Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy modal operators of second type over interval-valued intuitionistic fuzzy sets. Part 2

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Abstract: The basic definitions of the concept of interval-valued intuitionistic fuzzy set and of the operations over it are given. For a first time, modal operators from second type are defined and their basic properties are studied.
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1 Introduction

Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs; see, e.g., [1]) were introduced as extensions of Intuitionistic Fuzzy Sets (IFSs; see [1, 2]) by George Gargov (7 April 1947 – 9 November 1996) and the author in [5]. A lot of operations, relations and operators were defined over them.

During last years, the interest in IVIFSs increased and now they are object of intensive research and applications.

A of 2018, the author has started working actively over the development of the IVIFSs theory. In [3], new intuitionistic fuzzy operators from standard (first) modal type were introduced and some of their basic properties were discussed. In [4], new intuitionistic fuzzy operators from second type were introduced and some of their basic properties were discussed, too.

In the present paper, using the notations from papers [3, 4], we introduce a series of intuitionistic fuzzy modal type operators of second type over IVIFSs. They are analogues of the modal type operators of second type over IFSs, discussed in [2].

2 Preliminaries

An IVIFS A over universe E is an object of the form:

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\},$$

where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1.$$

For any two IVIFSs A and B the following relations hold:

$$\begin{array}{lll} A \subset B & \text{iff} & (\forall x \in E)(\inf M_A(x) \leq \inf M_B(x) \;\&\; \inf N_A(x) \geq \inf N_B(x) \\ & & \&\, \sup M_A(x) \leq \sup M_B(x) \;\&\; \sup N_A(x) \geq \sup N_B(x)), \\ A = B & \text{iff} & A \subset B \,\&\, B \subset A. \end{array}$$

For any two IVIFSs A and B the following operations hold:

$$\begin{array}{lll} \neg A & = & \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}, \\ A \cap B & = & \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))], \\ & & [\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}, \\ A \cup B & = & \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x) \sup M_B(x))], \\ & & [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\} \\ A + B & = & \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x). \inf M_B(x), \\ & & \sup M_A(x) + \sup M_B(x) - \sup M_A(x). \sup M_B(x)], \\ & & [\inf N_A(x). \inf N_B(x), \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\} \\ A.B & = & \{\langle x, [\inf M_A(x). \inf M_B(x), \sup M_A(x). \sup M_B(x)], \\ & & [\inf N_A(x) + \inf N_B(x) - \inf N_A(x). \inf N_B(x), \\ & & \sup N_A(x) + \sup N_B(x) - \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\} \\ A@B & = & \{\langle x, [(\inf M_A(x) + \inf M_B(x))/2, (\sup M_A(x) + \sup M_B(x))/2], \\ & & [(\inf N_A(x) + \inf N_B(x))/2, (\sup N_A(x)\rangle + \sup N_B(x))/2] \mid x \in E\} \end{array}$$

The standard intuitionistic fuzzy operators of modal type are defined over IVIFSs similarly to those, defined for IFSs:

$$\begin{array}{rl} \Box A = & \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\}, \\ \Diamond A = & \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\}. \end{array}$$

3 Main results

We describe standard interval-valued forms of some intuitionistic fuzzy modal-like operators of second type, following [2], where their IFS-forms are given. We study some basis properties of the defined operators. Below, we formulate only these operator properties that are valid for the currently discussed type of operators and are not valid for the next operator extensions.

Let the IVIFS

$$A = \{\langle x, [\inf M_A(x), \sup M_A(x)], [\inf N_A(x), \sup N_A(x)]\rangle \mid x \in E\},$$

be given.

Now, we introduce the first two new IVIFS-operators of second type $\boxplus$ and $\boxtimes$ as follows.

$$\boxplus A = \{\langle x, \left[\frac{\inf M_A(x)}{2}, \frac{\sup M_A(x)}{2}\right], \left[\frac{\inf N_A(x)+1}{2}, \frac{\sup N_A(x)+1}{2}\right]\rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \left[\frac{\inf M_A(x)+1}{2}, \frac{\sup M_A(x)+1}{2}\right], \left[\frac{\inf N_A(x)}{2}, \frac{\sup N_A(x)}{2}\right]\rangle | x \in E\}.$$

