# Intuitionistic Fuzzy Graph Interpretations of Multi-Person Multi-Criteria Decision Making

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## 1  Introduction

In Group Decision Making (GDM) a set of experts in a given field is involved in a decision process concerning the selection of the best alternative(s) among a set of predefined ones. An evaluation of the alternatives is performed independently by each decision maker: the experts express their evaluations on the basis of some decision scheme, which can be either implicitly assumed or explicitly specified in the form of a set of predefined criteria [11]. In both cases, the aim is to obtain an evaluation (performance judgement or rating) of the alternatives by each expert. In the case in which a set of predefined criteria is specified, a performance judgement is expressed by each expert for each criterion; this kind of decision problem is called multi-person multi-criteria decision making [11]: its aim is to compute a consensual judgement and a consensus degree for a majority of the experts on each alternative. As the main actors in a multi-person multi-criteria decision making activity are individuals with their inherent subjectivity, it often happens that performance judgements cannot be precisely assessed; the imprecision may arise from the nature of the characteristics of the alternatives, which can be either unquantifiable or unavailable; it may be also derive from the inability of the experts to formulate a precise evaluation [8, 9, 18]. To take into account this imprecision some fuzzy models of GDM have been proposed which relieve experts from quantifying qualitative concepts by directly managing performance judgements expressed linguistically [5, 6, 9, 10].

An important phase of a group decision process is the definition of a collective evaluation for each alternative: the main problem is to aggregate the experts' performance judgements to obtain an overall rating for each alternative. A consequent problem is to compare the experts' judgements to verify the consensus among them. In the case of unanimous consensus, the evaluation process ends with the selection of the best alternative(s). As in real situation humans rarely come to an unanimous agreement, in the literature some fuzzy approaches to evaluate a "soft" degree of consensus have been proposed [5, 12].

In this work, an intuitionistic fuzzy interpretation of multi-person multi-criteria decision making is proposed. Each expert is asked to evaluate at least a subset of the alternatives in terms of its performance with respect to each predefined criterion: the experts evaluations are expressed as a pair of numeric values, interpreted in the intuitionistic fuzzy framework: these numbers express a "positive" and a "negative" evaluation, respectively. Each expert is also assigned a pair of values, which express the expert's reliability (confidence in her/his evaluation with respect to each criterion). Distinct reliability values are associated with distinct criteria. The proposed formulation is based on the assumption of alternatives' independency.

In a previous paper on a fuzzy intuitionistic interpretation of multi expert multi criteria decision making , we assumed that the criteria that experts use are *linearly ordered* [4] In this paper we propose a procedure that gives the experts the possibility to use particular orders, i.e., orders represented by oriented graphs.

The following basic notation is adopted in the paper:

$E = \{E_1, E_2, ..., E_m\}$ is the set of the experts involved in the decision process;

$A = \{A_1, A_2, ..., A_p\}$ is the set of the considered alternatives;

$C = \{C_1, C_2, ..., C_q\}$ is the set of the criteria used for evaluating the alternatives.

Using the apparatus of the Intuitionistic Fuzzy Sets (IFSs, [2]) we shall present a possible construction of an aggregated evaluation. The possibility for using of elements of intuitionistic fuzziness in decision making was first discussed in [7, 13, 14, 15, 16, 17]. In these approaches the order of the criteria is linear. The idea for a non-linear order was introduced in [1] for a first time. In this paper we formally discuss it in detail for a first time. On the other hand, of course, it includes the standard order as a partial case. It is based on the concept of an intuitionistic fuzzy graph [2]. The paper is structured as follows: in section 2 some basic definitions of intuitionistic fuzzy sets are given and in section 3 the proposed method of multi-person multi-criteria decision making is described.

## 2  Some basic definitions

In this section we introduce some basic definitions, related to the Intuitionistic Fuzzy Sets (IFSs), Intuitionistic Fuzzy Relations (IFRs), and Index Matrices (IMs).

