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Entropy on IF-events

Marek Ďurica
Faculty of Natural Sciences
Matej Bel University
Tajovského 40
SK-974 01 Banská Bystrica
durica@fpv.umb.sk

Abstract

In this paper we study dynamical systems based on IF-sets. We show that the notion of Kolmogorov-Sinaj entropy can be extended on this systems.

Keywords: IF-sets, IF-partitions, IF-dynamical systems, IF-entropy.

1 Introduction

We start with classical dynamical systems $(\Omega, \mathcal{S}, P, T)$, where $(\Omega, \mathcal{S}, P)$ is a probability space and $T : \Omega \to \Omega$ is a measure preserving map, i.e. $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$. The entropy of the dynamical system is defined as follows (see [9],[10]). Consider measurable partition $\mathcal{A} = \{A_1, ..., A_k\}$, where $A_i \in \mathcal{S}; i = 1, ..., k, A_i \cap A_j = \emptyset; i \neq j, \bigcup_{i=1}^{k} A_i = \Omega$. Its entropy is the number

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(P(A_i)),$$

where $\varphi(x) = -x \log x$, if $x > 0$, and $\varphi(0) = 0$. If $\mathcal{A}$ is a measurable partition, then

$$T^{-1}(\mathcal{A}) = \{T^{-1}(A_1), ..., T^{-1}(A_k)\}$$

is a measurable partition, too. Moreover, if $\mathcal{A}, \mathcal{B}$ are two measurable partitions, then

$$\mathcal{A} \vee \mathcal{B} = \{A \cap B; A \in \mathcal{A}, B \in \mathcal{B}\}$$

generates a measurable partition. It can be proved that there exists

$$h(\mathcal{A}, T) = \lim_{n\to\infty} H(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})).$$

The entropy $h(T)$ of $(\Omega, \mathcal{S}, P, T)$ is defined as the supremum

$$h(T) = \sup\{h(\mathcal{A}, T); \mathcal{A} \text{ is a measurable partition}\}.$$

The notion of the entropy has been extended using fuzzy partitions instead of partitions (see [10]). Fuzzy partition is a set of functions $f_1, ..., f_k$ such that

$$\sum_{i=1}^{k} f_i = 1.$$

2 IF-dynamical system

Consider classical dynamical system $(\Omega, \mathcal{S}, P, T)$. Then we define the family of all fuzzy sets

$$\mathcal{T} = \{f : \Omega \to \langle 0, 1\rangle; f \text{ be } \mathcal{S}\text{-measurable function}\}.$$

and the family of all IF-sets (IF-events)

$$\mathcal{F} = \{(\mu_A, \nu_A); \mu_A + \nu_A \leq 1, \mu_A, \nu_A \in \mathcal{T}\}.$$

We shall consider an algebraic structure $(\mathcal{F}, +, \cdot, (1, 0))$ where $+$ is a partial binary operation on $\mathcal{F}$ defined by the formula

$$(\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B, \nu_A + \nu_B - 1), \text{ whenever } \mu_A + \mu_B \leq 1 \text{ and } 0 \leq \nu_A + \nu_B - 1 \leq 1$$

and $\cdot$ is a binary operation on $\mathcal{F}$ defined by the formula

$$(\mu_A, \nu_A) \cdot (\mu_B, \nu_B) = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B).$$

Recall that

$$\sum_{i=1}^{k} (\mu_{A_i}, \nu_{A_i}) = (\sum_{i=1}^{k} \mu_{A_i}, \sum_{i=1}^{k} \nu_{A_i} - (n-1))$$

and operations $+, \cdot$ fulfill the distributive law.

On $\mathcal{F}$, we define a state $m : \mathcal{F} \to [0, 1]$ by the formula

$$m((\mu_A, \nu_A)) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \int_\Omega (1 - \nu_A) dP,$$

where $\alpha \in [0, 1]$ and then we define a mapping $\tau : \mathcal{F} \to \mathcal{F}$, where $\tau((\mu_A, \nu_A)) = (\mu_A \circ T, \nu_A \circ T)$.