All of their properties are valid for their immediare extensions, that for a given real number $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 \leq \alpha_2$, and IVIFS A have the forms:

$$\begin{aligned}
\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)} A &= \{\langle x, [\alpha_1 \inf M_A(x), \alpha_2 \sup M_A(x)]\,, \\
&\quad [\alpha_1 \inf N_A(x) + 1 - \alpha_1, \alpha_2 \sup N_A(x) + 1 - \alpha_2]\rangle | x \in E\}, \\
\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)} A &= \{\langle x, [\alpha_1 \inf M_A(x) + 1 - \alpha_1, \alpha_2 \sup M_A(x) + 1 - \alpha_2]\,, \\
&\quad [\alpha_1 \inf N_A(x), \alpha_2 \sup N_A(x)]\rangle | x \in E\}.
\end{aligned}$$

Obviously, for every IVIFS A:

$$\boxplus_{\left(\begin{smallmatrix}0.5\\ 0.5\end{smallmatrix}\right)} A = \boxplus A,$$

$$\boxtimes_{\left(\begin{smallmatrix}0.5\\ 0.5\end{smallmatrix}\right)} A = \boxtimes A.$$

Therefore, the new operators $\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}$ and $\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}$ are generalizations of the first ones from this paper and of operators $\boxplus, \boxtimes, \boxplus_\alpha, \boxtimes_\alpha$ from [4].

The following assertions hold for the first two types of the next operators. We give the proof of only one of them (Theorem 1 (b)), while the rest assertions are proved in the same manner.

Theorem 1. For every IVIFS A and for every $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 \leq \alpha_2$:

(a) $\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)} A \subseteq A \subseteq \boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)} A,$

(b) $\neg\boxplus_{\binom{\alpha_1}{\alpha_2}}\neg A = \boxtimes_{\binom{\alpha_1}{\alpha_2}}A$,

(c) $\neg\boxtimes_{\binom{\alpha_1}{\alpha_2}}\neg A = \boxplus_{\binom{\alpha_1}{\alpha_2}}A$,

(d) $\boxplus_{\binom{\alpha_1}{\alpha_2}}\boxplus_{\binom{\alpha_1}{\alpha_2}}A \subseteq \boxplus_{\binom{\alpha_1}{\alpha_2}}A$,

(e) $\boxtimes_{\binom{\alpha_1}{\alpha_2}}\boxtimes_{\binom{\alpha_1}{\alpha_2}}A \supseteq \boxtimes_{\binom{\alpha_1}{\alpha_2}}A$.

Proof. We will prove (b). Let A be an IVIFS and $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 \leq \alpha_2$. Then

$$\neg\boxplus_{\binom{\alpha_1}{\alpha_2}}\neg A$$

$$= \neg\boxplus_{\binom{\alpha_1}{\alpha_2}}\neg\{\langle x, [\inf M_A(x), \sup M_A(x)], [\inf N_A(x), \sup N_A(x)]\rangle \mid x \in E\}$$

$$= \neg\boxplus_{\binom{\alpha_1}{\alpha_2}}\{\langle x, [\inf N_A(x), \sup N_A(x)], [\inf M_A(x), \sup M_A(x)]\rangle \mid x \in E\}$$

$$= \neg\{\langle x, [\alpha_1 \inf N_A(x), \alpha_2 \sup N_A(x)],$$

$$[\alpha_1 \inf M_A(x) + 1 - \alpha_1, \alpha_2 \sup M_A(x) + 1 - \alpha_2]\rangle | x \in E\},$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) + 1 - \alpha_1, \alpha_2 \sup M_A(x) + 1 - \alpha_2],$$

$$[\alpha_1 \inf N_A(x), \alpha_2 \sup N_A(x)]\rangle | x \in E\}$$

$$= \boxtimes_{\binom{\alpha_1}{\alpha_2}}A.$$

The rest assertions in Theorem 1, as well as from the subsequent theorems, are proved by analogy. □

Theorem 2. For every two IVIFSs A and B:

(a) $\boxplus_{\binom{\alpha_1}{\alpha_2}}(A+B) \subseteq \boxplus_{\binom{\alpha_1}{\alpha_2}}A + \boxplus_{\binom{\alpha_1}{\alpha_2}}B$,

(b) $\boxtimes_{\binom{\alpha_1}{\alpha_2}}(A+B) \supseteq \boxtimes_{\binom{\alpha_1}{\alpha_2}}A + \boxtimes_{\binom{\alpha_1}{\alpha_2}}B$,

(c) $\boxplus_{\binom{\alpha_1}{\alpha_2}}(A.B) \supseteq \boxplus_{\binom{\alpha_1}{\alpha_2}}A.\boxplus_{\binom{\alpha_1}{\alpha_2}}B$,

(d) $\boxtimes_{\binom{\alpha_1}{\alpha_2}}(A.B) \subseteq \boxtimes_{\binom{\alpha_1}{\alpha_2}}A.\boxtimes_{\binom{\alpha_1}{\alpha_2}}B$.

