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Normality and translation of $IFS(G \times Q)$ under norms

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Abstract: In this article, we introduce the idea of normality and translation of Q-intuitionistic fuzzy subgroups with respect to norms (*t*-norm *T* and *t*-conorm *C*) and some interesting results of them are given. Conditions for level cut subsets of them are explored and provided. Finally, we investigate them by using group homomorphisms are investigated.

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1 Introduction

The study of groups arose early in the nineteenth century in connection with the solution of equations. The theory of abstract groups plays an important part in present day mathematics and science. Groups arise in a bewildering number of apparently unconnected subjects. Thus they appear in crystallography and quantum mechanics, in geometry and topology, in analysis and algebra, in physics, chemistry and even in biology.

The notion of fuzzy sets was first introduced by Zadeh [28]. Rosenfeld [26] introduced the fuzzy sets in the realm of group theory. Since then many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader frame work of the fuzzy



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setting. Anthony and Sherwood [1] gave the definition of fuzzy subgroup based on t-norm. As a generalization of a fuzzy set, the concept of an intuitionistic fuzzy set was introduced by Atanassov [2, 3]. Solairaju and Nagarajan [27] introduced the notion of Q-fuzzy groups.

Norms were introduced in the framework of probabilistic metric spaces. However, they are widely applied in several other fields, e.g., in fuzzy set theory, fuzzy logic, and their applications. In previous works [4–25], by using norms, we investigated some properties of fuzzy algebraic structures, specially, we defined and investigated Q-fuzzy subgroups, anti Q-fuzzy subgroups, Level subsets and translations Q-fuzzy subgroups, level subsets and translations of anti Q-fuzzy subgroups and Q-intuitionistic fuzzy subgroups with respect to norms [4–7, 25].

In this paper we introduce the notion of normality and translation of Q-intuitionistic fuzzy subgroups with respect to norms (*t*-norm T and *t*-conorm C) and investigate some related properties. Level cut subset of them is introduced and the relation between this representation and them is discussed such that some of its properties are studied. Finally, some results of them by using group homomorphisms are investigated.

2 Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For more details we refer to [6, 7, 25].

Definition 2.1 (see [6]). A group is a non-empty set G on which there is a binary operation $(a, b) \rightarrow ab$ such that:

- (1) If a and b belong to G, then ab is also in G (closure),
- (2) a(bc) = (ab)c for all $a, b, c \in G$ (associativity),
- (3) There is an element $e \in G$ such that ae = ea = a for all $a \in G$ (identity),
- (4) If $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called Abelian if the binary operation is commutative, i.e., ab = ba for all $a, b \in G$. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \rightarrow a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \rightarrow ab$ for which the identity is denoted by e.

Definition 2.2 (see [6]). A function (or map) $f : G \to H$ from one group G to another H is a (group) homomorphism if the group operation is preserved in the sense that $f(g_1g_2) = f(g_1)f(g_2)$ and f is called an anti (group) homomorphism if $f(g_1g_2) = f(g_2)f(g_1)$ for all $g_1, g_2 \in G$.

Proposition 2.3 (see [7]). Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

(1) H is a subgroup of G.

(2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

A subgroup H of a group G is called normal if $ghg^{-1} \in H$ for any $g \in G$ and $h \in H$.

Definition 2.4 (see [7]). Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into [0, 1]. The set of all fuzzy subsets of G is called the [0, 1]-power set of G and is denoted $[0, 1]^G$.

Definition 2.5 (see [25]). For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.6 (see [25]). Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \rightarrow [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, *IFS*) in X such that $\mu_A, \nu_A \in [0, 1]^X$ and for all $x \in X$ we have $(\mu_A(x) + \nu_A(x)) \in [0, 1]$. In particular \emptyset_X and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0, 1)$ and $U_X(x) = (1, 0)$, respectively. We will denote the class of all *IFSs* in X as *IFS*(X).

Definition 2.7 (see [25]). Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be *IFSs* in X. Then

- (1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
- (2) Equality: A = B iff $A \subseteq B$ and $B \subseteq A$.

Definition 2.8 (see [6]). A *t*-norm *T* is a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ having the following four properties:

(T1) T(x, 1) = x (neutral element) (T2) $T(x, y) \le T(x, z)$ if $y \le z$ (monotonicity) (T3) T(x, y) = T(y, x) (commutativity) (T4) T(x, T(y, z)) = T(T(x, y), z) (associativity), for all $x, y, z \in [0, 1]$.

Corollary 2.9 (see [6]). Let *T* be a *t*-norm. Then for all $x \in [0, 1]$ (1) T(x, 0) = 0.

(2) T(0,0) = 0.

Example 2.10 (see [6]). (1) Standard intersection t-norm

$$T_m(x,y) = \min\{x,y\}$$

(2) Bounded sum t-norm

$$T_b(x, y) = \max\{0, x + y - 1\}.$$

(3) Algebraic product *t*-norm

$$T_p(x,y) = xy.$$

(4) Drastic *t*-norm

$$T_D(x,y) = \begin{cases} y, & \text{if } x = 1\\ x, & \text{if } y = 1\\ 0, & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum *t*-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\}, & \text{if } x+y > 1\\ 0, & \text{otherwise.} \end{cases}$$

(6) Hamacher product *t*-norm

$$T_{H_0}(x,y) = \begin{cases} 0, & \text{if } x = y = 0\\ \frac{xy}{x+y-xy}, & \text{otherwise.} \end{cases}$$

The drastic *t*-norm is the pointwise smallest *t*-norm and the minimum is the pointwise largest *t*-norm:

$$T_D(x,y) \le T(x,y) \le T_{\min}(x,y)$$

for all $x, y \in [0, 1]$.

