Notes on Intuitionistic Fuzzy Sets
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Intuitionistic Fuzzy Topological BH-Algebras

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Abstract: The intuitionistic fuzzification of a BH-algebra is considered and related results are investigated. The notion of equivalence relations on the family of all intuitionistic fuzzy BH-algebras of a BH-algebra is introduced, and then some properties are discussed. The concept of intuitionistic fuzzy topological BH-algebras is introduced, and some related results are obtained.
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1 Introduction

Y. Imai and K. Iseki [8] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [6, 7] Q.P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. In 1996, Jun, Roh and Kim introduced the notion of BH-algebra, which is a generalization of BCH-algebras [9]. In 2001, Q. Zhang, E.H. Roh and Y.B. Jun studied the fuzzy theory in BH-algebras [14]. C.H. Park introduced the notion of an interval-valued fuzzy BH-algebra in a BH-algebra and investigate related properties [10]. The concept of a fuzzy set, which was introduced in [13], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D. H. Foster [5] combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by A. Rosenfeld [11], to formulate the elements of a theory of fuzzy topological groups. After the introduction of fuzzy sets by L. A. Zadeh [13], several researchers were conducted on the generalizations of the notion of fuzzy sets. The idea of intuitionistic fuzzy set was first published by K. T. Atanassov [1] , as a generalization of the notion of fuzzy sets. In this paper, using the Atanassovs idea, we establish the notion of intuitionistic fuzzy BH-algebras, equivalence relations on the family of all intuitionistic fuzzy BH-algebras, and intuitionistic fuzzy topological BH-algebras which are generalization of the notion of fuzzy topological BH-algebras. We investigate several properties, and show that the BH-homomorphic image and preimage of an intuitionistic fuzzy topological BH-algebra is an intuitionistic fuzzy topological BH-algebra.

2 Preliminaries

Definition 2.1 ([9])**.** Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a BH-algebra if it satisfies the following axioms:

1. $x * x = 0$,

2. $x * y = 0$ and $y * x = 0$ imply $x = y$,

3. $x * 0 = x$

for all $x, y \in X$.

Definition 2.2. A non-empty set S of a BH-algebra X is called a *BH-subalgebra* of X if $x * y \in S$ for any $x, y \in X$.

Definition 2.3. A mapping $\theta : X \to Y$ of BH-algebras is called a *BH-homomorphism* if $\theta(x * y) = \theta(x) * \theta(y)$ for all $x, y \in X$.

Definition 2.4 ([5])**.** A *fuzzy topology* on a set X is a family τ of fuzzy sets in X which satisfies the following conditions:

1. for all $c \in [0, 1], k_c \in \tau$, where k_c has a constant membership function,
2. if $A, B \in \tau$, then $A \cap B \in \tau$,
3. if $A_j \in \tau$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \tau$.

The pair (X,τ) is called a *fuzzy topological space* and members of τ are called *open fuzzy sets*.

Definition 2.5. An *intuitionistic fuzzy set* (IFS for short) D in X is an object having the form

$$D = \{\langle x, \mu_D(x), \nu_D(x)\rangle | x \in X\}$$

where the functions $\mu_D : X \to [0,1]$ and $\nu_D : X \to [0,1]$ denote the degree of membership (namely $\mu_D(x)$) and the degree of nonmembership (namely $\nu_D(x)$) of each element $x \in X$ to the set D, respectively, and $0 \leq \mu_D(x) + \nu_D(x) \leq 1$ for each $x \in X$.

For the sake of simplicity, we shall use the notation $D = \langle x, \mu_D, \nu_D\rangle$ instead of

$$D = \{\langle x, \mu_D(x), \nu_D(x)\rangle | x \in X\}.$$

Let f be a mapping from a set X to a set Y. If

$$B = \{\langle y, \mu_B(y), \nu_B(y)\rangle | y \in Y\}$$

is an IFS in Y, then the *preimage* of B under f denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)\rangle | x \in X\},$$

and if

$$D = \{\langle x, \mu_D(x), \nu_D(x)\rangle | x \in X\}$$

is an IFS in X, then the image of D under f, denoted by $f(D)$, is the IFS in Y defined by

$$f(D) = \{\langle y, f_{\sup}\mu_D(y), f_{\inf}\nu_D(y)\rangle | y \in Y\},$$

where

$$f_{\sup}(\mu_D)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_D(x), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{\inf}(\nu_D)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_D(x), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

3 Intuitionistic fuzzy BH-algebras

Definition 3.1. Let X be a BH -algebra. An IFS

$$D = \langle x, \mu_D, \nu_D\rangle$$

in X is called an intuitionistic fuzzy BH-algebra if it satisfies:

$$\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\}$$

and

$$\nu_D(x * y) \leq \max\{\nu_D(x), \nu_D(y)\} \text{ for all } x, y \in X.$$

Example 3.2. Consider a BH-algebra $X = \{0, a, b, c\}$ with the following cayley table:

$*$	0	a	b	c
0	0	c	0	b
a	a	0	0	0
b	b	b	0	c
c	c	c	b	0

Let

$$D = \langle x, \mu_D(x), \nu_D(x) \rangle$$

be an IFS in X by $\mu_D(0) = 0.7, \mu_D(a) = 0.2, \mu_D(b) = 0.5, \mu_D(c) = 0.4$ and $\nu_D(0) = 0.2, \nu_D(a) = 0.8, \nu_D(b) = 0.3, \nu_D(c) = 0.5$. Then

$$D = \langle x, \mu_D, \nu_D \rangle$$

is an intuitionistic fuzzy BH-algebra.

Proposition 3.3. If an IFS

$$D = \langle x, \mu_D, \nu_D \rangle$$

in X is an intuitionistic fuzzy BH-algebra of X, then

$$\mu_D(0) \geq \mu_D(x)$$

and

$$\nu_D(0) \leq \nu_D(x) \text{ for all } x \in X.$$

Proof. Let $x \in X$. Then

$$\mu_D(0) = \mu_D(x * x) \geq \min\{\mu_D(x), \mu_D(x)\} = \mu_D(x)$$

and

$$\nu_D(0) = \nu_D(x * x) \leq \max\{\nu_D(x), \nu_D(x)\} = \nu_D(x).$$ □

Theorem 3.4. If $\{D_i | i \in \wedge\}$ is an arbitrary family of intuitionistic fuzzy BH-algebras of X, then $\cap D_i$ is an intuitionistic fuzzy BH-algebra of X, where

$$\cap D_i = \{\langle x, \wedge\mu_{D_i}(x), \vee\nu_{D_i}(x)\rangle | x \in X\}.$$

Proof. Let $x, y \in$ X. Then

$$\wedge\mu_{D_i}(x * y) \geq \wedge(\min\{\mu_{D_i}(x), \mu_{D_i}(y)\}) = \min\{\wedge\mu_{D_i}(x), \wedge\mu_{D_i}(y)\}$$

and

$$\vee\nu_{D_i}(x * y) \leq \vee(\max\{\nu_{D_i}(x), \nu_{D_i}(y)\}) = \max\{\vee\nu_{D_i}(x), \vee\nu_{D_i}(y)\}.$$

Hence,

$$\cap D_i = \langle x, \wedge\mu_{D_i}, \vee\nu_{D_i} \rangle$$

is an intuitionistic fuzzy BH-algebra of X. □

Theorem 3.5. If an IFS $D = \langle x, \mu_D, \nu_D \rangle$ in X is an intuitionistic fuzzy BH-algebra of X, then so is $\overline{D}$, where $\overline{D} = \{\langle x, \mu_D(x), 1 - \mu_D(x) \rangle | x \in X\}$.

Proof. It is sufficient to show that $\overline{\mu}_D$ satisfies the second condition in Definition 3.1.

Let $x, y \in X$. Then

$$\begin{aligned} \mu_{\mathcal{D}}(x * y) &= 1 - \mu_D(x * y) \\ &\leq 1 - \min\{\mu_D(x), \mu_D(y)\} \\ &= \max\{1 - \mu_D(x), 1 - \mu_D(y)\} \\ &= \max\{\mu_{\mathcal{D}}(x), \mu_{\mathcal{D}}(y)\}. \end{aligned}$$

Hence $\mathcal{D}$ is an intuitionistic fuzzy BH-algebra of X. □

Theorem 3.6. If an IFS $D = \langle x, \mu_D, \nu_D \rangle$ in X is an intuitionistic fuzzy BH-algebra of X, then the sets

$$X_\mu := \{x \in X | \mu_D(x) = \mu_D(0)\}$$

and

$$X_\nu := \{x \in X | \nu_D(x) = \nu_D(0)\}$$

are BH-subalgebras of X.

Proof. Let $x, y \in X_\mu$. Then $\mu_D(x) = \mu_D(0) = \mu_D(y)$, and so $\mu_D(x*y) = \min\{\mu_D(x), \mu_D(y)\} = \mu_D(0)$. By using Proposition 3.3, we know that $\mu_D(x * y) = \mu_D(0)$ or equivalently $x * y \in X_\mu$. Now let $x, y \in X_\nu$. Then $\nu_D(x * y) \leq \max\{\nu_D(x), \nu_D(y)\} = \nu_D(0)$, and by applying Proposition 3.3 we conclude that $\nu_D(x * y) = \nu_D(0)$ and hence $x * y \in X_\nu$. □

Definition 3.7. Let $D = \langle x, \mu_D, \nu_D \rangle$ be an IFS in X and let $t \in [0, 1]$. Then the set $U(\mu_D, t) := \{x \in X | \mu_D(x) \geq t\}$ (resp. $L(\nu_D, t) := \{x \in X | \nu_D(x) \leq t\}$) is called a μ-level t-cut (resp. ν-level t-cut) of D.