Moreover, the following assertions are also true.

Theorem 3. For every IVIFS A and for every two real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$:

(a) $\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}\boxplus_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}A = \boxplus_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A$,

(b) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}\boxtimes_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}A = \boxtimes_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A$,

(c) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}\boxplus_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}A \supseteq \boxplus_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A$.

The second extension has the forms:

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A = \{\langle x, [\alpha_1 \inf M_A(x), \alpha_2 \sup M_A(x)],$$
$$[\alpha_1 \inf N_A(x) + \beta_1, \alpha_2 \sup N_A(x) + \beta_2]\rangle | x \in E\},$$

$$\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A = \{\langle x, [\alpha_1 \inf M_A(x) + \beta_1, \alpha_2 \sup M_A(x) + \beta_2],$$
$$[\alpha_1 \inf N_A(x), \alpha_2 \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_2 + \beta_2 \in [0, 1]$ and $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$.

Obviously, for every IVIFS A:

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A = \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \perp\\ \alpha_2 & 1-\alpha_2\end{smallmatrix}\right)}A,$$

$$\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A = \boxtimes_{\left(\begin{smallmatrix}\alpha_2 & 1-\alpha_2\\ \alpha_2 & \perp\end{smallmatrix}\right)}A.$$

The following assertions hold for the new operators.

Theorem 4. For every IVIFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2 \in [0, 1]$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$:

(a) $\neg\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}\neg A = \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A$,

(b) $\neg\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}\neg A = \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A$.

Theorem 5. For every IVIFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$, $\alpha_1 + \beta_1$, $\alpha_2 + \beta_2, \gamma_1 + \delta_1, \gamma_2 + \delta_2 \in [0, 1]$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2$,:

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}\boxtimes_{\left(\begin{smallmatrix}\gamma_1 & \delta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A \subseteq \boxtimes_{\left(\begin{smallmatrix}\gamma_1 & \delta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A.$$

Now, we introduce the third extension of the above operators. They have the forms:

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)}A$$
$$= \{\langle x, [\alpha_1 \inf M_A(x), \alpha_2 \sup M_A(x)], [\beta_1 \inf N_A(x) + \gamma_1, \beta_2 \sup N_A(x) + \gamma_2]\rangle | x \in E\},$$

$$\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A$$
$$= \{\langle x, [\alpha_1 \inf M_A(x) + \gamma_1, \alpha_2 \sup M_A(x) + \gamma_2], [\beta_1 \inf N_A(x), \beta_2 \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]$, $\max(\alpha_i, \beta_i) + \gamma_i \leq 1$ for $i = 1,2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2$.

Obviously, for every IVIFS A:

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)} A = \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \alpha_1 & \beta_1\\ \alpha_2 & \alpha_1 & \beta_2\end{smallmatrix}\right)} A,$$

$$\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)} A = \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \alpha_1 & \beta_1\\ \alpha_2 & \alpha_1 & \beta_2\end{smallmatrix}\right)} A.$$

The following assertions hold for the new operators.

Theorem 6. For every IVIFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]$, $\max(\alpha_i, \beta_i) + \gamma_i \leq 1$ for $i = 1,2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2$:

(a) $\neg \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} \neg A = \boxtimes_{\left(\begin{smallmatrix}\beta_1 & \alpha_1 & \gamma_1\\ \beta_2 & \alpha_2 & \gamma_2\end{smallmatrix}\right)} A,$

(b) $\neg \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} \neg A = \boxplus_{\left(\begin{smallmatrix}\beta_1 & \alpha_1 & \gamma_1\\ \beta_2 & \alpha_2 & \gamma_2\end{smallmatrix}\right)} A.$

Theorem 7. For every two IVIFSs A and B and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]$, $\max(\alpha_i, \beta_i) + \gamma_i \leq 1$ for $i = 1,2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2$:

(a) $\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} (A \cap B) = \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A \cap \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} B,$

(b) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} (A \cap B) = \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A \cap \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} B,$

(c) $\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} (A \cup B) = \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A \cup \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} B,$

(d) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} (A \cup B) = \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A \cup \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} B,$

(e) $\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} (A@B) = \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A @ \boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} B,$

(f) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} (A@B) = \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A @ \boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} B.$

Theorem 8. For every IVIFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]$, $\max(\alpha_i, \beta_i) + \gamma_i \leq 1$ for $i = 1,2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2$:

(a) $\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} C(A) = C(\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A),$

(b) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} C(A) = C(\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A)$,

(c) $\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} I(A) = I(\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A)$,

(d) $\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} I(A) = I(\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A)$.