Lemma 2.11 (see [6]). Let T be a t-norm. Then

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.12 (see [7]). A *t*-conorm *C* is a function $C : [0,1] \times [0,1] \rightarrow [0,1]$ having the following four properties:

 $\begin{array}{l} ({\rm C1})\ C(x,0) = x \\ ({\rm C2})\ C(x,y) \leq C(x,z) \ {\rm if}\ y \leq z \\ ({\rm C3})\ C(x,y) = C(y,x) \\ ({\rm C4})\ C(x,C(y,z)) = C(C(x,y),z) \ , \end{array}$ for all $x,y,z \in [0,1].$

Corollary 2.13 (see [7]). Let C be a t-conorm. Then for all $x \in [0, 1]$ (1) C(x, 1) = 1. (2) C(0, 0) = 0.

Example 2.14 (see [7]).

(1) Standard union t-conorm

$$C_m(x,y) = \max\{x,y\}.$$

(2) Bounded sum *t*-conorm

$$C_b(x, y) = \min\{1, x + y\}.$$

(3) Algebraic sum *t*-conorm

$$C_p(x,y) = x + y - xy.$$

(4) Drastic *t*-conorm

$$C_D(x,y) = \begin{cases} y, & \text{if } x = 0\\ x, & \text{if } y = 0\\ 1, & \text{otherwise,} \end{cases}$$

dual to the drastic *t*-norm.

(5) Nilpotent maximum t-conorm, dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\}, & \text{if } x + y < 1\\ 1, & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity)

$$C_{H_2}(x,y) = \frac{x+y}{1+xy}$$

is a dual to one of the Hamacher *t*-norms. Note that all *t*-conorms are bounded by the maximum and the drastic *t*-conorm:

$$C_{\max}(x,y) \le C(x,y) \le C_D(x,y)$$

for any *t*-conorm *C* and all $x, y \in [0, 1]$.

Recall that t-norm T (respectively, t-conorm C) is idempotent if for all $x \in [0, 1]$, T(x, x) = x (respectively, C(x, x) = x).

Lemma 2.15 (see [7]). Let C be a t-conorm. Then

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.16 (see [25]). Let (G, .) be a group and Q be a non-empty set. An intuitionistic fuzzy set $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ is said to be a Q-intuitionistic fuzzy subgroup of G with respect to norms (*t*-norm T and *t*-conorm C) if the following conditions are satisfied: (1)

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) \supseteq A(T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))),$$

(2)

$$A(x^{-1},q) = (\mu_A(x^{-1},q),\nu_A(x^{-1},q)) \supseteq A(x,q) = (\mu_A(x,q),\nu_A(x,q)),$$

which means:

(a)
$$\mu_A(xy,q) \ge T(\mu_A(x,q),\mu_A(y,q)),$$

(b) $\nu_A(xy,q) \le C(\nu_A(x,q),\nu_A(y,q)),$
(c) $\mu_A(x^{-1},q) \ge \mu_A(x,q),$
(d) $\nu_A(x^{-1},q) \le \nu_A(x,q),$
for all $x, y \in G$ and $q \in Q.$

Throughout this paper the set of all Q-intuitionistic fuzzy subgroups of G with respect to norms (t-norm T and t-conorm C) will be denoted by QIFSN(G).

Proposition 2.17 (see [25]). Let T and C be idempotent. Then

$$A = (\mu_A, \nu_A) \in QIFSN(G)$$

if and only if

$$A(xy^{-1}, q) \supseteq A(T(\mu_A(x, q), \mu_A(y, q)), C(\nu_A(x, q), \nu_A(y, q)))$$

for all $x, y \in G$ and $q \in Q$.

3 Main results

Proposition 3.1. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\alpha, \beta \in [0, 1]$. If T, C are idempotent, then

$$A_{\alpha,\beta} = \{ x \in G : A(x,q) \supseteq (\alpha,\beta) \}$$

is a subgroup of G.

Proof. Let $x, y \in A_{\alpha,\beta}$. Then $A(x,q) \supseteq (\alpha,\beta)$ and $A(y,q) \supseteq (\alpha,\beta)$, thus $\mu_A(x,q), \mu_A(y,q) \ge \alpha$ and $\nu_A(x,q), \nu_A(y,q) \le \beta$. As

$$\mu_A(xy^{-1}, q) \ge T(\mu_A(x, q), \mu_A(y^{-1}, q)) \ge T(\mu_A(x, q), \mu_A(y, q)) \ge T(\alpha, \alpha) = \alpha$$

and

$$\nu_A(xy^{-1}, q) \le C(\nu_A(x, q), \nu_A(y^{-1}, q)) \le C(\nu_A(x, q), \nu_A(y, q)) \le C(\beta, \beta) = \beta$$

so $\mu_A(xy^{-1},q) \ge \alpha$ and $\nu_A(xy^{-1},q) \le \beta$, which implies that $xy^{-1} \in A_{\alpha,\beta}$. Thus Proposition 2.3 gives us that $A_{\alpha,\beta}$ is a subgroup of G.

Proposition 3.2. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\alpha_i, \beta_i \in [0, 1]$ for i = 1, 2 such that $A(e_G, q) \supseteq (\alpha_i, \beta_i)$ for i = 1, 2 with $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$. Then $A_{\alpha_1,\beta_1} = A_{\alpha_2,\beta_2}$ iff there is no $x \in G$ such that $(\alpha_2, \beta_2) \subset A(x, q) \subset (\alpha_1, \beta_1)$.