Since

$$m(\tau((\mu_A, \nu_A))) = (1 - \alpha) \int_\Omega \mu_A \circ T dP + \alpha \int_\Omega (1 - \nu_A \circ T) dP =$$

$$= (1 - \alpha) \int_\Omega \mu_A dP \circ T^{-1} + \alpha (1 - \int_\Omega \nu_A dP \circ T^{-1}) =$$

$$= (1 - \alpha) \int_\Omega \mu_A dP + \alpha \int_\Omega (1 - \nu_A) dP = m((\mu_A, \nu_A)),$$

then $(\mathcal{F}, m, \tau)$ is an IF-dynamical system.

3 IF-partitions

Definition 3.1. *By an IF-partition we shall mean a finite collection* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *of IF-sets such that*

$$\sum_{i=1}^{k} (\mu_{A_i}, \nu_{A_i}) = (1, 0).$$

If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *and* $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ *are two IF-partitions, then we define*

$$\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \cdot (\mu_{B_j}, \nu_{B_j}); i = 1, ..., k, j = 1, ..., l\}$$

and we write $\mathcal{B} \geq \mathcal{A}$ *(we say* $\mathcal{B}$ *is a refinement of* $\mathcal{A}$*), if there exist a partition* $I(1), ..., I(k)$ *of the set* $\{1, ..., l\}$ *such that*

$$(\mu_{A_i}, \nu_{A_i}) = \sum_{j \in I(i)} (\mu_{B_j}, \nu_{B_j})$$

for every $i = 1, ..., k$*.*

Proposition 3.2. *If* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *and* $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ *are two IF-partitions, then* $\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ..., \tau((\mu_{A_k}, \nu_{A_k}))\}$ *and* $\mathcal{A} \vee \mathcal{B}$ *are IF-partitions, too. Further* $\mathcal{A} \vee \mathcal{B} \geq \mathcal{A}$*.*

Proof.

$$\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ..., \tau((\mu_{A_k}, \nu_{A_k}))\} =$$
$$= \{(\mu_{A_1} \circ T, \nu_{A_1} \circ T), ..., (\mu_{A_k} \circ T, \nu_{A_k} \circ T)\}$$

Then

$$\sum_{i=1}^{k} \tau((\mu_{A_i}, \nu_{A_i})) = \sum_{i=1}^{k} (\mu_{A_i} \circ T, \nu_{A_i} \circ T) =$$

$$= (\left(\sum_{i=1}^{k} \mu_{A_i}\right) \circ T, \left(\sum_{i=1}^{k} \nu_{A_i} - (n-1)\right) \circ T) = (1 \circ T, 0 \circ T) = (1, 0).$$

Further, $\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \cdot (\mu_{B_j}, \nu_{B_j}); i = 1, ..., k, j = 1, ..., l\}$. Therefore

$$\sum_{i=1}^{k} \sum_{j=1}^{l} (\mu_{A_i}, \nu_{A_i}) \cdot (\mu_{B_j}, \nu_{B_j}) = \sum_{i=1}^{k} \sum_{j=1}^{l} (\mu_{A_i} \mu_{B_j}, \nu_{A_i} + \nu_{B_j} - \nu_{A_i} \nu_{B_j}) =$$

$$= (\sum_{i=1}^{k} \sum_{j=1}^{l} \mu_{A_i} \mu_{B_j}, \sum_{i=1}^{k} \sum_{j=1}^{l} \nu_{A_i} + \sum_{i=1}^{k} \sum_{j=1}^{l} \nu_{B_j} - \sum_{i=1}^{k} \sum_{j=1}^{l} \nu_{A_i} \nu_{B_j} - (mn-1)) =$$

$$= (\left(\sum_{i=1}^{k} \mu_{A_i}\right) \left(\sum_{j=1}^{l} \mu_{B_j}\right), \sum_{j=1}^{l} \left(\sum_{i=1}^{k} \nu_{A_i}\right) + \sum_{i=1}^{k} \left(\sum_{j=1}^{l} \nu_{B_j}\right) -$$

$$-\left(\sum_{i=1}^{k}\nu_{A_i}\right)\left(\sum_{j=1}^{l}\nu_{B_j}\right)-(mn-1))=$$