Theorem 3.8. If an IFS $D = \langle x, \mu_D, \nu_D \rangle$ in X is an intuitionistic fuzzy BH-algebra of X, then the μ-level t-cut and ν-level t-cut of D are BH-subalgebras of X for every $t \in [0, 1]$ such that $t \in \text{Im}(\mu_D) \cap \text{Im}(\nu_D)$, which are called a μ-level BH-subalgebra and a ν-level BH-subalgebra respectively.

Proof. Let $x, y \in U(\mu_D, t)$. Then $\mu_D(x) \geq t$ and $\mu_D(y) \geq t$. It follows that $\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\} \geq t$ so that $x * y \in U(\mu_D, t)$. Hence, $U(\mu_D, t)$ is a BH-subalgebra of X. Now let $x, y \in L(\nu_D, t)$. Then $\nu_D(x * y) \leq \max\{\nu_D(x), \nu_D(y)\} \leq t$ and so $x * y \in L(\nu_D, t)$. Therefore, $L(\nu_D, t)$ is a BH-subalgebra of X. □

Theorem 3.9. Let $D = \langle x, \mu_D, \nu_D \rangle$ be an IFS in X such that the sets $U(\mu_D, t)$ and $L(\nu_D, t)$ are BH-subalgebras of X. Then $D = \langle x, \mu_D, \nu_D \rangle$ is an intuitionistic fuzzy BH-algebra of X.

Proof. Assume that there exist $x_0, y_0 \in X$ such that $\mu_D(x_0 * y_0) < \min\{\mu_D(x_0), \mu_D(y_0)\}$.

Let

$$t_0 := 1/2(\mu_D(x_0 * y_0) + \min\{\mu_D(x_0), \mu_D(y_0)\}).$$

Then

$$\mu_D(x_0 * y_0) < t_0 < \min\{\mu_D(x_0), \mu_D(y_0)\}$$

and so $x_0 * y_0 \notin U(\mu_D, t_0)$, but $x_0, y_0 \in U(\mu_D, t_0)$. This is a contradiction, and therefore

$$\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\} \text{ for all } x, y \in X.$$

Now suppose that $\nu_D(x_0 * y_0) > \max\{\nu_D(x_0), \nu_D(y_0)\}$ for some $x_0, y_0 \in X$. Taking

$$S_0 := 1/2(\nu_D(x_0 * y_0) + \max\{\nu_D(x_0), \nu_D(y_0)\}),$$

then

$$\max\{\nu_D(x_0), \nu_D(y_0)\} < S_0 < \nu_D(x_0 * y_0).$$

It follows that $x_0, y_0 \in L(\nu_D, S_0)$ and $x_0 * y_0 \notin L(\nu_D, S_0)$, a contradiction. Hence

$$\nu_D(x * y) \leq \max\{\nu_D(x), \nu_D(y)\} \text{ for all } x, y \in X.$$

This completes the proof. □

Theorem 3.10. Any BH-subalgebra of X can be realized as both a μ-level BH-subalgebra and a ν-level BH-subalgebra of some intuitionistic fuzzy BH-algebra of X.

Proof. Let S be a BH-subalgebra of X and let μ_D and ν_D be fuzzy sets in X defined by

$$\mu_D(x) := \begin{cases} t, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_D(x) = \begin{cases} s, & \text{if } x \in S \\ 1, & \text{otherwise} \end{cases}$$

for all $x \in X$ where t and s are fixed numbers in $(0, 1)$ such that $t + s < 1$.

Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$. Hence $\mu_D(x * y) = \min\{\mu_D(x), \mu_D(y)\}$and $\nu_D(x * y) = \max\{\nu_D(x), \nu_D(y)\}$. If at least one of x and y does not belong to S, then at least one of $\mu_D(x)$ and $\mu_D(y)$ is equal to 0, and at least one of $\nu_D(x)$ and $\nu_D(y)$ is equal to 1. It follows that

$$\mu_D(x * y) \geq 0 = \min\{\mu_D(x), \mu_D(y)\}, \nu_D(x * y) \leq 1 = \max\{\nu_D(x), \nu_D(y)\}.$$

Hence $D = \langle x, \mu_D, \nu_D \rangle$ is an intuitionistic fuzzy BH-algebra of X. Obviously,

$$U(\mu_D, t) = S = L(\nu_D, s).$$

This completes the proof. □

Theorem 3.11. Let α be a BH-homomorphism of a BH-algebra X into a BH-algebra Y and B an intuitionistic fuzzy BH-algebra of Y. Then $\alpha^{-1}(B)$ is an intuitionistic fuzzy BH-algebra of X.