Proof. Let $A_{\alpha_1,\beta_1} = A_{\alpha_2,\beta_2}$ and there exists an $x \in G$ such that $(\alpha_2,\beta_2) \subset A(x,q) \subset (\alpha_1,\beta_1)$. Then $A_{\alpha_1,\beta_1} \subseteq A_{\alpha_2,\beta_2}$ and so $x \in A_{\alpha_2,\beta_2}$ but $x \notin A_{\alpha_1,\beta_1}$ and this is a contradiction to $A_{\alpha_1,\beta_1} = A_{\alpha_2,\beta_2}$. Thus there is no $x \in G$ such that $(\alpha_2,\beta_2) \subset A(x,q) \subset (\alpha_1,\beta_1)$.

Conversely, if there is no $x \in G$ such that $(\alpha_2, \beta_2) \subset A(x,q) \subset (\alpha_1, \beta_1)$, then $A_{\alpha_1,\beta_1} = A_{\alpha_2,\beta_2}$.

Proposition 3.3. Let $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ and $A_{\alpha,\beta}$ be a subgroup of G for all $\alpha, \beta \in [0, 1]$ and $A(e_G, q) \supseteq (\alpha, \beta)$. Then $A = (\mu_A, \nu_A) \in QIFSN(G)$.

Proof. Let $x, y \in G$ and $q \in Q$ with $A(x,q) = (\alpha_1, \beta_1)$ and $A(y,q) = (\alpha_2, \beta_2)$ and then $x \in A_{\alpha_1,\beta_1}$ and $y \in A_{\alpha_2,\beta_2}$. Now we investigate the following conditions.

(1) If $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$, then $y \in A_{\alpha_1,\beta_1}$ and as A_{α_1,β_1} is a subgroup of G so $xy, x^{-1} \in A_{\alpha_1,\beta_1}$. Now

$$\mu_{A}(xy,q) \geq \alpha_{1} = T(\alpha_{1},\alpha_{2}) = T(\mu_{A}(x,q),\mu_{A}(y,q)),$$

$$\nu_{A}(xy,q) \leq \beta_{1} = C(\beta_{1},\beta_{2}) = C(\nu_{A}(x,q),\nu_{A}(y,q)),$$

$$\mu_{A}(x^{-1},q) \geq \alpha_{1} = \mu_{A}(x,q),$$

$$\nu_{A}(x^{-1},q) \leq \beta_{1} = \nu_{A}(x,q).$$

Thus $A = (\mu_A, \nu_A) \in QIFSN(G)$.

(2) If $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$, or if $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$, then the proof is similar to (1).

(3) If $\alpha_2 < \alpha_1$ and $\beta_2 > \beta_1$, then $x \in A_{\alpha_2,\beta_2}$ and as A_{α_2,β_2} is a subgroup of G so $xy, x^{-1} \in A_{\alpha_2,\beta_2}$. Now

$$\mu_A(xy,q) \ge \alpha_2 = T(\alpha_2,\alpha_1) = T(\alpha_1,\alpha_2) = T(\mu_A(x,q),\mu_A(y,q)),\\\nu_A(xy,q) \le \beta_2 = C(\beta_2,\beta_1) = C(\beta_1,\beta_2) = C(\nu_A(x,q),\nu_A(y,q)),$$

$$\mu_A(x^{-1}, q) \ge \alpha_2 = \mu_A(x, q),$$

$$\nu_A(x^{-1}, q) \le \beta_2 = \nu_A(x, q).$$

Then $A = (\mu_A, \nu_A) \in QIFSN(G)$.

(4) If $\alpha_1 = \alpha_2$ and $\beta_2 > \beta_1$, or if $\beta_1 = \beta_2$ and $\alpha_1 > \alpha_2$, then the proof is similar to (3).

(5) If $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$, then it is trivial.

Proposition 3.4. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\alpha_i, \beta_i \in [0, 1]$ for i = 1, 2. If A_{α_1, β_1} and A_{α_2, β_2} are two subgroups in G, then $A_{\alpha_1, \beta_1} \cap A_{\alpha_2, \beta_2}$ will be a subgroup in G.

Proof. Let $A(e_G, q) \supseteq (\alpha_1, \beta_1)$ and $A(e_G, q) \supseteq (\alpha_2, \beta_2)$ and $x \in G, q \in Q$. Then

- (1) If $(\alpha_1, \beta_1) \subset A(x, q) \subset (\alpha_2, \beta_2)$, then $A_{\alpha_2, \beta_2} \subseteq A_{\alpha_1, \beta_1}$ and so $A_{\alpha_1, \beta_1} \cap A_{\alpha_2, \beta_2} = A_{\alpha_2, \beta_2}$ and as A_{α_2, β_2} is a subgroup in G, so $A_{\alpha_1, \beta_1} \cap A_{\alpha_2, \beta_2}$ will be a subgroup in G.
- (2) If $(\alpha_2, \beta_2) \subset A(x, q) \subset (\alpha_1, \beta_1)$, then $A_{\alpha_1, \beta_1} \subseteq A_{\alpha_2, \beta_2}$ and so $A_{\alpha_2, \beta_2} \cap A_{\alpha_1, \beta_1} = A_{\alpha_1, \beta_1}$ and as A_{α_1, β_1} is a subgroup in G, so $A_{\alpha_2, \beta_2} \cap A_{\alpha_1, \beta_1}$ will be a subgroup in G.
- (3) If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, then $A_{\alpha_1, \beta_1} = A_{\alpha_2, \beta_2}$ and then $A_{\alpha_1, \beta_1} \cap A_{\alpha_2, \beta_2}$ will be a subgroup in G.

Corollary 3.5. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\{\alpha_i, \beta_i\}_{i \in I} \in [0, 1]$. If A_{α_i, β_i} are subgroups in G, then $\cap A_{\alpha_i, \beta_i}$ will be a subgroup in G.