A natural extension of the last two operators is the operator

$$\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) + \gamma_1, \alpha_2 \sup M_A(x) + \gamma_2],$$

$$[\beta_1 \inf N_A(x) + \delta_1, \beta_2 \sup N_A(x) + \delta_2]\rangle | x \in E\},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in [0,1]$, $\max(\alpha_i, \beta_i) + \gamma_i + \delta_i \leq 1$ for $i = 1,2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2$.

It is the fourth type of operators from the current type.

Obviously, for every IVIFS A:

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A = \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & 0 & \gamma_1\\ \alpha_2 & \beta_2 & 0 & \gamma_2\end{smallmatrix}\right)} A,$$

$$\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)} A = \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & 0\\ \alpha_2 & \beta_2 & \gamma_2 & 0\end{smallmatrix}\right)} A.$$

The following assertions hold for the new operator.

Theorem 9. For every IFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in [0,1]$, $\max(\alpha_i, \beta_i) + \gamma_i + \delta_i \leq 1$ for $i = 1,2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2$:

(a) $\neg \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} \neg A = \boxdot_{\left(\begin{smallmatrix}\beta_1 & \alpha_1 & \delta_1 & \gamma_1\\ \beta_2 & \alpha_2 & \delta_2 & \gamma_2\end{smallmatrix}\right)} A$,

(b) $\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} (A \cap B) = \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} A \cap \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} B$,

(c) $\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} (A \cup B) = \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} A \cup \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} B$,

(d) $\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} (A @ B) = \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} A @ \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} B$,

(e) $\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} \Box A \supseteq \Box \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} A$,

(f) $\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} \Diamond A \subseteq \Diamond \boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{smallmatrix}\right)} A$.

Theorem 10. For every IVIFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in [0,1]$, $\max(\alpha_i,\ \beta_i) + \gamma_i + \delta_i \leq 1$ for $i = 1, 2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2$; $\varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2, \eta_1, \eta_2, \theta_1, \theta_2 \in [0,1]$, $\max(\varepsilon_i, \zeta_i) + \eta_i + \theta_i \leq 1$ for $i = 1, 2$ and $\varepsilon_1 \leq \varepsilon_2, \zeta_1 \leq \zeta_2, \eta_1 \leq \eta_2, \theta_1 \leq \theta_2$:

$$\boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{smallmatrix}\right)}(\boxed{\bullet}_{\left(\begin{smallmatrix} \varepsilon_1 & \zeta_1 & \eta_1 & \theta_1 \\ \varepsilon_2 & \zeta_2 & \eta_2 & \theta_2 \end{smallmatrix}\right)}A)$$

$$= \boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1\varepsilon_1 & \beta_1\zeta_1 & \alpha_1\eta_1 + \gamma_1 & \beta_1\theta_1 + \delta_1 \\ \alpha_2\varepsilon_2 & \beta_2\zeta_2 & \alpha_2\eta_2 + \gamma_2 & \beta_2\theta_2 + \delta_2 \end{smallmatrix}\right)}A.$$

Theorem 11. For every IVIFS A and for every $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in [0,1]$, $\max(\alpha_i,\ \beta_i) + \gamma_i + \delta_i \leq 1$ for $i = 1, 2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2$:

(a) $\boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{smallmatrix}\right)}C(A) = C(\boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{smallmatrix}\right)}A)$,

(b) $\boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{smallmatrix}\right)}I(A) = I(\boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{smallmatrix}\right)}A)$.