**2.1.** Let a set $E$ be fixed. An IFS $A$ in $E$ is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

**2.2.** Let $I$ be a fixed set of indices and $\mathcal{R}$ be the set of real numbers. By IM with index sets $K$ and

$L$ $(K, L \subset I)$ we will mean the object (see [3]):

$$[K, L, \{a_{k_i, l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1, l_1} & a_{k_1, l_2} & \dots & a_{k_1, l_n} \\ k_2 & a_{k_2, l_1} & a_{k_2, l_2} & \dots & a_{k_2, l_n} \\ \vdots & & & & \\ k_m & a_{k_m, l_1} & a_{k_m, l_2} & \dots & a_{k_m, l_n} \end{array}$$

where $K = \{k_1, k_2, ..., k_m\}, L = \{l_1, l_2, ..., l_n\}$, for $1 \le i \le m$, and for $1 \le j \le n : a_{k_i, l_j} \in \mathcal{R}$. Given two IMs $A = [K, L, \{a_{k_i, l_j}\}]$ and $B = [P, Q, \{b_{p_r, q_s}\}]$ the usual matrix operations "addition" and "multiplication" are defined.

**2.3.** Below we will consider applications of IFSs, IFRs and IMs to graph theory. Let the oriented graph $G = (V, A)$ be given, where $V$ is a set of vertices and $A$ is a set of arcs. Every graph arc connects two graph vertices.

Following [2] the concept of an *Intuitionistic Fuzzy Graph (IFG)* will be introduced.

Let $E_1$ and $E_2$ be two sets; let $x \in E_1$ and $y \in E_2$ and let operation $\times$ denote the standard Cartesian product operation. Let the operation $o \in \{\times_1, \times_2, \dots, \times_5\}$.

The set

$$G^* = \{\langle\langle x, y\rangle, \mu_G(x, y), \nu_G(x, y)\rangle \, \langle x, y\rangle \in E_1 \times E_2\}$$

is called an $o$-IFG (or briefly, an IFG) if the functions $\mu_G : E_1 \times E_2 \to [0, 1]$ and $\nu_G : E_1 \times E_2 \to [0, 1]$ define the degree of membership and the degree of non-membership, respectively, of the element $\langle x, y\rangle \in E_1 \times E_2$ to the set $G \subset E_1 \times E_2$.

These functions have the forms of the corresponding components of the $o$-Cartesian product over IFSs, and for all $\langle x, y\rangle \in E_1 \times E_2$:

$$0 \le \mu_G(x, y) + \nu_G(x, y) \le 1.$$

For simplicity, below we will write $G$ instead of $G^*$.

It can be easily seen (see [2]) that $G$ can be modified to the following form:

$$G = [V_I \cup \overline{V}, \overline{V} \cup V_O, \{a_{i,j}\}],$$

where $V_I, V_O$ and $\overline{V}$ are respectively the sets of the graph input, output and internal vertices. At

least one arc leaves every vertex of the first type, but none enters; at least one arc enters each vertex of the second type but none leaves it; every vertex of the third type has at least one arc ending in it and at least one arc starting from it.

Operations "$\cup$", "$\cap$" and "." over IFGs are the same as the operations over IMs.

## 3 About the experts who order alternatives

Let there be $m$ experts: $E_1, E_2, \ldots, E_m$, $p$ alternatives for expert estimation $A_1, A_2, \ldots, A_p$ and $q$ estimation criteria $C_1, C_2, \ldots, C_q$. Let each expert has her/his own (current) reliability score $\langle \delta_i, \varepsilon_i \rangle \in [0,1]^2$ and her/his own (current) number of participations in expert investigations $\gamma_i$ (these two values correspond to her/his last estimation). Expert's reliabiliy scores can be interpreted, e.g., as

$$
\left\{
\begin{array}{c}
\delta_i = \dfrac{\sum\limits_{j=1}^{q} \delta_{i,j}}{q} \\[2em]
\varepsilon_i = \dfrac{\sum\limits_{j=1}^{q} \varepsilon_{i,j}}{q}
\end{array}
\right. ,
$$

where $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$ are elements of the IM

$$
T = 
\begin{array}{c|cccc}
 & C_1 & C_2 & \ldots & C_q \\
\hline
E_1 & & & & \\
 & & \langle \delta_{i,j}, \varepsilon_{i,j} \rangle & & \\
E_2 & & & & \\
 & & (1 \le i \le m, & & \\
\vdots & & & & \\
 & & 1 \le j \le q) & & \\
E_m & & & &
\end{array}
$$

and $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$ is the rating of the $i$-th expert with respect to the $j$-th criterion (we assume that $i$-th expert's knowledge reliability may differ over different criteria; the case when the expert is equally good a specialist with respect to the different criteria is a special one).

To illustrate a possible computation of an expert's reliability score we give an example: a sport's expert gave 10 prognoses for the end of 10 football matches. In 5 of the cases he pointed the winner, in 3 of the cases he failed and in the rest two

cases he did not engage with final opinion about the result. That is why we determine his reliability score as $\langle 0.5, 0.3 \rangle$.