$$=(1,m(n-1)+n(m-1)-(n-1)(m-1)-(mn-1))=(1,0).$$

Finally, let us mention that $\mathcal{A}\vee\mathcal{B}$ is indexed by $\{(i,j);i=1,...,n;j=1,...,m\}$. Therefore, if we put $I(i)=\{(i,1),...,(i,m)\}$, then by the equalities

$$(1,0)=\sum_{j=1}^{l}(\mu_{B_j},\nu_{B_j})$$

we obtain

$$(\mu_{A_i},\nu_{A_i})=(\mu_{A_i},\nu_{A_i})\cdot(1,0)=(\mu_{A_i},\nu_{A_i})\cdot\left(\sum_{j=1}^{l}(\mu_{B_j},\nu_{B_j})\right)=$$

$$=\sum_{j=1}^{l}\big((\mu_{A_i},\nu_{A_i})\cdot(\mu_{B_j},\nu_{B_j})\big)=\sum_{(k,j)\in I(i)}((\mu_{A_k},\nu_{A_k})\cdot(\mu_{B_j},\nu_{B_j}))$$

for every $i=1,...,k$. It follows $\mathcal{A}\vee\mathcal{B}\geq\mathcal{A}$.

□

4 Entropy of IF-partitions

Definition 4.1. *If* $\mathcal{A}=\{(\mu_{A_1},\nu_{A_1}),...,(\mu_{A_k},\nu_{A_k})\}$ *and* $\mathcal{B}=\{(\mu_{B_1},\nu_{B_1}),...,(\mu_{B_l},\nu_{B_l})\}$ *are two IF-partitions, then we define the entropy of an IF-partition*

$$H(\mathcal{A})=\sum_{i=1}^{k}\varphi\big(m((\mu_{A_i},\nu_{A_i}))\big),$$

where $\varphi(x)=-x\log x$*, if* $x>0$, $\varphi(0)=0$ *and the conditional entropy*

$$H(\mathcal{A}|\mathcal{B})=\sum_{i=1}^{k}\sum_{j=1}^{l}m((\mu_{B_j},\nu_{B_j}))\varphi\left(\frac{m((\mu_{A_i},\nu_{A_i})\cdot(\mu_{B_j},\nu_{B_j}))}{m((\mu_{B_j},\nu_{B_j}))}\right).$$

Proposition 4.2. *If* $\mathcal{B}\leq\mathcal{C}$*, then* $H(\mathcal{A}|\mathcal{C})\leq H(\mathcal{A}|\mathcal{B})$.

Proof. Put $(\mu_{B_j},\nu_{B_j})=\sum_{t\in I(j)}(\mu_{C_t},\nu_{C_t})$, where $\{I(1),...,I(k)\}$ is the corresponding partition and put (for fixed j) $\alpha_t=m((\mu_{C_t},\nu_{C_t}))/m((\mu_{B_j},\nu_{B_j}))$. Then

$$\sum_{t\in I(j)}\alpha_t=\frac{1}{m((\mu_{B_j},\nu_{B_j}))}m\left(\sum_{t\in I(j)}(\mu_{C_t},\nu_{C_t})\right)=1,$$

hence by the concavness of φ

$$\sum_{t\in I(j)} \alpha_t \varphi(x_t) \leq \varphi\left(\sum_{t\in I(j)} \alpha_t x_t\right).$$

Therefore

$$H(\mathcal{A}|\mathcal{C}) = \sum_{i=1}^{n}\sum_{t=1}^{k} m((\mu_{C_t},\nu_{C_t}))\varphi\left(\frac{m((\mu_{A_i},\nu_{A_i})\odot(\mu_{C_t},\nu_{C_t}))}{m((\mu_{C_t},\nu_{C_t}))}\right) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} m((\mu_{B_j},\nu_{B_j}))\sum_{t\in I(j)} \frac{m((\mu_{C_t},\nu_{C_t}))}{m((\mu_{B_j},\nu_{B_j}))}\varphi\left(\frac{m((\mu_{A_i},\nu_{A_i})\odot(\mu_{C_t},\nu_{C_t}))}{m((\mu_{C_t},\nu_{C_t}))}\right) \leq$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{m} m((\mu_{B_j},\nu_{B_j}))\varphi\left(\sum_{t\in I(j)} \frac{m((\mu_{A_i},\nu_{A_i})\odot(\mu_{C_t},\nu_{C_t}))}{m((\mu_{B_j},\nu_{B_j}))}\right) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} m((\mu_{B_j},\nu_{B_j}))\varphi\left(\frac{m((\mu_{A_i},\nu_{A_i})\odot\sum_{t\in I(j)}(\mu_{C_t},\nu_{C_t}))}{m((\mu_{B_j},\nu_{B_j}))}\right) = H(\mathcal{A}|\mathcal{B}).$$