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\mu_{\alpha^{-1}}(B)(x*y) &= \mu_B(\alpha(x*y)) \\ &= \mu_B(\alpha(x)*\alpha(y)) \\ &\geq \min\{\mu_B(\alpha(x)), \mu_B(\alpha(y))\} \\ &= \min\{\mu_{\alpha^{-1}}(B)(x), \mu_{\alpha^{-1}}(B)(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_{\alpha^{-1}}(B)(x*y) &= \nu_B(\alpha(x*y)) \\ &= \nu_B(\alpha(x)*\alpha(y)) \\ &\leq \max\{\nu_B(\alpha(x)), \nu_B(\alpha(y))\} \\ &= \max\{\nu_{\alpha^{-1}}(B)(x), \nu_{\alpha^{-1}}(B)(y)\}.\end{aligned}$$

Hence $\alpha^{-1}(B)$ is an intuitionistic fuzzy BH-algebra in X. □

Theorem 3.12. Let α be a BH-homomorphism of a BH -algebra X onto a BH-algebra Y. If $D = \langle x, \mu_D, \nu_D\rangle$ is an intuitionistic fuzzy BH-algebra of X, then $\alpha(D) = \langle y, \alpha_{\text{sup}}\mu_D), \alpha_{\text{inf}}(\nu_D)\rangle$ is an intuitionistic fuzzy BH-algebra of Y.

Proof. Let $D = \langle x, \mu_D, \nu_D\rangle$be an intuitionistic fuzzy topological BH-algebra in X and let $y_1, y_2 \in Y$. Noticing that

$$\{x_1 * x_2 | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \subseteq \{x \in X | x \in \alpha^{-1}(y_1 * y_2)\},$$

we have

$$\begin{aligned}\alpha_{\text{sup}}(\mu_D)(y_1 * y_2) &= \sup\{\mu_D(x) | x \in \alpha^{-1}(y_1 * y_2)\} \\ &\geq \sup\{\mu_D(x_1 * x_2) | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\ &\geq \sup\{\min\{\mu_D(x_1), \mu_D(x_2)\} | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\ &= \min\{\sup\{\mu_D(x_1) | x_1 \in \alpha^{-1}(y_1)\}, \sup\{\mu_D(x_2) | x_2 \in \alpha^{-1}(y_2)\}\} \\ &= \min\{\alpha_{\text{sup}}(\mu_D)(y_1), \alpha_{\text{sup}}(\mu_D)(y_2)\}\end{aligned}$$

and

$$\begin{aligned}\alpha_{\text{inf}}(\nu_D)(y_1 * y_2) &= \inf\{\nu_D(x) | x \in \alpha^{-1}(y_1 * y_2)\} \\ &\leq \inf\{\nu_D(x_1 * x_2) | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\ &\leq \inf\{\max\{\nu_D(x_1), \nu_D(x_2)\} | x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \\ &= \max\{\inf\{\nu_D(x_1) | x_1 \in \alpha^{-1}(y_1)\}, \inf\{\nu_D(x_2) | x_2 \in \alpha^{-1}(y_2)\}\} \\ &= \max\{\alpha_{\text{inf}}(\nu_D)(y_1), \alpha_{\text{inf}}(\nu_D)(y_2)\}\end{aligned}$$

Hence $\alpha(D) = \langle y, \alpha_{\sup}(\mu_D), \alpha_{\inf}(\nu_D)\rangle$ is an intuitionistic fuzzy BH-algebra in Y. Let $\Omega(X)$ denote the family of all intuitionistic fuzzy BH-algebras of X and let $t \in [0,1]$. Define binary relations $\tilde{\mu}$ and $\tilde{\nu}$ on $\Omega(X)$ as follows :