Each expert is allowed to indicate which criteria he/she will use in the evaluation. To this aim each expert orders the criteria (or a part of them, if he/she thinks that some of them are not necessary), on the vertices of a graph. The highest vertices of this graph will correspond to the most relevant criteria according to the expert. The second top-down vertices interpret the criteria that are weaker than the first ones. There are no arcs between vertices which are incomparable due to some criterion. Therefore, each of the experts not only orders the criteria that he/she will use (it is possible, omitting some of them), but his/her order is not a linear one. As a result, we will obtain $m$ different graphs. Now, we shall transform these graphs into an IFGs, labelling each arc of the $i$-th expert's graph with a pair of values, corresponding to his/her expert's reliability score.

By using the operation $\cup$ over the IFGs (see section **2.3**), we obtain a new IFG. Let us note it by $G$. It represents all expert opinions about how to order the criteria. Now, its arcs have intuitionistic fuzzy weights being the disjunctions of the weights, of the same arcs in the separate IFGs. Of course, the new graph may not be well ordered, while the expert graphs are well ordered. Now, we reconfigure the IFG $G$ as follows. If there is a cycle between two vertices $V_1$ and $V_2$, i.e., there are vertices $U_1, U_2, \ldots, U_u$ and vertices $W_1, W_2, \ldots, W_w$, such that $V_1, U_1, U_2, \ldots, U_u, V_2$ and $V_2, W_1, W_2, \ldots, W_w, V_1$ are simple paths in the graph, then we calculate the weights of both paths as conjunctions of the weights of the arcs which take part in the respective paths. The path that has smallest weight must be cut in two, removing its arc with smallest weight. If both arcs have equal weights, these arcs will be removed. Therefore, the new graph is already cycle-free. Now, we can determine the priorities (weights) of the vertices of the IFG, i.e., the priorities of the criteria. Let them be $\varphi_1, \varphi_2, \ldots, \varphi_q$. For example, they can have values $\frac{s-1}{t}$ for the vertices from the $s$-th level bottom-top of the IFG with $t+1$ levels for $t \ge 1$. We shall use these values below.

This procedure will be used in a next authors' re-

search, but with other form of the algorithm for decision making using the so constructed IFGs more actively. Here we use the above construction only to propose the experts' possibility to work with non-linearly ordered criteria and obtain priorities of these criteria.

Having in mind that the $i$-th expert can use only a part of the criteria and can estimate only a part of the alternatives, we can construct the IM of her/his estimations in the form

$$S_i = \begin{array}{c|c} & \begin{array}{cccc} A_{l_1} & A_{l_2} & \ldots & A_{l_{p_i}} \end{array} \\ \hline \begin{array}{c} C_{i_1} \\ C_{i_2} \\ \vdots \\ C_{i_{q_i}} \end{array} & \begin{array}{c} \langle \alpha^i_{i_j,l_k} \beta^i_{i_j,l_k} \rangle \\ \\ (1 \leq i_j \leq q_i \leq q, \\ \\ 1 \leq l_k \leq p_i \leq p) \end{array} \end{array}$$

where: $\alpha^i_{i_j,k}, \beta^i_{i_j,k} \in [0,1]$, $\alpha^i_{i_j,k} + \beta^i_{i_j,k} \leq 1$ and $\langle \alpha^i_{j,k}, \beta^i_{j,k} \rangle$ is the $i$-th expert estimation for the $k$-th alternative about the $j$-th criterion; $C_{i_1}, ..., C_{i_{q_i}}$ and $A_{l_1}, ..., A_{l_{p_i}}$ are only those of the criteria and alternatives which the $i$-th expert prefers. Let us assume that in cases when pair $\langle \alpha^i_{j,k}, \beta^i_{j,k} \rangle$ does not exist, we will work with pair $\langle 0, 1 \rangle$.