□

Proposition 4.3. *$H(\mathcal{B}\vee\mathcal{C}|\mathcal{A}) = H(\mathcal{B}|\mathcal{A}) + H(\mathcal{C}|\mathcal{B}\vee\mathcal{A})$ for any IF-partitions $\mathcal{A},\mathcal{B},\mathcal{C}$.*

Proof. By the definition

$$H(\mathcal{B}\vee\mathcal{C}|\mathcal{A}) = \sum_{i,j,k} m((\mu_{A_i},\nu_{A_i}))\cdot$$

$$\cdot\varphi\left(\frac{m((\mu_{B_j},\nu_{B_j})\odot(\mu_{C_k},\nu_{C_k})\odot(\mu_{A_i},\nu_{A_i}))m((\mu_{B_j},\nu_{B_j})\odot(\mu_{A_i},\nu_{A_i}))}{m((\mu_{A_i},\nu_{A_i}))m((\mu_{B_j},\nu_{B_j})\odot(\mu_{A_i},\nu_{A_i}))}\right) =$$

$$= \sum_{i,j,k} m((\mu_{B_j},\nu_{B_j})\odot(\mu_{C_k},\nu_{C_k})\odot(\mu_{A_i},\nu_{A_i}))\log\frac{m((\mu_{B_j},\nu_{B_j})\odot(\mu_{C_k},\nu_{C_k})\odot(\mu_{A_i},\nu_{A_i}))}{m((\mu_{B_j},\nu_{B_j})\odot(\mu_{A_i},\nu_{A_i}))} +$$

$$+ \sum_{i,j,k} m((\mu_{B_j},\nu_{B_j})\odot(\mu_{C_k},\nu_{C_k})\odot(\mu_{A_i},\nu_{A_i}))\log\frac{m((\mu_{B_j},\nu_{B_j})\odot(\mu_{A_i},\nu_{A_i}))}{m((\mu_{A_i},\nu_{A_i}))} =$$

$$= H(\mathcal{C}|\mathcal{B}\vee\mathcal{A}) + H(\mathcal{B}|\mathcal{A}).$$

□

5 IF-entropy

Proposition 5.1. *For any IF-partition $\mathcal{D}$ there exists*

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{D})\right).$$

Proof. Let $\mathcal{A} = \{(1,0)\}$. Then $H(\mathcal{B}|\mathcal{A}) = H(\mathcal{B})$, hence by Proposition 4.2 and 4.3

$$H(\mathcal{B} \vee \mathcal{C}) = H(\mathcal{B} \vee \mathcal{C}|\mathcal{A}) = H(\mathcal{B}|\mathcal{A}) + H(\mathcal{C}|\mathcal{B} \vee \mathcal{A}) =$$
$$= H(\mathcal{B}) + H(\mathcal{C}|\mathcal{B}) \le H(\mathcal{B}) + H(\mathcal{C})$$

for every $\mathcal{B}, \mathcal{C}$. Put $a_n = H(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{D}))$. Then $a_{n+m} \le a_n + a_m$ for every $n, m \in \mathbf{N}$ and this property guarantees the existence of $\lim_{n\to\infty}(1/n)a_n = \lim_{n\to\infty}(1/n)H(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{D}))$.