$$A\tilde{\mu}B \Leftrightarrow U(\mu_A, t) = U(\mu_B, t)$$

and

$$A\tilde{\nu}B \Leftrightarrow L(\nu_A, t) = L(\nu_B, t),$$

respectively, for $A = \langle x, \mu_A, \nu_A\rangle$ and $B = \langle x, \mu_B, \nu_B\rangle \in \Omega(X)$. Then clearly $\tilde{\mu}$ and $\tilde{\nu}$ are equivalence relations on $\Omega(X)$. For any $A = \langle x, \mu_A, \nu_A\rangle \in \Omega(X)$, let $[A]_\mu$ (resp. $[A]_\nu$)denote the equivalence class of $A = \langle x, \mu_A, \nu_A\rangle$ modulo $\tilde{\mu}$ (resp. $\tilde{\nu}$), and denote by $\Omega(X)|\tilde{\mu}$ (resp. $\Omega(X)|\tilde{\nu}$) the collection of all equivalence classes of A modulo $\tilde{\mu}$ (resp. $\tilde{\nu}$). $\Omega(X)|\tilde{\mu} := \{[A]_\mu | A = \langle x, \mu_A, \nu_A\rangle \in \Omega(X)\}$ (resp. $\Omega(X)|\tilde{\nu} := \{[A]_\nu | A = \langle x, \mu_A, \nu_A\rangle \in \Omega(X)\}$. Now, let S(X) denote the family of all BH-subalgebras of X and let $t \in [0,1]$. Define maps α_t and β_t from $\Omega(X)$ to $S(X) \cup \{\phi\}$ by $\alpha_t(A) = U(\mu_A, t)$ and $\beta_t(A) = L(\nu_A, t)$, respectively, for all A = $\langle x, \mu_A, \nu_A\rangle \in \Omega(X)$. Then α_t and β_t are clearly well-defined. □

Theorem 3.13. For any $t \in (0,1)$ the maps α_t and β_t are surjective from $\Omega(X)$ to $S(X) \cup \{\phi\}$.

Proof. Let $t \in (0,1)$. Note that $0\tilde{=}\langle x, 0, 1\rangle$ is in $\Omega(X)$, where 0 and 1 are fuzzy sets in X defined by 0(x)=0 and 1(x)=1 for all $x \in X$. Obviously $\alpha_t(\tilde{0})$ = U(0,t) = ϕ = L(1,t) = $\beta_t(\tilde{0})$. Let G($\neq \phi$) $\in$ S(X). For $\tilde{G} = \langle x, \chi_G, \chi_G\rangle \in \Omega(X)$, we have $\alpha_t(\tilde{G}) = U(\chi_G, t)$ = G and $\beta_t(\tilde{G}) = L(\chi_G', t)$ = G. Hence α_t and β_t are surjective. □

Theorem 3.14. The quotient sets $\Omega(X)|\tilde{\mu}$ and $\Omega(X)|\tilde{\nu}$ are equipotent to $S(X) \cup \{\phi\}$ for every $t \in (0,1)$.

Proof. For $t \in$ (0,1) let α_t^* (resp. β_t^*) be a map from $\Omega(X)|\tilde{\mu}$ (resp. $\Omega(X)|\tilde{\nu}$) to $S(X) \cup \{\phi\}$ defined by $\alpha_t^*([A]_\mu) = \alpha_t[A]$ (resp. $\beta_t^*([A]_\nu) = \beta_t(A)$) for all $A = \langle x, \mu_A, \nu_A\rangle \in \Omega(X)$.

If $U(\mu_A, t) = U(\mu_B, t)$ and $L(\nu_A, t) = L(\nu_B$,t) for $A = \langle x, \mu_A, \nu_A\rangle$ and $B = \langle x, \mu_B, \nu_B\rangle \in \Omega(X)$, then $A\tilde{\mu}B$ and $A\tilde{\nu}B$; hence $[A]_\mu = [B]_\mu$ and $[A]_\nu = [B]_\nu$. Therefore, the maps α_t^* and β_t^* are injective. Now let $G(\neq \phi) \in S(X)$. For $G\tilde{=}\langle x, \chi_G, \chi_G\rangle \in \Omega(X)$, we have $\alpha_t^*([\tilde{G}]_\mu) = \alpha_t(\tilde{G}) = U(\chi_G, t) = G$ and $\beta_t^*([\tilde{G}]\nu) = \beta_t(\tilde{G}) = L(\chi_G, t) = G$. Finally, for $0\tilde{=}\langle x, 0, 1\rangle \in \Omega(X)$ we get

$$\alpha_t^*([\tilde{0}]\mu) = \alpha_t(\tilde{0}) = U(0, t) = \phi$$

and $\beta_t^*([\tilde{0}]\nu)$ = $\beta_t(\tilde{0})$ = L(1,t) = ϕ. This show that α_t^* and β_t^* are surjective, and we are done. For any $t \in [0,1]$, we define another relation $\mathfrak{R}^t$ on $\Omega(X)$ as follows :

$$(A, B) \in \mathfrak{R}^t \Leftrightarrow U(\mu_A, t) \cap L(\nu_A, t) = U(\mu_B, t) \cap L(\nu_B, t).$$

For any $A = \langle x, \mu_A, \nu_A\rangle$, $B = \langle x, \mu_B, \nu_B\rangle \in \Omega(X)$. Then the relation $\mathfrak{R}^t$ is also an equivalence relation on $\Omega(X)$. □

Theorem 3.15. For any $t \in (0,1)$, the map $\phi_t : \Omega(X) \to S(X) \cup \{\phi\}$defined by $\phi_t(A) = \alpha_t(A) \cap \beta_t(A)$ for each A = $\langle x, \mu_A, \nu_A\rangle \in \Omega(X)$ is surjective.