Now, we can construct an IM containing the aggregated estimations with the form

$$S = \begin{array}{c|c} & \begin{array}{cccc} A_1 & A_2 & \ldots & A_p \end{array} \\ \hline \begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_q \end{array} & \begin{array}{c} \langle \alpha_{j,k} \beta_{j,k} \rangle \\ \\ (1 \leq j \leq q, \\ \\ 1 \leq k \leq p) \end{array} \end{array}$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ can be calculated by different formulae. Some examples of such formulae are the following:

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_i . \alpha^i_{j,k}}{m} \\ \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_i . \beta^i_{j,k}}{m} \end{cases}$$

(here only the average degrees of expert reliability participate),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_{i,j} . \alpha^i_{j,k}}{m} \\ \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_{i,j} . \beta^i_{j,k}}{m} \end{cases}$$

(here estimated by the corresponding criteria, only the experts' degrees of reliability participate), (here there participate only the experts' degrees of reliability estimated by the corresponding criteria),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\alpha}^i_{j,k}}{m} \\ \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\beta}^i_{j,k}}{m} \end{cases},$$

where $\overline{\alpha}^i_{j,k}$ and $\overline{\beta}^i_{j,k}$ can be calculated by various formulae, according to the particular goals and the expert's knowledge. For example, such formulae can be:

$$\begin{cases} \overline{\alpha}^i_{j,k} = \gamma_i . \dfrac{\alpha^i_{j,k} . \delta_{i,j} + \beta^i_{j,k} . \varepsilon_{i,j}}{\gamma_i + 1} \\ \\ \overline{\beta}^i_{j,k} = \gamma_i . \dfrac{\alpha^i_{j,k} . \varepsilon_{i,j} + \beta^i_{j,k} . \delta_{i,j}}{\gamma_i + 1} \end{cases}$$

or

$$\begin{cases} \overline{\alpha}^i_{j,k} = \alpha^i_{j,k} . \dfrac{\delta_{i,j} + 1 - \varepsilon_{i,j}}{2} \\ \\ \overline{\beta}^i_{j,k} = \beta^i_{j,k} . \dfrac{\varepsilon_{i,j} + 1 - \delta_{i,j}}{2} \end{cases}.$$

The first formula takes into account not only the rating of each expert by the different criteria, but also the number of times he has made a prognosis (his first time is neglected, for the lack of former experience). Obviously, the so constructed elements of the IM satisfy the inequality: $\alpha_{j,k} + \beta_{j,k} \leq 1$. This IM contains the average experts' estimations taking into account the experts' ratings. As we noted above, each of the criteria $C_j (1 \leq j \leq q)$ has itself a priority, denoted by

$\varphi_j \in [0,1]$. For every alternative $A_k$ we can determine the global estimation $\langle \alpha_k, \beta_k \rangle$, where

$$\begin{cases} \alpha_k = \dfrac{\sum\limits_{j=1}^{q} \varphi_j . \alpha_{j,k}}{q} \\[3ex] \beta_k = \dfrac{\sum\limits_{j=1}^{q} \varphi_j . \beta_{j,k}}{q} \end{cases} .$$

Let alternatives (processes) have the following (objective) values about the different criteria after the end of the expert estimations:

|       | $A_1$ | $A_2$ | $\ldots$ | $A_p$ |
|-------|-------|-------|----------|-------|
| $C_1$ |       |       |          |       |
| $C_2$ |       | $\langle a_{j,k} b_{j,k} \rangle$ | | |
|       |       | $(1 \le j \le q,$ | | |
| $\vdots$ |    |       |          |       |
|       |       | $1 \le k \le p)$ | | |
| $C_q$ |       |       |          |       |

where: $a_{j,k}, b_{j,k} \in [0,1]$ and $a_{j,k} + b_{j,k} \le 1$. Then the expert's new rating, $\langle \delta_i, \varepsilon_i \rangle$, and new number of participations in expert investigations, $\gamma_i'$ will be:

$$\gamma_i' = \gamma_i + 1,$$

and

$$\begin{cases} \delta_i' = \dfrac{\gamma_i . \delta_i + \dfrac{c_M - c_i}{2}}{\gamma_i'}, \\[3ex] \varepsilon_i' = \dfrac{\gamma_i . \varepsilon_i - \dfrac{c_M - c_i}{2}}{\gamma_i'}, \end{cases},$$

where:

$$c_i = \frac{\sum\limits_{j=1}^{q} \sum\limits_{k=1}^{p} ((\alpha_{j,k} - a_{j,k})^2 + (\beta_{j,k} - b_{j,k})^2)^{1/2}}{p.q},$$

and

$$c_M = \frac{\sum\limits_{i=1}^{n} c_i}{n}.$$

Other formulae for the expert's rating are also possible and will be discussed in a next research.