□

Definition 5.2. *For every IF-partition $\mathcal{A}$ we define*

$$h(\mathcal{A}, \tau) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right)$$

and then the entropy of IF-dynamical system $(\mathcal{F}, m, \tau)$ (shortly IF-entropy) by the equality

$$h(\tau) = \sup\{h(\mathcal{A}, \tau); \mathcal{A} \textit{ be an IF-partition}\}.$$

Proposition 5.3.

$$h(\mathcal{C}, \tau) = h\left(\bigvee_{j=0}^{k} \tau^j(\mathcal{C}), \tau\right).$$

for every $k \in \mathbf{N}$ and any IF-partition $\mathcal{C}$.

Proof. We obtain immediately

$$h\left(\bigvee_{j=0}^{k} \tau^j(\mathcal{C}), \tau\right) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i\left(\bigvee_{j=0}^{k} \tau^j(\mathcal{C})\right)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{t=0}^{n+k-1} \tau^t(\mathcal{C})\right) = \lim_{p\to\infty} \frac{p}{p-k} \frac{1}{p} H\left(\bigvee_{t=0}^{p-1} \tau^t(\mathcal{C})\right) = h(\mathcal{C}, \tau).$$

□

Theorem 5.4.

$$h(\mathcal{A}, \tau) = \lim_{n\to\infty} H\left(\mathcal{A}|\bigvee_{i=1}^{n} \tau^i(\mathcal{A})\right).$$

Proof. By Proposition 4.3

$$H\left(\bigvee_{i=0}^{k}\tau^i(\mathcal{A})\right) = H\left(\mathcal{A}\vee\tau\left(\bigvee_{i=0}^{k-1}\tau^i(\mathcal{A})\right)\right) =$$

$$= H\left(\tau\left(\bigvee_{i=0}^{k-1}\tau^i(\mathcal{A})\right)\right) + H\left(\mathcal{A}|\tau\left(\bigvee_{i=0}^{k-1}\tau^i(\mathcal{A})\right)\right) =$$

$$H\left(\bigvee_{i=0}^{k-1}\tau^i(\mathcal{A})\right) + H\left(\mathcal{A}|\bigvee_{i=1}^{k}\tau^i(\mathcal{A})\right).$$

Now, by induction we obtain

$$H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) = H(\mathcal{A}) + \sum_{k=1}^{n-1} H\left(\mathcal{A}|\bigvee_{i=1}^{k}\tau^i(\mathcal{A})\right). \tag{1}$$

By Proposition 4.2 we obtain that $\left(H\left(\mathcal{A}|\bigvee_{i=1}^{n}\tau^i(\mathcal{A})\right)\right)_n$ is decreasing, so that

$$\lim_{n\to\infty} H\left(\mathcal{A}|\bigvee_{i=1}^{n}\tau^i(\mathcal{A})\right).$$

exists. But then there exists also the limit of the Cesaro means

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} H\left(\mathcal{A}|\bigvee_{i=1}^{k}\tau^i(\mathcal{A})\right).$$

By (1) we obtain

$$\lim_{n\to\infty} H\left(\mathcal{A}|\bigvee_{i=1}^{n}\tau^i(\mathcal{A})\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} H\left(\mathcal{A}|\bigvee_{i=1}^{k}\tau^i(\mathcal{A})\right) =$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[H\left(\bigvee_{i=1}^{n}\tau^i(\mathcal{A})\right) - H(\mathcal{A})\right] = \lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{n-1}\tau^j(\mathcal{A})\right) = h(\mathcal{A},\tau).$$

□

Theorem 5.5. *$h(\mathcal{A},\tau) \le h(\mathcal{C},\tau) + H(\mathcal{A}|\mathcal{C})$ for any IF-partitions $\mathcal{A},\mathcal{C}$.*

Proof. Since $H(\mathcal{B}\vee\mathcal{D}) = H(\mathcal{B}) + H(\mathcal{D}|\mathcal{B})$ (see Proposition 4.3),

$$H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) \le H\left[\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right)\vee\left(\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right)\right] =$$

$$= H\left(\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right) + H\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})|\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right).$$