Proof. Let $t \in (0,1)$. For $0\tilde{} = \langle x, 0, 1\rangle \in \Omega(X)$, we get

$$\phi_t(\tilde{0}) = \alpha_t(\tilde{0}) \cap \beta_t(\tilde{0}) = \cup(0,t) \cap L(1,t) = \phi.$$

For any $H \in \Omega(X)$, there exists $H\tilde{=}\langle x, \chi_H, \chi_H\rangle \in \Omega(X)$ such that

$$\phi_t(\tilde{H}) = \alpha_t(\tilde{H}) \cap \beta_t(\tilde{H}) = \cup(\chi_H, t) \cap L(\chi_H, t) = H.$$

This completes the proof. □

Theorem 3.16. For any $t \in (0,1)$, the quotient set $\Omega(X)|\mathfrak{R}^t$ is equipotent to $S(X) \cup \{\phi\}$.

Proof. Let $t \in$ (0,1) and let $\phi_t^* : \Omega(X)|\mathfrak{R}^t \to S(X)\cup\{\phi\}$ be a map defined by $\phi_t^*([A]\mathfrak{R}^t) = \phi_t(A)$ for all $[A]_{\mathfrak{R}^t} \in \Omega(X)|\mathfrak{R}^t$. Assume that $\phi_t^*([A]_{\mathfrak{R}^t}) = \phi_t^*([B]_{\mathfrak{R}^t})$ for any $[A]_{\mathfrak{R}^t}, [B]_{\mathfrak{R}^t} \in \Omega(X)|\mathfrak{R}^t$. Then

$\alpha_t(A) \cap \beta_t(A) = \alpha_t(B) \cap \beta_t(B)$. i.e., $U(\mu_A, t) \cap L(\nu_A, t) = U(\mu_B, t) \cap L(\nu_B, t)$. Hence $(A, B) \in \mathfrak{R}^t$, and so $[A]_{\mathfrak{R}^t} = [B]_{\mathfrak{R}^t}$. Therefore ϕ_t^* is injective. Now for

$$0\tilde{} = \langle x, 0, 1\rangle \in \Omega(X)$$

we have

$$\phi_t^*([\tilde{0}]\mathfrak{R}^t) = \phi_t(\tilde{0}) = \alpha_t(\tilde{0}) \cap \beta_t(\tilde{0}) = U(0,t) \cap L(1,t) = \phi.$$

For

$$H\tilde{} = \langle x, \chi_H, \chi_H\rangle \in \Omega(X)$$

we get

$$\phi_t^*([\tilde{H}]\mathfrak{R}^t) = \phi_t(\tilde{H}) = \alpha_t(\tilde{H}) \cap \beta_t(\tilde{H}) = \cup(\chi_H, t) \cap L(\chi_H, t) = H.$$

Thus ϕ_t^* is surjective. This completes the proof. □

4 Intuitionistic fuzzy topological BH-algebras

In [4], Coker generalized the concept of fuzzy topological space, first initiated by Chang [3], to the case of intuitionistic fuzzy sets as follows.

Definition 4.1 ([4])**.** An intuitionistic fuzzy topology (IFT) on a non-empty set X is a family Φ of IFSs in X satisfying the following axioms:

(T1) $\tilde{0}, \tilde{1} \in \Phi$,
(T2) $G_1 \cap G_2 \in \Phi$ for any $G_1, G_2 \in \Phi$,
(T3)$\cup_{i \in J} G_i \in \Phi$ for any family $\{G_i : i \in J\} \subseteq \Phi$.

In this case the pair (X,Φ) is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in Φ is called an intuitionistic fuzzy open set (IFOS for short) in X.

Definition 4.2. Let (X, Φ) and (Y, Ψ) be two IFTSs. A mapping $f : X \to Y$ is said to be intuitionistic fuzzy continuous if the preimage of each IFS in Ψ is an IFS in Ψ; and f is said to be intuitionistic fuzzy open if the image of each IFS in Φ is an IFS in Ψ.

