16] for further details).

4.1 Scalar Cardinalities of Fuzzy Sets

We restrict ourselves to the class FFS of all finite fuzzy sets in $\boldsymbol{M}$.

Definition 4.1. A function $\sigma\colon \mathrm{FFS} \to [0, \infty)$ is called a *scalar cardinality* if for each $a, b \in [0, 1]$, $x, y \in \boldsymbol{M}$ and $A, B \in \mathrm{FFS}$ the following axioms are satisfied:

$\sigma(1/x) = 1,$ (*coincidence*)

$a \le b \Rightarrow \sigma(a/x) \le \sigma(b/y),$ (*monotonicity*)

$A \cap B = 1_\varnothing \Rightarrow \sigma(A \cup B) = \sigma(A) + \sigma(B).$ (*additivity*)

If these postulates are fulfilled, one says that $\sigma(A)$ is a *scalar cardinality of* A. As usual, $\cap$ ($\cup$, resp.) denotes the standard operation of intersection (sum, resp.) realized via $\wedge$ ($\vee$, resp.). As stated in the following characterization theorem, axiomatically defined scalar cardinalities from Definition 4.1 are exactly a natural generalization of the sigma count $sc_A = \Sigma_{x \in \mathrm{supp}(A)} A(x)$ of A (see e.g. [17]). Therefore, they will be called *generalized sigma counts*.

Theorem 4.2. σ *is a scalar cardinality iff there exists a nondecreasing function* $f\colon [0, 1] \to [0, 1]$ *such that* $f(0) = 0$, $f(1) = 1$ *and*

$$\sigma(A) = \sum_{x \in \mathrm{supp}(A)} f(A(x)) \quad \textit{for each } A.$$

Each function f satisfying the conditions of Theorem 4.2 is said to be a *cardinality pattern*. It expresses our understanding of the scalar cardinality of a singleton. Simplest examples of cardinality patterns and resulting scalar cardinalities are given below.

(Ex. 1) $f_{1,t}(a) = (1$ if $a \ge t$, else $0)$, $t \in (0, 1]$.
It leads to

$$\sigma(A) = |A_t|.$$

A_t denotes the t-cut set of A ($A_t = \{x\colon A(x) \ge t\}$).

(Ex. 2) $f_{2,t}(a) = (1$ if $a > t$, else $0)$, $t \in [0, 1)$.
Then

$$\sigma(A) = |A^t|$$

with A^t denoting the sharp t-cut set of A, i.e. $A^t = \{x\colon A(x) > t\}$.

(Ex. 3) $f_{3,p}(a) = a^p$, $p > 0$. It gives

$$\sigma(A) = \sum_{x \in \mathrm{supp}(A)} (A(x))^p.$$

More advanced instances of cardinality patterns are, say, normed (additive) generators of nonstrict Archimedean t-conorms (see [15, 16]). By the way, the identity function in $[0,1]$ is the normed generator of $\boldsymbol{s}_{Ł}$ and, thus, one can say that sigma counts are scalar cardinalities generated by the normed generator of the Łukasiewicz t-conorm.

Let us write $\sigma_f(A)$ instead of $\sigma(A)$ in order to emphasize which cardinality pattern is involved. Generalized sigma counts $\sigma_f(A)$ can be used to define *relative scalar cardinalities* in a very flexible way (cf. [17]). Namely, let

$$\sigma_f(A \mid B) = \sigma_f(A \cap_{\boldsymbol{t}} B) / \sigma_f(B)$$

with a cardinality pattern f and a t-norm $\boldsymbol{t}$. This proportion of elements of A which are in B fulfils all axioms of (conditional) probability. In particular, we have

$$\sigma_f(A \mid 1_{\boldsymbol{M}}) = \frac{1}{|\boldsymbol{M}|} \sum_{x \in \mathrm{supp}(A)} f(A(x)). \tag{4.1}$$

4.2 Group Decision Making

We like to present a general algorithm of group decision making which forms a generalization of the method proposed by Kacprzyk [5] and developed by Szmidt and Kacprzyk [8] by adding a hesitation factor (cf. also [9]). What we propose is the use of arbitrary relative scalar cardinalities combined with the use of the alternative approach to hesitation from Section 3. Consequently, one obtains a method which is more flexible in comparison with that from [8] and which unifies various specific variants of doing considered in [5, 8]. Because of page limit, our discussion is restricted to an outline of the direct approach to group decision making without a social fuzzy preference relation. Details as well as an appropriate generalization in the case of the indirect approach, involving a social fuzzy preference relation, will be presented in [7].