Proposition 3.6. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\alpha_i, \beta_i \in [0, 1]$ for i = 1, 2. If A_{α_1, β_1} and A_{α_2, β_2} are two subgroups in G, then $A_{\alpha_1, \beta_1} \cup A_{\alpha_2, \beta_2}$ will be a subgroup in G.

Proof. Let $A(e_G, q) \supseteq (\alpha_1, \beta_1)$ and $A(e_G, q) \supseteq (\alpha_2, \beta_2)$ and $x \in G, q \in Q$. Then

- (1) If $(\alpha_1, \beta_1) \subset A(x, q) \subset (\alpha_2, \beta_2)$, then $A_{\alpha_2, \beta_2} \subseteq A_{\alpha_1, \beta_1}$ and so $A_{\alpha_1, \beta_1} \cup A_{\alpha_2, \beta_2} = A_{\alpha_1, \beta_1}$ and as A_{α_1, β_1} is a subgroup in G, so $A_{\alpha_1, \beta_1} \cup A_{\alpha_2, \beta_2}$ will be a subgroup in G.
- (2) If $(\alpha_2, \beta_2) \subset A(x, q) \subset (\alpha_1, \beta_1)$, then $A_{\alpha_1, \beta_1} \subseteq A_{\alpha_2, \beta_2}$ and so $A_{\alpha_2, \beta_2} \cup A_{\alpha_1, \beta_1} = A_{\alpha_2, \beta_2}$ and as A_{α_2, β_2} is a subgroup in G, so $A_{\alpha_2, \beta_2} \cup A_{\alpha_1, \beta_1}$ will be a subgroup in G.
- (3) If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, then $A_{\alpha_1, \beta_1} = A_{\alpha_2, \beta_2}$ and then $A_{\alpha_1, \beta_1} \cup A_{\alpha_2, \beta_2}$ will be a subgroup in G.

Corollary 3.7. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\{\alpha_i, \beta_i\}_{i \in I} \in [0, 1]$. If A_{α_i, β_i} are subgroups in G, then $\cup A_{\alpha_i, \beta_i}$ will be a subgroup in G.

Proposition 3.8. Let T, C be idempotent norms. Then any subgroup H of a group G can be realized as a level subgroup of $A = (\mu_A, \nu_A) \in QIFSN(G)$.

Proof. Let $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ defined by

$$A(x,q) = (\mu_A(x,q), \nu_A(x,q)) = \begin{cases} (\alpha,0) & \text{if } x \in H \text{ and } q \in Q \text{ and } 0 < \alpha < 1\\ (0,\beta) & \text{if } x \notin H \text{ and } q \in Q \text{ and } 0 < \beta < 1. \end{cases}$$

We show that $A = (\mu_A, \nu_A) \in QIFSN(G)$.

Let $x, y \in G$ and $q \in Q$ and now we consider the following conditions.

(1) If $x, y \in H$, then as H is a subgroup of G so $xy^{-1} \in H$. Thus $\mu_A(x,q) = \mu_A(y,q)$ = $\mu_A(xy^{-1},q) = \alpha$ and $\nu_A(x,q) = \nu_A(y,q) = \nu_A(xy^{-1},q) = 0$. Then

$$\mu_A(xy^{-1},q) = \alpha \ge \alpha = T(\alpha,\alpha) = T(\mu_A(x,q),\mu_A(x,q))$$

and

$$\nu_A(xy^{-1},q) = 0 \le 0 = C(0,0) = C(\nu_A(x,q),\nu_A(x,q)),$$

and from Proposition 2.17 we get that $A = (\mu_A, \nu_A) \in QIFSN(G)$. (2) If $x \in H$ and $y \notin H$, then $xy^{-1} \notin H$ and then $\mu_A(x,q) = \alpha$ and $\mu_A(y,q) = \mu_A(xy^{-1},q) = 0$ and $\nu_A(x,q) = 0$ and $\nu_A(y,q) = \nu_A(xy^{-1},q) = \beta$. Thus

$$\mu_A(xy^{-1}, q) = 0 \ge 0 = T(\alpha, 0) = T(\mu_A(x, q), \mu_A(y, q))$$

and

$$\nu_A(xy^{-1}, q) = \beta \le \beta = C(0, \beta) = C(\nu_A(x, q), \nu_A(y, q))$$

and from Proposition 2.17 we obtain that $A = (\mu_A, \nu_A) \in QIFSN(G)$.

(3) If $x, y \notin H$, then $\mu_A(x,q) = \mu_A(y,q) = 0$ and $\nu_A(x,q) = \nu_A(y,q) = \beta$, then xy^{-1} may or may not belong to H.

• If $xy^{-1} \in H$, then

$$\mu_A(xy^{-1}, q) = \alpha \ge 0 = T(0, 0) = T(\mu_A(x, q), \mu_A(y, q))$$

and

$$\nu_A(xy^{-1}, q) = 0 \le \beta = C(\beta, \beta) = C(\nu_A(x, q), \nu_A(y, q))$$

thus as Proposition 2.17 we get $A = (\mu_A, \nu_A) \in QIFSN(G)$.

• If $xy^{-1} \notin H$, then

$$\mu_A(xy^{-1},q) = 0 \ge 0 = T(0,0) = T(\mu_A(x,q),\mu_A(y,q))$$

and

$$\nu_A(xy^{-1},q) = \beta \le \beta = C(\beta,\beta) = C(\nu_A(x,q),\nu_A(y,q))$$

as Proposition 2.17 we will have that $A = (\mu_A, \nu_A) \in QIFSN(G)$.

Thus in all the cases $A = (\mu_A, \nu_A) \in QIFSN(G)$.