The extended form of all above operators is the operator

$$\boxed{\circ}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{smallmatrix}\right)}A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \inf N_A(x) + \gamma_2],$$

$$[\beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \inf M_A(x) + \delta_2]\rangle | x \in E\},$$

where $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0,1]$, and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2, \varepsilon_1 \geq \varepsilon_2, \zeta_1 \leq \zeta_2$, and for $i = 1, 2$:

$$\max(\alpha_i - \zeta_i, \beta_i - \varepsilon_i) + \gamma_i + \delta_i \leq 1, \tag{1}$$

$$\min(\alpha_i - \zeta_i, \beta_i - \varepsilon_i) + \gamma_i + \delta_i \geq 0. \tag{2}$$

Obviously, for every IVIFS A:

$$\boxed{\bullet}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{smallmatrix}\right)}A = \boxed{\circ}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & 0 & 0 \end{smallmatrix}\right)}A.$$

The following assertions hold for the operator $\boxed{\circ}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{smallmatrix}\right)}$.

Theorem 12. For every IVIFS A and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0,1]$, and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2, \varepsilon_1 \geq \varepsilon_2, \zeta_1 \leq \zeta_2$, for which (1) and (2) are valid, the equality

$$\neg\boxed{\circ}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_1 & \zeta_1 \end{smallmatrix}\right)}\neg A = \boxed{\circ}_{\left(\begin{smallmatrix} \beta_1 & \alpha_1 & \delta_1 & \gamma_1 & \zeta_1 & \varepsilon_1 \\ \beta_2 & \alpha_2 & \delta_2 & \gamma_2 & \zeta_2 & \varepsilon_2 \end{smallmatrix}\right)}A$$

holds.

Theorem 13. For every two IVIFSs A and B and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0,1]$, and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2, \varepsilon_1 \geq \varepsilon_2, \zeta_1 \leq \zeta_2$, for which (1) and (2) are valid, the equality

$$\boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}}(A@B)$$

$$= \boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} A @ \boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} B$$

holds.

We must note that equalities

$$\boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}}(A \cap B)$$

$$= \boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} A \cap \boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} B,$$

$$\boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}}(A \cup B)$$

$$= \boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} A \cup \boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} B,$$

which are valid for operator $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}$, are not always valid now.

Theorem 14. For every IVIFS A and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0,1]$, and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2, \varepsilon_1 \geq \varepsilon_2, \zeta_1 \leq \zeta_2$, for which (1) and (2) are valid, for every $\eta_1, \theta_1, \iota_1, \kappa_1, \lambda_1, \mu_1, \eta_2, \theta_2, \iota_2, \kappa_2, \lambda_2, \mu_2 \in [0,1]$, and $\eta_1 \leq \eta_2, \theta_1 \leq \theta_2, \iota_1 \leq \iota_2, \kappa_1 \leq \kappa_2, \lambda_1 \geq \lambda_2, \mu_1 \leq \mu_2$, for which

$$\max(\eta_i - \mu_i, \theta_i - \lambda_i) + \iota_i + \kappa_i \leq 1,$$

$$\min(\eta_i - \mu_i, \theta_i - \lambda_i) + \iota_i + \kappa_i \geq 0$$

are valid, the equality

$$\boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{pmatrix}} \left(\boxed{\circ}_{\begin{pmatrix} \eta_1 & \theta_1 & \iota_1 & \kappa_1 & \lambda_1 & \mu_1 \\ \eta_2 & \theta_2 & \iota_2 & \kappa_2 & \lambda_2 & \mu_2 \end{pmatrix}} A \right)$$

$$= \boxed{\circ}_{\begin{pmatrix} \alpha_1\eta_1 + \varepsilon_1\mu_1 & \beta_1\theta_1 + \zeta_1\lambda_1 & \alpha_1\iota_1 - \varepsilon_1\kappa_1 + \gamma_1 & \beta_1\kappa_1 - \zeta_1\iota_1 + \delta_1 & \alpha_1\lambda_1 + \varepsilon_1\theta_1 & \beta_1\mu_1 + \zeta_1\eta_1 \\ \alpha_2\eta_2 + \varepsilon_2\mu_2 & \beta_2\theta_2 + \zeta_2\lambda_2 & \alpha_2\iota_2 - \varepsilon_2\kappa_2 + \gamma_2 & \beta_2\kappa_2 - \zeta_2\iota_2 + \delta_2 & \alpha_2\lambda_2 + \varepsilon_2\theta_2 & \beta_2\mu_2 + \zeta_2\eta_2 \end{pmatrix}} A$$

holds.

4 Conclusion

In [4], inf- and sup-components have equal parameters (real numbers). In the present research, these parameters for the inf- and sup-components have different values. Their specific properties were studied. In a next paper, the author will introduce a next generalizations of all the operators, describing here and in [4] for a first time. In near future, the author plans to extend the results from the present paper to a book.

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