Let $\boldsymbol{P} = \{p_1, p_2, \ldots, p_m\}$ be a set of $m \ge 1$ *individuals*, and let $\boldsymbol{S} = \{s_1, s_2, \ldots, s_n\}$ with $n \ge 2$ be a set of *options* (*alternatives*). Suppose that each individual p_k formulates his/her own preferences over $\boldsymbol{S}$ and expresses them by means of a binary fuzzy relation $R_k \colon \boldsymbol{S} \times \boldsymbol{S} \to [0,1]$. It can be represented as an $n \times n$ matrix $R_k = [r_{ij}^k]$ with $i, j = 1, 2, \ldots, n$. The number $r_{ij}^k \in [0,1]$ is a *degree to which p_k prefers s_i to s_j*; for each i, we put $r_{ii}^k = 0$. One accepts that there is a relationship between r_{ij}^k and r_{ji}^k, namely

$$r_{ij}^k \le \nu(r_{ji}^k) \tag{4.2}$$

with a strong (i.e. continuous, strictly decreasing and involutive) negation ν having a unique fixed point $a^* \in (0,1)$. Throughout, for brevity, we assume that $k = 1, \ldots, m$ and $i, j = 1, \ldots, n$ unless otherwise specified. Finally, let Q denote a relative linguistic quantifier of "most"-type (see [17]), i.e. $Q\colon [0,1] \to [0,1]$ is nondecreasing, $Q(0) = 0$ and $Q(a) = 1$ for $a > t$, where $t < 1$ (cf. Theorem 4.2). Our task is to find a *solution* understood as a fuzzy set S_Q of options such that a soft majority, i.e. Q individuals, is not against them. We like to construct a suitable general, flexible procedure

$$\{R_1, R_2, \ldots, R_m\} \to S_Q.$$

For the sake of notational convenience, put $P = 1_{\boldsymbol{P}}$, $\boldsymbol{S}_j = \boldsymbol{S} \setminus \{s_j\}$ and $S_j = 1_{\boldsymbol{S}_j}$.

Case 1: $\boldsymbol{t}$ is a strict t-norm or $\boldsymbol{t} = \wedge$. We then take $\nu = \nu_{Ł}$, i.e. $a^* = 0.5$.

Variant 1a. Assume all R_k's are reciprocal, which means that in (4.2) all inequalities with $i \ne j$ do collapse to equalities, and thus we have

$$r_{ij}^k + r_{ji}^k = 1 \quad \text{whenever } i \ne j. \tag{4.3}$$

We propose the following algorithm of finding S_Q.

Step 1. Construct fuzzy sets $R_{k,j}$ of options which p_k prefers to s_j. So, $R_{k,j} \colon \boldsymbol{S}_j \to [0,1]$ with

$$R_{k,j}(s_i) = r_{ij}^k \quad \text{for each } s_i \in \boldsymbol{S}_j.$$

The number

$$r_{*j}^k = \nu(\sigma_f(R_{k,j} \mid S_j)) = \nu\Big(\frac{1}{n-1} \sum_{\substack{i=1 \\ i \ne j}}^{n} f(r_{ij}^k)\Big) \tag{4.4}$$

with a cardinality pattern f is a degree to which p_k is not against s_j.

Step 2. Construct fuzzy sets I_j of individuals being not against an option s_j, i.e. $I_j \colon \boldsymbol{P} \to [0,1]$ with

$$I_j(p_k) = r_{*j}^k.$$

Let

$$d_j = \sigma_{f*}(I_j \mid P) = \frac{1}{m}\sum_{k=1}^{m} f*(r_{*j}^k) \qquad (4.5)$$

with a cardinality pattern $f*$ which is possibly different from f.

Step 3. Compute $Q(d_j)$ for $j = 1, \ldots, n$. $Q(d_j)$ is a degree to which Q individuals are not against s_j.