Definition 3.9. We say that $A = (\mu_A, \nu_A) \in QIFSN(G)$ is a normal if $\mu_A(xyx^{-1}, q) = \mu_A(y, q)$ and $\nu_A(xyx^{-1}, q) = \nu_A(y, q)$ for all $x, y \in G$ and $q \in Q$. We denote by NQIFSN(G) the set of all normal *Q*-intuitionistic fuzzy subgroups of *G* with respect to norms (*t*-norm *T* and *t*-conorm *C*).

Proposition 3.10. If $A = (\mu_A, \nu_A) \in NQIFSN(G)$, then $A_{\alpha,\beta}$ is a normal subgroup of G for all $\alpha, \beta \in [0, 1]$ and $A(e_G, q) \supseteq (\alpha, \beta)$.

Proof. Let $A = (\mu_A, \nu_A) \in NQIFSN(G)$ then from Proposition 3.1 we will have that $A_{\alpha,\beta}$ is a subgroup of G. Now let $x \in A_{\alpha,\beta}$ and $y \in G$ and $q \in Q$ then $\mu_A(x,q) \ge \alpha$ and $\nu_A(x,q) \le \beta$. Thus

$$\mu_A(yxy^{-1},q) = \mu_A(x,q) \ge \alpha$$

and

$$\nu_A(yxy^{-1},q) = \nu_A(x,q) \le \beta$$

and then $yxy^{-1} \in A_{\alpha,\beta}$ and thus $A_{\alpha,\beta}$ will be a normal subgroup of G.

Definition 3.11. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $\alpha, \beta \in [0, 1 - \sup\{A(x, q) : x \in G, \\ \emptyset_X(x) = (0, 1) \subset A(x, q) \subset U_X(x) = (1, 0)\}]$. Then

$$T^A_{(\alpha,\beta)} = (T^A_\alpha, T^A_\beta) = (\mu_A, \nu_A) + (\alpha, \beta) = (\mu_A + \alpha, \nu_A + \beta) : G \times Q \to [0,1]$$

is called a translation of A if

$$T^{A}_{(\alpha,\beta)}(x,q) = (\mu_{A}(x,q) + \alpha, \nu_{A}(x,q) + \beta)$$

for all $x \in G$.

Also we say that $T^A_{(\alpha,\beta)}$ is normal if $T^A_{(\alpha,\beta)}(xyx^{-1},q) = T^A_{(\alpha,\beta)}(y,q)$ for all $x, y \in G$.

Proposition 3.12. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $T^A_{(\alpha,\beta)}$ be a translation of A. Then: (1) $T^A_{(\alpha,\beta)}(x^{-1},q) = T^A_{(\alpha,\beta)}(x,q)$ for all $x \in G$ and $q \in Q$ and $\alpha, \beta \in [0,1]$.

(2) If T, C are idempotent norms, then $T^A_{(\alpha,\beta)}(e_G,q) \supseteq T^A_{(\alpha,\beta)}(x,q)$ for all $x \in G$ and $q \in Q$ and $\alpha, \beta \in [0,1]$.

Proof. Let $x \in G$ and $q \in Q$ and $\alpha \in [0, 1]$. Then (1)

$$T^{A}_{\alpha}(x,q) = \mu_{A}(x,q) + \alpha$$

$$= \mu_{A}((x^{-1})^{-1},q) + \alpha$$

$$\geq \mu_{A}(x^{-1},q) + \alpha$$

$$= T^{A}_{\alpha}(x^{-1},q)$$

$$= \mu_{A}(x^{-1},q) + \alpha$$

$$\geq \mu_{A}(x,q) + \alpha$$

$$= T^{A}_{\alpha}(x,q)$$

thus $T^A_\alpha(x^{-1},q)=T^A_\alpha(x,q).$ Also

$$T^{A}_{\beta}(x,q) = \nu_{A}(x,q) + \beta$$

= $\nu_{A}((x^{-1})^{-1},q) + \beta$
 $\leq \nu_{A}(x^{-1},q) + \beta$
= $T^{A}_{\beta}(x^{-1},q)$
= $\nu_{A}(x^{-1},q) + \beta$
 $\leq \nu_{A}(x,q) + \beta$
= $T^{A}_{\beta}(x,q),$

then $T^A_\beta(x^{-1},q) = T^A_\beta(x,q).$ Therefore

$$T^{A}_{(\alpha,\beta)}(x^{-1},q) = (T^{A}_{\alpha}(x^{-1},q), T^{A}_{\beta}(x^{-1},q)) = (T^{A}_{\alpha}(x,q), T^{A}_{\beta}(x,q)) = T^{A}_{(\alpha,\beta)}(x,q).$$

$$T^{A}_{\alpha}(e_{G},q) = \mu_{A}(e_{G},q) + \alpha$$

$$= \mu_{A}(xx^{-1},q) + \alpha$$

$$\geq T(\mu_{A}(x,q),\mu_{A}(x^{-1},q)) + \alpha$$

$$\geq T(\mu_{A}(x,q),\mu_{A}(x,q)) + \alpha$$

$$= \mu_{A}(x,q) + \alpha$$

$$= T^{A}_{\alpha}(x,q)$$

so $T^A_{\alpha}(e_G,q) \geq T^A_{\alpha}(x,q)$. Also

(2)

$$T^{A}_{\beta}(e_{G},q) = \nu_{A}(e_{G},q) + \beta$$

= $\nu_{A}(xx^{-1},q) + \beta$
 $\leq C(\nu_{A}(x,q),\nu_{A}(x^{-1},q)) + \beta$
 $\leq C(\nu_{A}(x,q),\nu_{A}(x,q)) + \beta$
= $\nu_{A}(x,q) + \beta$
= $T^{A}_{\beta}(x,q)$

so $T^A_\beta(e_G,q) \leq T^A_\beta(x,q)$. Thus

$$T^{A}_{(\alpha,\beta)}(e_{G},q) = (T^{A}_{\alpha}(e_{G},q), T^{A}_{\beta}(e_{G},q)) \supseteq (T^{A}_{\alpha}(x,q), T^{A}_{\beta}(x,q)) = T^{A}_{(\alpha,\beta)}(x,q).$$

Proposition 3.13. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $T^A_{(\alpha,\beta)}$ be a translation of A. If T, C are idempotent norms and $T^A_{(\alpha,\beta)}(xy^{-1},q) = T^A_{(\alpha,\beta)}(e_G,q)$, then $T^A_{(\alpha,\beta)}(x,q) = T^A_{(\alpha,\beta)}(y,q)$ for all $x, y \in G$ and $q \in Q$ and $\alpha, \beta \in [0,1]$.