Further $H(\mathcal{D} \vee \mathcal{E}|\mathcal{B}) \leq H(\mathcal{D}|\mathcal{B}) + H(\mathcal{E}|\mathcal{B}), H(\mathcal{D}|\mathcal{B} \vee \mathcal{E}) \leq H(\mathcal{D}|\mathcal{B})$ (see Proposition 4.2 and 4.3), hence

$$H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}) | \bigvee_{j=0}^{n-1} \tau^j(\mathcal{C})\right) \leq \sum_{i=0}^{n-1} H\left(\tau^i(\mathcal{A}) | \bigvee_{j=0}^{n-1} \tau^j(\mathcal{C})\right) \leq$$

$$\leq \sum_{i=0}^{n-1} H\left(\tau^i(\mathcal{A})|\tau^i(\mathcal{C})\right) = nH(\mathcal{A}|\mathcal{C}).$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right) \leq \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \tau^j(\mathcal{C})\right) + H(\mathcal{A}|\mathcal{C})$$

and finally

$$h(\mathcal{A}, \tau) \leq h(\mathcal{C}, \tau) + H(\mathcal{A}|\mathcal{C}).$$

□

Corollary 5.6. *Put* $\mathcal{C}_n = \bigvee_{i=0}^{n} \tau^i(\mathcal{C})$. *Then*

$$h(\mathcal{A}, \tau) \leq h(\mathcal{C}, \tau) + H(\mathcal{A}|\mathcal{C}_n)$$

for any IF-partition $\mathcal{A}$.

Proof. See Theorem 5.5 and Proposition 5.3.

□

6 IF-entropy and generators

Now, we want to prove a variant of the Kolmogorov-Sinaj theorem (see e.g. [10]) for our entropy on IF-dynamical systems. Recall that by $E(f|\mathcal{S}_0)$ we denote the expected value of a random variable (measurable function) with respect to a sub-σ-algebra $\mathcal{S}_0$ of the σ-algebra $\mathcal{S}$. If $\mathcal{S}_0$ is finite with atoms $U_1, ..., U_t$, then

$$E(f|\mathcal{S}_0) = \sum_{i=1}^{t} \left(\frac{1}{P(U_i)} \int_{U_i} f dP\right) \chi_{U_i}.$$

Proposition 6.1. *Let* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *be a IF-partition and* $\mathcal{B} = \{U_1, ..., U_t\}$ *be a crisp partition of* Ω. *Let* $\sigma(\mathcal{B})$ *be the* σ*-algebra generated by* $\mathcal{B}$. *Then*

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{k} \int_{\Omega} \varphi\Big(E\big((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})|\sigma(\mathcal{B})\big)\Big) dP.$$

Proof. Put $\mathcal{B} = \{(\chi_{U_1}, 1-\chi_{U_1}), ..., (\chi_{U_t}, 1-\chi_{U_t})\}$ instead of $\mathcal{B} = \{U_1, ..., U_t\}$. Then we have

$$m((\chi_{U_j}, 1-\chi_{U_j})) = (1-\alpha)\int_\Omega \chi_{U_j} dP + \alpha \int_\Omega (1-(1-\chi - U_j))dP =$$

$$= (1-\alpha)P(U_j) + \alpha P(U_j) = P(U_j); j = 1, ..., t$$

and

$$m((\mu_{A_i}, \nu_{A_i}) \cdot (\chi_{U_j}, 1-\chi_{U_j})) = m((\mu_{A_i}\chi_{U_j}, \nu_{A_i} + (1-\chi_{U_j}) - \nu_{A_i}(1-\chi_{U_j}))) =$$

$$= (1-\alpha)\int_\Omega \mu_{A_i}\chi_{U_j} dP + \alpha \int_\Omega (1-(1-\chi_{U_j} + \nu_{A_i}\chi_{U_j}))dP =$$

$$= \int_{U_j} (1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})dP; i = 1, ..., k, j = 1, ..., t.$$