Step 4. Put

$$S_Q = Q(d_1)/s_1 + \ldots + Q(d_n)/s_n. \qquad (4.6)$$

Let us mention a few important particular cases of the solution S_Q from (4.6):

(i) If $f = f_{1,a*}$ and $f* = id$, S_Q collapses to the fuzzy Q-core from [5].

(ii) Taking $f = f_{1,\alpha}$ with $\alpha \le 0.5$ and $f* = id$, S_Q becomes the fuzzy α/Q-core from [5], a fuzzy set of options such that Q individuals are not sufficiently (to degree $\ge 1 - \alpha$) against them.

(iii) $f(a) = (1$ if $a \ge a*$, else $a/a*)$ and $f* = id$. S_Q is then the fuzzy s/Q-core from [5], a fuzzy set of options such that Q individuals are not strongly against them.

Variant 1b. Suppose at least some of the matrices R_k's are not reciprocal, i.e. we generally have

$$r_{ij}^k + r_{ji}^k \le 1. \qquad (4.7)$$

To each R_k we assign an $n \times n$ symmetric *hesitation matrix* $H_k = [\, h_{ij}^k \,]$ of elements from $[0, 1]$ such that (cf. (2.3))

$$r_{ij}^k + r_{ji}^k + h_{ij}^k = 1 \quad \text{whenever} \quad i \ne j \qquad (4.8)$$

and $h_{ii}^k = 0$. The number h_{ij}^k expresses a hesitation margin of p_k as to his/her preference between s_i and s_j. It is now possible that

$$r_{ij}^k,\, r_{ji}^k < \alpha \quad \text{for some} \quad i \ne j \qquad (4.9)$$

with $\alpha \le 0.5$. The intensity of the preference between s_i and s_j formulated by p_k is then more or less low and, paradoxically, s_i and s_j are thus rather equally good for p_k if his/her choice is restricted to $\{s_i, s_j\}$. Consequently, we modify R_k by putting

$$r_{ij}^k = r_{ji}^k = 1 \qquad (4.10)$$

whenever (4.9) holds. These modified R_k's will be used in the algorithm given below.

Step 1. As in (4.4), compute

$$r_{*j}^k = \nu(\sigma_f(R_{k,j} \mid S_j)) = \nu\Big(\frac{1}{n-1}\sum_{\substack{i=1 \\ i \ne j}}^{n} f(r_{ij}^k)\Big).$$

Step 2. Similarly to (4.5), compute

$$d_j = \sigma_{f*}(I_j \mid P) = \frac{1}{m}\sum_{k=1}^{m} f*(r_{*j}^k).$$

Step 3. Construct fuzzy sets $H_{k,j}$ of options such that p_k is hesitant as to his/her preference between them and s_j, i.e. $H_{k,j}: \boldsymbol{S}_j \to [0, 1]$ with

$$H_{k,j}(s_i) = h_{ij}^k \quad \text{for each} \ s_i \in \boldsymbol{S}_j.$$

Let

$$h_{*j}^k = \sigma_g(H_{k,j} \mid S_j) = \frac{1}{n-1}\sum_{\substack{i=1 \\ i \ne j}}^{n} g(h_{ij}^k). \qquad (4.11)$$

This is a degree to which p_k is hesitant as to his/her preference between s_j and another option.

Step 4. Construct fuzzy sets L_j of individuals who hesitate as to their preferences between s_j and another option, i.e. $L_j: \boldsymbol{P} \to [0, 1]$ with $L_j(p_k) = h_{*j}^k$. Define

$$e_j = \sigma_{g*}(L_j \mid P) = \frac{1}{m}\sum_{k=1}^{m} g*(h_{*j}^k). \qquad (4.12)$$

Step 5. Compute $Q(d_j)$'s and $Q(d_j \,\boldsymbol{s}_{\text{Ł}}\, e_j)$'s.

Step 6.

$$S_Q = [Q(d_1), Q(d_1 \,\boldsymbol{s}_{\text{Ł}}\, e_1)]/s_1 + \ldots + [Q(d_n), Q(d_n \,\boldsymbol{s}_{\text{Ł}}\, e_n)]/s_n. \qquad (4.13)$$

S_Q is now an interval-valued fuzzy set. The above procedure, allowing us to use four different cardinality patterns, gives in general a bit redundant flexibility in this respect. The use of $f* = g = g*$ seems to be in practice sufficient in many cases (see e.g. [8] in which $f = f_{1,a*}$ and $f* = g = g* = id$).