Proof. Let $x, y \in G$ and $q \in Q$ and $\alpha, \beta \in [0, 1]$. Then

$$\begin{aligned} T^{A}_{\alpha}(e_{G},q) &= \mu_{A}(x,q) + \alpha \\ &= \mu_{A}(xy^{-1}y,q) + \alpha \\ &\geq T(\mu_{A}(xy^{-1},q),\mu_{A}(y,q)) + \alpha \\ &= T(\mu_{A}(xy^{-1},q),\pi_{\alpha}(y,q)) + \alpha) \\ &= T(T^{A}_{\alpha}(xy^{-1},q),T^{A}_{\alpha}(y,q)) \\ &= T(T^{A}_{\alpha}(e_{G},q),T^{A}_{\alpha}(y,q)) \\ &\geq T(T^{A}_{\alpha}(y,q),T^{A}_{\alpha}(y,q)) \\ &= T^{A}_{\alpha}(y,q) \\ &= \mu_{A}(y,q) + \alpha \\ &= \mu_{A}(yx^{-1}x,q) + \alpha \\ &\geq T(\mu_{A}(yx^{-1},q),\mu_{A}(x,q)) + \alpha \\ &= T(\mu_{A}(yx^{-1},q),\pi^{A}_{\alpha}(x,q)) \\ &= T(T^{A}_{\alpha}(xy^{-1},q),T^{A}_{\alpha}(x,q)) \\ &= T(T^{A}_{\alpha}(xy^{-1})^{-1},q),T^{A}_{\alpha}(x,q)) \end{aligned}$$

$$= T(T^{\mu}_{\alpha}(xy^{-1},q),T^{\mu}_{\alpha}(x,q))$$

$$= T(T^{\mu}_{\alpha}(e_{G},q),T^{\mu}_{\alpha}(x,q))$$

$$\geq T(T^{A}_{\alpha}(x,q),T^{A}_{\alpha}(x,q))$$

$$= T^{A}_{\alpha}(x,q)$$

so $T^A_{\alpha}(e_G,q) \ge T^A_{\alpha}(x,q)$. Also

$$T^{A}_{\beta}(e_{G},q) = \nu_{A}(e_{G},q) + \beta$$

$$= \nu_{A}(xx^{-1},q) + \beta$$

$$\leq C(\nu_{A}(x,q),\nu_{A}(x^{-1},q)) + \beta$$

$$\leq C(\nu_{A}(x,q),\nu_{A}(x,q)) + \beta$$

$$= \nu_{A}(x,q) + \beta$$

$$= T^{A}_{\beta}(x,q)$$

and thus $T^A_{\alpha}(x,q) = T^A_{\alpha}(y,q).$ Also

$$\begin{split} T^{A}_{\beta}(e_{G},q) &= \mu_{A}(x,q) + \beta \\ &= \mu_{A}(xy^{-1}y,q) + \beta \\ &\leq C(\nu_{A}(xy^{-1},q),\nu_{A}(y,q)) + \beta \\ &= C(\nu_{A}(xy^{-1},q),T^{A}_{\beta}(y,q)) \\ &= C(T^{A}_{\beta}(xy^{-1},q),T^{A}_{\beta}(y,q)) \\ &= C(T^{A}_{\beta}(e_{G},q),T^{A}_{\beta}(y,q)) \\ &\leq C(T^{A}_{\beta}(y,q),T^{A}_{\beta}(y,q)) \\ &= T^{A}_{\beta}(y,q) \\ &= \nu_{A}(y,q) + \beta \\ &= \nu_{A}(yx^{-1}x,q) + \beta \\ &\leq C(\nu_{A}(yx^{-1},q),\nu_{A}(x,q)) + \beta \\ &= C(T^{A}_{\beta}(yx^{-1},q),T^{A}_{\beta}(x,q)) \\ &= C(T^{A}_{\beta}(xy^{-1},q),T^{A}_{\beta}(x,q)) \\ &= C(T^{\mu}_{\beta}(xy^{-1},q),T^{\mu}_{\beta}(x,q)) \\ &= C(T^{\mu}_{\beta}(x,q),T^{\mu}_{\beta}(x,q)) \\ &= C(T^{\mu}_{\beta}(x,q),T^{A}_{\beta}(x,q)), \\ &= T^{A}_{\beta}(x,q) \end{split}$$

then $T^A_\beta(x,q) = T^A_\beta(y,q)$. Therefore

$$T^A_{(\alpha,\beta)}(x,q) = (T^A_\alpha(x,q), T^A_\beta(x,q)) = (T^A_\alpha(y,q), T^A_\beta(y,q)) = T^A_{(\alpha,\beta)}(y,q).$$

Proposition 3.14. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $T^A_{(\alpha,\beta)}$ be a translation of A. Then $T^A_{(\alpha,\beta)} \in QIFSN(G)$ for all $\alpha, \beta \in [0,1]$.