## 4   Conclusion

In this paper an intuitionistic fuzzy model of a multi person multi criteria decision making activity has been proposed. The proposed approach uses two main kind of information: the experts' reliability scores (objective data), and their alternatives' evaluations (subjective data), and derives an "objective" estimation about the current event. The proposed model can be for example successfully used in decision areas involving evaluations of the public opinion about currently flowing processes and tendencies in the society, or evaluating the ratings of the politicians, the media and other similar phenomena. For instance, when we are evaluating analogous programs in competing televisions, we dispose of a fixed set of criteria, among which we take the most relevant (in our opinion). These criteria might be grouped in different categories - related to the program itself, or to the media, or to the variety of themes the program covers. If there exists some casual relation between two of the chosen parameters in our evaluation, it seems natural to grade them from top to bottom in the graph representation of the problem. If both factors are independent but equal in weight, their place is next to on the same hierarchy level in the graph. In a future research we shall discuss a possible generalisation of this model to the case when criteria are not linearly ordered.

## References

[1] O. Asparoukhov, K. Atanassov, B. Magaev, Application of intuitionistic fuzzy graphs and intuitionistic fuzzy relation- methods for ordering expert estimation criteria, Proceedings of the VI IFSA World Congress, Sao Paulo, Brazil, July 1995, Vol. 1, 535-537.

[2] K. Atanassov, Intuitionistic Fuzzy Sets, Springer Physica-Verlag, Berlin, 1999.

[3] K. Atanassov, Generalized index matrices. – Compt. Rend. de l'Academie Bulgare des Sciences, Vol.40, 1987, No 11, 15-18.

[4] K. Atanassov, G. Pasi and R. Yager. Intuitionistic fuzzy interpretations of multi-person multi-criteria decision making. Pro-

ceedings of 2002 First International Symposium Intelligent Systems, Vol. 1, 115-119.

[5] G. Bordogna, M. Fedrizzi, and G. Pasi, A linguistic modeling of consensus in Group Decision Making based on OWA operators, IEEE Trans. on System Man and Cybernetics, 27 (1), 126-132, 1997.

[6] S.J. Chen, and C.L.Hwang, Fuzzy Multiple Attribute Decision Making - Methods and Applications, Springer-Verlag, 1992.

[7] S.K. De, R. Biswas, A.R. Roy, Multicriteria decision making using intuitionistic fuzzy set theory, The Journal of Fuzzy Mathematics, Vol. 6, No. 4, 1998, 837-842.

[8] P.C. Fishburn, Utiliy and Decision Making under Uncertainty, Mathematical social sciences, Vol. 8, 253-285, 1984.

[9] J.C. Fodor, and M. Rubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publisher, Dordrecht, 1994.

[10] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, A Linguistic Decision Process in Group Decision Making, Group Decision and Negotiation, Vol. 5, 165-176, 1996.

[11] C.L.Hwang, and M.J. Lin, Group Decision Making under Multiple Criteria. Methods and Applications, Lecture Notes in economics and Mathematical Systems, Springer-Verlag, 1987.

[12] J.Kacprzyk, H. Nurmi, M. Fedrizzi eds., Consensus under fuzziness, Kluwer Academic Publishers, Dordrecht, 1997.

[13] E. Szmidt, J. Kacprzyk, Group decision making under intuitionistic fuzzy preference relations, Proceedings of Seventh International Conference Information Processing and Management of Uncertainty in Knowledge-Based Systems, Paris, July 6-10, 1998, 172- 178.

[14] E. Szmidt, J. Kacprzyk, Intuitionistic fuzzy sets for more realistic group decision making, Proceedings of International Conference Transitions to Advanced Market Institutions and Economies, Warsaw, June 18-21, 1997, 430-433.

[15] E. Szmidt, J. Kacprzyk, Decision making in an intuitionistic fuzzy environment, Proceedings of EUROFUSE-SIC'99, Budapest, 25-28 May 1999, 292-297.

[16] E. Szmidt , J. Kacprzyk, Intuitionistic fuzzy sets in group decision making, Notes on IFS, Vol. 2, 1996, No. 1, 15-32.

[17] Temponi C., Shannon A., Atanassov K., Ban A., An idea for an intuitionistic fuzzy approach to decision making. Proceedings of the Third International Conference on Intuitionistic Fuzzy Sets (J. Kacprzyk and K. Atanassov, Eds.), Vol. 1; Notes on Intuitionistic Fuzzy Sets, Vol. 5 (1999), No. 3, 6-10.

[18] Yager, R. R., Non-numeric multi-criteria multi-person decision making, International Journal of Group Decision Making and Negotiation 2, 81-93, 1993.