Case 2: $\boldsymbol{t}$ is nonstrict Archimedean and $\nu = \nu_{\boldsymbol{t}}$.

Variant 2a. Assume

$$r_{ij}^k = \nu(r_{ji}^k) \quad \text{whenever} \quad i \ne j. \qquad (4.14)$$

Clearly, (4.14) collapses to (4.3) for $\boldsymbol{t} = \boldsymbol{t}_{\text{Ł}}$, and implies $r_{ij}^k \ge a*$ and/or $r_{ji}^k \ge a*$. The algorithm of finding S_Q is identical to (4.4)-(4.6) in Variant 1a.

Variant 2b. Assume

$$r_{ij}^k \le \nu(r_{ji}^k)$$

for each k, i and j. Again, to each R_k we assign a symmetric $n\times n$ hesitation matrix $H_k = [\,h_{ij}^k\,]$ with $h_{ii}^k = 0$ and (cf. (3.3), (4.8))

$$h_{ij}^k = \nu(\, r_{ij}^k\ \boldsymbol{t}^{\circ}\ r_{ji}^k\,) \quad \text{whenever} \quad i \neq j. \tag{4.15}$$

If, say, $\boldsymbol{t} = \boldsymbol{t}_{S,2}$, then (see (3.6))

$$h_{ij}^k = (1 - (\,r_{ij}^k)^2 - (\,r_{ji}^k)^2)^{1/2}. \tag{4.16}$$

For $i \neq j$, one has (see (3.4), (3.5))

$$r_{ij}^k\ \boldsymbol{t}^{\circ}\ r_{ji}^k\ \boldsymbol{t}^{\circ}\ h_{ij}^k = 1$$

and

$$h_{ij}^k = 0 \Leftrightarrow r_{ij}^k = \nu(\,r_{ji}^k).$$

If $r_{ij}^k,\ r_{ji}^k < \alpha$ for $i \neq j$ with $\alpha \leq a^*$, we modify R_k by putting $r_{ij}^k = r_{ji}^k = 1$. S_Q is now generated as follows (cf. Variant 1b).

Step 1 - Step 4. Follow (4.4), (4.5), (4.11), (4.12).

Step 5. Compute $Q(d_j)$'s and $Q(d_j\ \boldsymbol{t}^{\circ} e_j)$'s.

Step 6. $S_Q = [Q(d_1),\, Q(d_1\ \boldsymbol{t}^{\circ} e_1)]\,/\,s_1 + \ldots$
$+ [Q(d_n),\, Q(d_n\ \boldsymbol{t}^{\circ} e_n)]\,/\,s_n$.

The following example with $m = n = 4$ is considered in [8]:

$$R_1 = \begin{bmatrix} 0 & 0.3 & 0.7 & 0.4 \\ 0.7 & 0 & 0.6 & 0.9 \\ 0.3 & 0.4 & 0 & 0.5 \\ 0.4 & 0 & 0.3 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0.4 & 0.6 & 0.5 \\ 0.6 & 0 & 0.7 & 0.7 \\ 0.4 & 0.3 & 0 & 0.4 \\ 0.3 & 0.1 & 0.3 & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 & 0.5 & 0.7 & 0.3 \\ 0.5 & 0 & 0.8 & 0.7 \\ 0.3 & 0.2 & 0 & 0.5 \\ 0.4 & 0.1 & 0.2 & 0 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0 & 0.4 & 0.7 & 0.6 \\ 0.6 & 0 & 0.4 & 0.6 \\ 0.3 & 0.6 & 0 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0 \end{bmatrix}$$

and $Q(x) = [1$ if $x \geq 0.8$, $2x - 0.6$ if $x \in (0.3, 0.8)$, 0 if $x \leq 0.3]$. The algorithm from [8] of group decision making with a hesitation factor, which collapses to that from Variant 1b with $\alpha = 0.5$, $f = f_{1,0.5}$ and $f^* = g = g^* = id$, leads to the solution

$$S_Q = [4/10,\ 17/30]\,/\,s_1 + [1, 1]\,/\,s_2 + \\ [0,\ 7/60]\,/\,s_3 + [0, 0]\,/\,s_4.$$