Proof. Let $x, y \in G$ and $q \in Q$ and $\alpha, \beta \in [0, 1]$. Then (1) $T^A(x; x) = u_{\alpha}(x; y) + \alpha$

$$T^{A}_{\alpha}(xy,q) = \mu_{A}(xy,q) + \alpha$$

$$\geq T(\mu_{A}(x,q),\mu_{A}(y,q)) + \alpha$$

$$= T(\mu_{A}(x,q) + \alpha,\mu_{A}(y,q) + \alpha)$$

$$= T(T^{A}_{\alpha}(x,q),T^{A}_{\alpha}(y,q)),$$

thus $T^A_{\alpha}(xy,q) \ge T(T^A_{\alpha}(x,q),T^A_{\alpha}(y,q)).$ (2)

$$T^{A}_{\alpha}(x^{-1},q) = \mu_{A}(x^{-1},q) + \alpha \ge \mu_{A}(x,q) + \alpha = T^{A}_{\alpha}(x,q).$$

(3) $T^{A}_{\beta}(xy,q) = \nu_{A}(xy,q) + \beta$ $\leq C(\nu_{A}(x,q),\nu_{A}(y,q)) + \beta$ $= C(\nu_{A}(x,q) + \beta,\nu_{A}(y,q) + \beta)$ $= C(T^{A}_{\beta}(x,q),T^{A}_{\beta}(y,q)),$

then $T^A_\beta(xy,q) \leq C(T^A_\beta(x,q),T^A_\beta(y,q)).$

(4)

$$T^{A}_{\beta}(x^{-1},q) = \nu_{A}(x^{-1},q) + \beta \le \nu_{A}(x,q) + \beta = T^{A}_{\beta}(x,q).$$

Then from (1)–(4) we get that $T^A_{(\alpha,\beta)} = (T^A_{\alpha}, T^A_{\beta}) \in QIFSN(G)$ for all $\alpha, \beta \in [0,1]$.

Proposition 3.15. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $T^A_{(\alpha,\beta)}$ be a translation of A. If T, C are idempotent norms, then $H = \{x \in G : T^A_{(\alpha,\beta)}(x,q) = T^A_{(\alpha,\beta)}(e_G,q)\}$ is a subgroup of G for all $\alpha, \beta \in [0,1]$.

Proof. Let $x, y \in H$ and $q \in Q$ and $\alpha, \beta \in [0, 1]$. Then $T^A_{(\alpha, \beta)}(x, q) = T^A_{(\alpha, \beta)}(y, q) = T^A_{(\alpha, \beta)}(e_G, q)$. Now

$$\begin{split} T^{A}_{\alpha}(xy^{-1},q) &\geq T(T^{A}_{\alpha}(x,q),T^{A}_{\alpha}(y^{-1},q)) \\ &\geq T(T^{A}_{\alpha}(x,q),T^{A}_{\alpha}(y,q)) \\ &= T(T^{A}_{\alpha}(e_{G},q),T^{A}_{\alpha}(e_{G},q)) \\ &= T^{A}_{\alpha}(e_{G},q) \\ &= T^{A}_{\alpha}((xy^{-1})(xy^{-1})^{-1},q) \\ &\geq T(T^{A}_{\alpha}(xy^{-1},q),T^{A}_{\alpha}((xy^{-1})^{-1},q)) \\ &\geq T(T^{A}_{\alpha}(xy^{-1},q),T^{A}_{\alpha}(xy^{-1},q)) \\ &= T^{A}_{\alpha}(xy^{-1},q), \end{split}$$

therefore $T^A_{\alpha}(xy^{-1},q) = T^A_{\alpha}(e_G,q).$

Also

$$\begin{split} T^{A}_{\beta}(xy^{-1},q) &\leq C(T^{A}_{\beta}(x,q),T^{A}_{\beta}(y^{-1},q)) \\ &\leq C(T^{A}_{\alpha}(x,q),T^{A}_{\alpha}(y,q)) \\ &= C(T^{A}_{\beta}(e_{G},q),T^{A}_{\beta}(e_{G},q)) \\ &= T^{A}_{\beta}(e_{G},q) \\ &= T^{A}_{\beta}((xy^{-1})(xy^{-1})^{-1},q) \\ &\leq C(T^{A}_{\beta}(xy^{-1},q),T^{A}_{\beta}((xy^{-1})^{-1},q)) \\ &\leq C(T^{A}_{\beta}(xy^{-1},q),T^{A}_{\beta}(xy^{-1},q)) \\ &= T^{A}_{\beta}(xy^{-1},q), \end{split}$$

thus $T^{A}_{\beta}(xy^{-1}, q) = T^{A}_{\beta}(e_{G}, q).$

Now as $T^A_{(\alpha,\beta)}(xy^{-1},q) = (T^A_{\alpha}(xy^{-1},q), T^A_{\beta}(xy^{-1},q)) = T^A_{(\alpha,\beta)}(e_G,q)$ so $xy^{-1} \in H$ and Proposition 2.3 gives us that $H = \{x \in G : T^A_{(\alpha,\beta)}(x,q) = T^A_{(\alpha,\beta)}(e_G,q)\}$ will be a subgroup of G for all $\alpha, \beta \in [0,1]$.

Proposition 3.16. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $T^A_{(\alpha,\beta)}$ be a translation of A. If $T^A_{(\alpha,\beta)}(xy^{-1},q) = (1,0)$, then $T^A_{(\alpha,\beta)}(x,q) = T^A_{(\alpha,\beta)}(y,q)$ for all $x, y \in G$ and $q \in Q$ and $\alpha, \beta \in [0,1]$.