Definition 4.3. Let D be an IFS in an IFTS (X, Ψ). Then the induced intuitionistic fuzzy topology (IIFT for short) on D is the family of IFSs in D which are the intersection with D of IFOSs in X. The IIFT is denoted by Ψ_D, and the pair (D, Ψ_D) is called an intuitionistic fuzzy subspace of (X, Ψ).

Definition 4.4. Let (D, Φ_D) and (B, Ψ_B) be intuitionistic fuzzy subspaces of IFTSs (X, Φ) and (Y, Ψ), respectively, and let $f : X \to Y$ be a mapping. Then f is a mapping of D into B if $f(D) \subset B$. Furthermore, f is said to be relatively intuitionistic fuzzy continuous if for each IFS $V_B \in \Psi_B$, the intersection $f^{-1}(V_B) \cap D$ is an IFS in Φ_D ; and f is said to be relatively intuitionistic fuzzy open if for each IFS $U_D \in \Phi_D$, the image $f(U_D)$ is an IFS in Ψ_B.

Proposition 4.5. Let (D, Φ_D) and (B, Ψ_B) be intuitionistic fuzzy subspaces of IFTSs (X, Φ) and (Y, Ψ) respectively, and let f be an intuitionistic fuzzy continuous mapping of X into Y such that $f(D) \subset B$. Then f is relatively intuitionistic fuzzy continuous mapping of D into B.

Proof. Let V_B be an IFS in Ψ_B. Then there exists $V \in \Psi$ such that $V_B = V \cap B$. Since f is intuitionistic fuzzy continuous, it follows that $f^{-1}(V)$ is an IFS in Φ. Hence

$$f^{-1}(V_B) \cap D = f^{-1}(V \cap B) \cap D = f^{-1}(V) \cap f^{-1}(B) \cap D = f^{-1}(V) \cap D$$

is an IFS in Φ_D. This completes the proof.

For any BH-algebra X and any element $a \in X$ we use a_r to denote the selfmap of X defined by $a_r(x) = x * a$ for all $x \in X$. □

Definition 4.6. Let X be BH-algebra, Φ an IFT on X and D an intuitionistic fuzzy BH- algebra with IIFT Φ_D. Then D is called an intuitionistic fuzzy topological BH-algebra if for each $a \in$ X the mapping

$$a_r : (D, \Phi_D) \to (D, \Phi_D), x \mapsto x * a,$$

is relatively intuitionistic fuzzy continuous.

Theorem 4.7. Given BH-algebras X and Y, and a BH-homomorphism $\alpha : X \to Y$, let Φ and Ψ be the IFTs on X and Y, respectively such that $\Phi = \alpha^{-1}(\Psi)$. If B is an intuitionistic fuzzy topological BH-algebra in Y, then $\alpha^{-1}(B)$ is an intuitionistic fuzzy topological BH-algebra in X.

Proof. Let $a \in X$ and let U be an IFS in $\Phi_{\alpha^{-1}}(B)$. Since α is an intuitionistic fuzzy continuous mapping of (X, Φ) into (Y, Ψ), it follows from Proposition 4.5 that α is a relatively intuitionistic fuzzy continuous mapping of $(\alpha^{-1}(B), \Phi_{\alpha^{-1}}(B))$ into (B, Ψ_B). Note that there exists an IFS V in Ψ_B such that $\alpha^{-1}(V) = U$. Then

$$\begin{aligned}\mu_{a_r^{-1}}(U)(x) &= \mu_U(a_r(x)) = \mu_U(x * a) = \mu_{\alpha^{-1}(V)}(x * a)\\ &= \mu_V(\alpha(x * a)) = \mu_V(\alpha(x) * \alpha(a))\end{aligned}$$

and

$$\begin{aligned}\nu_{a_r^{-1}}(U)(x) &= \nu_U(a_r(x)) = \nu_U(x * a) = \nu_{\alpha^{-1}(V)}(x * a)\\ &= \nu_V(\alpha(x * a)) = \nu_V(\alpha(x) * \alpha(a))\end{aligned}$$

Since B is an intuitionistic fuzzy topological BH-algebra in Y, the mapping

$$b_r : (B, \Psi_B) \to (B, \Psi_B), y \mapsto y * b$$

is relatively intuitionistic fuzzy continuous for every $b \in Y$. Hence,

$$\begin{aligned} \mu_{a_r^{-1}}(U)(x) &= \mu_V(\alpha(x) * \alpha(a)) = \mu_V(\alpha(a)r(\alpha(x))) \\ &= \mu_{\alpha(a)r(V)^{-1}}(\alpha(x)) = \mu^{-1}_{\alpha(\alpha(a)r^{-1}(V))}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{a_r^{-1}}(U)(x) &= \nu_V(\alpha(x) * \alpha(a)) = \nu_V(\alpha(a)r(\alpha(x))) \\ &= \nu_{\alpha(a)r(V)^{-1}}(\alpha(x)) = \nu^{-1}_{\alpha(\alpha(a)r^{-1}(V))}(x) \end{aligned}$$

Therefore

$$a_r^{-1}(U) = \alpha^{-1}(\alpha(a)r^{-1}(V)),$$

and so

$$a_r^{-1}(U) \cap \alpha^{-1}(B) = \alpha^{-1}(\alpha(a)r^{-1}(V)) \cap \alpha^{-1}(B)$$

is an IFS in $\Phi_{\alpha^{-1}}(B)$.