By the definition

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{k}\sum_{j=1}^{t} m((\chi_{U_j}, 1-\chi_{U_j}))\varphi\left(\frac{m((\mu_{A_i}, \nu_{A_i}) \cdot (\chi_{U_j}, 1-\chi_{U_j}))}{m((\chi_{U_j}, 1-\chi_{U_j}))}\right) =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{t} \int_{U_j} \varphi\left(\frac{m((\mu_{A_i}, \nu_{A_i}) \cdot (\chi_{U_j}, 1-\chi_{U_j}))}{P(U_j)}\right) dP =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{t} \int_{U_j} \varphi\left(\frac{1}{P(U_j)}\int_{U_j} (1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})dP\right) dP =$$

$$= \sum_{i=1}^{k} \int_\Omega \left(\sum_{j=1}^{t} \varphi\left(\frac{1}{P(U_j)}\int_{U_j} (1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})dP\right)\chi_{U_j}\right) dP =$$

$$= \sum_{i=1}^{k} \int_\Omega \varphi\left(\sum_{j=1}^{t} \left(\frac{1}{P(U_j)}\int_{U_j} (1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})dP\right)\chi_{U_j}\right) dP =$$

$$= \sum_{i=1}^{k} \int_\Omega \varphi\Big(E\big((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})|\sigma(\mathcal{B})\big)\Big)dP.$$

□

Proposition 6.2. *Let* $(\mathcal{B}_n)_{n=1}^\infty$ *be an increasing sequence of crisp partitions such that* $\sigma(\bigcup_{n=1}^\infty \mathcal{B}_n) = \mathcal{S}$. *Let* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *be an IF-partition. Then*

$$\lim_{n\to\infty} H(\mathcal{A}|\mathcal{B}_n) = \sum_{i=1}^{k} \int_\Omega \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i}))dP =$$

$$= \int_\Omega \left(\sum_{i=1}^{k} \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i}))\right) dP.$$

Proof. Since $\sigma(\mathcal{B}_n) \nearrow \mathcal{S}$, by the martingale convergence theorem

$$E\big((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})|\sigma(\mathcal{B}_n)\big) \nearrow E((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})|\mathcal{S}) =$$

$$= (1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i}); i = 1, ..., k$$

since $(1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})$ are $\mathcal{S}$-measurable. Since φ is a continuous function, we obtain

$$\varphi\Big(E\big((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})|\sigma(\mathcal{B}_n)\big)\Big) \nearrow \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})); i = 1, ..., k.$$

Finally, by the Lebesgue dominated convergence theorem, the linearity of the integral and Proposition 6.1

$$\lim_{n\to\infty} H(\mathcal{A}|\mathcal{B}_n) = \lim_{n\to\infty} \sum_{i=1}^{k} \int_{\Omega} \varphi\Big(E\big((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})|\sigma(\mathcal{B}_n)\big)\Big) dP =$$

$$= \sum_{i=1}^{k} \int_{\Omega} \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i}))dP = \int_{\Omega} \left(\sum_{i=1}^{k} \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})) \right) dP.$$

□

Theorem 6.3. *Let* $\mathcal{C} = \{C_1, ..., C_t\}$ *be a measurable partition of* Ω *being a generator, i.e.* $\sigma(\bigcup_{i=0}^{\infty} U^i(\mathcal{C})) = \mathcal{S}$. *Then for every IF-partition* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *there holds*

$$h(\mathcal{A}, \tau) \leq h(\mathcal{C}, \tau) + \int_{\Omega} \left(\sum_{i=1}^{k} \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})) \right) dP.$$

Proof. Put $\mathcal{B}_n = \bigvee_{i=0}^{n} \tau^i(\mathcal{C})$. Since $\sigma(\bigcup_{i=0}^{\infty} \tau^i(\mathcal{C})) = \mathcal{S}$, we obtain $\sigma(\mathcal{B}_n) \nearrow \mathcal{S}$. By Theorem 5.5 and Proposition 5.3 (see Corollary 5.6)

$$h(\mathcal{A}, \tau) \leq h(\mathcal{C}, \tau) + H(\mathcal{A}|\mathcal{B}_n).$$

Finally, Proposition 6.2 implies

$$h(\mathcal{A}, \tau) \leq h(\mathcal{C}, \tau) + \int_{\Omega} \left(\sum_{i=1}^{k} \varphi((1-\alpha)\mu_{A_i} + \alpha(1-\nu_{A_i})) \right) dP.$$

□

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