The same values of both α and cardinality patterns applied in the method from Variant 2b with $\boldsymbol{t} = \boldsymbol{t}_{S,2}$ and $\nu = \nu_{\boldsymbol{t}}$ do give $[\,d_j] = [0.85, 0.97, 0.56, 0]$ and (see (4.16))

$$H_1 = \begin{bmatrix} 0 & 0.65 & 0.65 & 0.82 \\ 0.65 & 0 & 0.69 & 0.44 \\ 0.65 & 0.69 & 0 & 0.81 \\ 0.82 & 0.44 & 0.81 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0.69 & 0.69 & 0.81 \\ 0.69 & 0 & 0.65 & 0.71 \\ 0.69 & 0.65 & 0 & 0.87 \\ 0.81 & 0.71 & 0.87 & 0 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0 & 0.71 & 0.65 & 0.87 \\ 0.71 & 0 & 0.57 & 0.71 \\ 0.65 & 0.57 & 0 & 0.84 \\ 0.87 & 0.71 & 0.84 & 0 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 & 0.69 & 0.65 & 0.79 \\ 0.69 & 0 & 0.69 & 0.79 \\ 0.65 & 0.69 & 0 & 0.91 \\ 0.79 & 0.79 & 0.91 & 0 \end{bmatrix}.$$

Hence $[\,e_j] = [0.72, 0.66, 0.73, 0.78]$ and

$$S_Q = [1, 1]\,/\,s_1 + [1, 1]\,/\,s_2 + [0.52, 1]\,/\,s_3 + [0, 0.96]\,/\,s_4.$$

The solution is now less "selective". But $a^* \approx 0.71$ and, thus, the values of the R_k's do have in this case a quite different meaning in comparison with that in the first part of this example.

References

[1] K. Atanassov, *Intuitionistic Fuzzy Sets. Theory and Applications*, Physica-Verlag, Heidelberg New York, 1999.

[2] K. Atanassov and S. Stoeva, Intuitionistic fuzzy sets, in: *Proc. Polish Symp. Interval and Fuzzy Mathematics*, Poznań, 1983, 23-26.

[3] G. Deschrijver and E. E. Kerre, A generalisation of operators on intuitionistic fuzzy sets using triangular norms and conorms, Notes on IFS 8 (2002) 19-27.

[4] S. Gottwald, *A Treatise on Many-Valued Logics*, Research Studies Press, Baldock, Hertfordshire, 2001.

[5] J. Kacprzyk, Group decision making with a fuzzy linguistic majority, Fuzzy Sets abd Systems 18 (1986) 105-118.

[6] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer, Dordrecht, 2000.

[7] A. Pankowska and M. Wygralak, On triangular norm-based intuitionistic fuzzy sets, in preparation.

[8] E. Szmidt and J. Kacprzyk, Group decision making under intuitionistic fuzzy preference relations, in: *Proc. 7th IPMU Conf.*, Paris, 1998, 172-178.

[9] E. Szmidt and J. Kacprzyk, Decision making in an intuitionistic fuzzy environment, in: *Proc. Joint 4th EUROFUSE and 2nd SIC Conf.*, Budapest, 1999, 292-297.

[10] E. Szmidt and J. Kacprzyk, Entropy for intuitionistic fuzzy sets, Fuzzy Sets and Systems 118 (2001) 467-477.

[11] G. Takeuti and S. Titani, Intuitionistic fuzzy logic and intuitionistic fuzzy set theory, J. Symb.Logic, 49 (1984) 851-866.

[12] G. Takeuti and S. Titani, Fuzzy logic and fuzzy set theory, Arch. Math. Logic, 32 (1992) 1-32.

[13] G. J. Wang and Y. Y. He, Intuitionistic fuzzy sets and L-fuzzy sets, Fuzzy Sets and Systems 110 (2000) 271-274.

[14] M. Wygralak, An axiomatic approach to scalar cardinalities of fuzzy sets, Fuzzy Sets and Systems 110 (2000) 175-179.

[15] M. Wygralak, *Cardinalities of Fuzzy Sets*, Springer-Verlag, Berlin Heidelberg 2003.

[16] M. Wygralak, Cardinalities of fuzzy sets with triangular norms, in: *Proc. EUSFLAT'2003 Conf.*, Zittau, 2003.

[17] L. A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Comput. and Math. with Appl. 9 (1983) 149-184.