This completes the proof. □

Theorem 4.8. Given BH-algebras X and Y, and a BH-isomorphism α of X to Y, let Φ and Ψ be the IFTs on X and Y respectively such that $\alpha(\Phi) = \Psi$. If D is an intuitionistic fuzzy topological BH-algebra in X, then $\alpha(D)$ is an intuitionistic fuzzy topological BH-algebra in Y.

Proof. It is sufficient to show that the mapping

$$b_r : (\alpha(D), \Psi_{\alpha(D)}) \to (\alpha(D), \Psi_{\alpha(D)}), y \mapsto y * b$$

is relatively intuitionistic fuzzy continuous for each $b \in Y$. Let U_D be an IFS in Ψ_D. Then there exists an IFS U in Φ such that $U_D = U \cap D$. Since α is one-one, it follows that

$$\alpha(U_D) = \alpha(U \cap D) = \alpha(U) \cap \alpha(D)$$

which is an IFS in $\Psi_{\alpha(D)}$. This shows that α is relatively intuitionistic fuzzy open.

Let $V_\alpha(D)$ be an IFS in $\Psi_\alpha(D)$. The surjectivity of α implies that for each $b \in Y$ there exists $a \in X$ such that $b = \alpha(a)$. Hence

$$\begin{aligned} \mu^{-1}\alpha(br(V\alpha(D))(x) &= \mu_{\alpha^{-1}(\alpha(a)r^{-1}(V\alpha(D))}(x) \\ &= \mu_{\alpha(a)r^{-1}(V\alpha(D))}(\alpha(x)) \\ &= \mu_{V\alpha(D)}(\alpha(a)r(\alpha(x))) \\ &= \mu_{V\alpha(D)}(\alpha(x) * \alpha(a)) \\ &= \mu_{V\alpha(D)}(\alpha(x * a)) \\ &= \mu_{\alpha^{-1}(V\alpha(D))}(x * a) \\ &= \mu_{\alpha^{-1}(V\alpha(D))}(a_r(x)) \\ &= \mu_{a_r^{-1}(\alpha^{-1}(V\alpha(D))}(x) \end{aligned}$$

$$
\begin{aligned}
\nu^{-1}\alpha(br(V\alpha(D))(x) &= \nu_{\alpha^{-1}(\alpha(a)r^{-1}(V\alpha(D))}(x) \\
&= \nu_{\alpha(a)r^{-1}(V\alpha(D))}(\alpha(x)) \\
&= \nu_{V\alpha(D)}(\alpha(a)r(\alpha(x))) \\
&= \nu_{V\alpha(D)}(\alpha(x)*\alpha(a)) \\
&= \nu_{V\alpha(D)}(\alpha(x*a)) \\
&= \nu_{\alpha^{-1}(V\alpha(D))}(x*a) \\
&= \nu_{\alpha^{-1}(V\alpha(D))}(a_r(x)) \\
&= \nu_{a_r^{-1}(\alpha^{-1}(V\alpha(D))}(x)
\end{aligned}
$$

Therefore

$$\alpha^{-1}(b_r^{-1}(V_{\alpha(D)}) \;=\; a_r^{-1}(\alpha^{-1}(V_{\alpha(D)})).$$

By hypothesis, the mapping

$$a_r : (D, \Phi_D) \to (D, \Phi_D), x \mapsto x * a$$

is relatively intuitionistic fuzzy continuous and α is a relatively intuitionistic fuzzy continuous map:

$$(D, \Phi_D) \to (\alpha(D), \Psi_{\alpha(D)}).$$

Thus

$$\alpha^{-1}(b_r^{-1}(V_{\alpha(D)})) \cap D \;=\; a_r^{-1}(\alpha^{-1}(V_{\alpha(D)})) \cap D$$

is an IFS in Φ_D. Since α is relatively intuitionistic fuzzy open,

$$\alpha(\alpha^{-1}(b_r^{-1}(V_{\alpha(D)})) \cap D) \;=\; b_r^{-1}(V_{\alpha(D)}) \cap \alpha(D).$$

is an IFS in $\Psi_{\alpha(D)}$. This completes the proof. □

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