Proof. Let $x, y \in G$ and $q \in Q$ and $\alpha, \beta \in [0, 1]$. Then

$$\begin{split} T^A_{\alpha}(x,q) &= T^A_{\alpha}(xy^{-1}y,q) \\ &\geq T(T^A_{\alpha}(xy^{-1},q),T^A_{\alpha}(y,q)) \\ &= T(1,T^A_{\alpha}(y,q)) \\ &= T^A_{\alpha}(y,q) \\ &= T^A_{\alpha}(y^{-1},q) \\ &= T^A_{\alpha}(x^{-1}xy^{-1},q) \\ &\geq T(T^A_{\alpha}(x^{-1},q),T^A_{\alpha}(xy^{-1},q)) \\ &\geq T(T^A_{\alpha}(x,q),T^A_{\alpha}(xy^{-1},q)) \\ &= T(T^A_{\alpha}(x,q),1) \\ &= T^A_{\alpha}(x,q). \end{split}$$

Thus $T^A_{\alpha}(x,q) = T^A_{\alpha}(y,q).$

Also

$$\begin{split} T^{A}_{\beta}(x,q) &= T^{A}_{\beta}(xy^{-1}y,q) \\ &\leq C(T^{A}_{\beta}(xy^{-1},q),T^{A}_{\beta}(y,q)) \\ &= C(0,T^{A}_{\beta}(y,q)) \\ &= T^{A}_{\beta}(y,q) \\ &= T^{A}_{\beta}(y^{-1},q) \\ &= T^{A}_{\beta}(x^{-1}xy^{-1},q) \end{split}$$

$$\leq C(T_{\beta}^{A}(x^{-1},q),T_{\beta}^{A}(xy^{-1},q)) \\\leq C(T_{\beta}^{A}(x,q),T_{\beta}^{A}(xy^{-1},q)) \\= C(T_{\alpha}^{A}(x,q),0) \\= T_{\beta}^{A}(x,q).$$

Then $T^A_\beta(x,q) = T^A_\beta(y,q)$. Therefore

$$T^{A}_{(\alpha,\beta)}(x,q) = (T^{A}_{\alpha}(x,q), T^{A}_{\beta}(x,q)) = (T^{A}_{\alpha}(y,q), T^{A}_{\beta}(y,q)) = T^{A}_{(\alpha,\beta)}(y,q).$$

Definition 3.17. Let $f : G \to H$ be a group homomorphism such that $A = (\mu_A, \nu_A) \in QIFSN(G) : G \times Q \to [0, 1]$ and $B = (\mu_B, \nu_B) \in QIFSN(H) : H \times Q \to [0, 1]$. Let $T^A_{(\alpha,\beta)} = (T^A_\alpha, T^A_\beta)$ be a translation of A and $T^B_{(\alpha,\beta)} = (T^B_\alpha, T^B_\beta)$ be a translation of B. Define a fuzzy image by

$$\begin{split} &f(T^A_{(\alpha,\beta)})(y,q) \\ &= (f(T^A_{\alpha}(y,q),f(T^A_{\beta}))(y,q)) \\ &= \begin{cases} (\sup\{T^A_{\alpha}(x,q) \mid x \in G, f(x) = y\}, \inf\{T^A_{\beta}(x,q) \mid x \in G, f(x) = y\}) & \text{if } f^{-1}(y) \neq \varnothing, \\ (0,1) & \text{if } f^{-1}(y) = \varnothing. \end{cases} \end{split}$$

Also $f^{-1}(T^B_{(\alpha,\beta)})(x,q) = (f^{-1}(T^B_{\alpha})(x,q), f^{-1}(T^B_{\beta})(x,q)) = (T^B_{\alpha}(f(x),q), T^B_{\beta}(f(x),q)).$

Proposition 3.18. Let f be an epimorphism from group G into group H. If $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $T^A_{(\alpha,\beta)}$ is a translation of A, then $f(T^A_{(\alpha,\beta)}) \in QIFSN(H)$.

$$\begin{array}{l} \textit{Proof. Let } h_1, h_2 \in H \text{ and } q \in Q. \text{ Then} \\ (1) \\ f(T^A_{\alpha})(h_1h_2,q) = \sup\{T^A_{\alpha}(g_1g_2,q) \mid g_1,g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ & \geq \sup\{T(T^A_{\alpha}(g_1,q),T^A_{\alpha}(g_2,q)) \mid g_1,g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ & = T(\sup\{T^A_{\alpha}(g_1,q) \mid g_1 \in G, f(g_1) = h_1\}, \sup\{T^A_{\alpha}(g_2,q) \mid g_2 \in G, f(g_2) = h_2\}) \\ & = T(f(T^A_{\alpha})(h_1,q), f(T^A_{\alpha})(h_2,q)), \\ \text{thus } f(T^A_{\alpha})(h_1h_2,q) \geq T(f(T^A_{\alpha})(h_1,q), f(T^A_{\alpha})(h_2,q)). \end{array}$$

(2)

$$f(T_{\alpha}^{A})(h_{1}^{-1},q) = \sup\{T_{\alpha}^{A}(g_{1}^{-1},q) \mid g_{1} \in G, f(g_{1}^{-1}) = h_{1}^{-1}\}$$

$$\geq \sup\{T_{\alpha}^{A}(g_{1},q) \mid g_{1} \in G, f(g_{1},q) = h_{1}\} = f(T_{\alpha}^{A})(h_{1},q)$$

so $f(T^A_{\alpha})(h_1^{-1},q) \ge f(T^A_{\alpha})(h_1,q).$ (3)

$$\begin{split} f(T^A_\beta)(h_1h_2,q) &= \inf\{T^A_\beta(g_1g_2,q) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &\leq \inf\{C(T^A_\beta(g_1,q), T^A_\beta(g_2,q)) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &= C(\inf\{T^A_\beta(g_1,q) \mid g_1 \in G, f(g_1) = h_1\}, \inf\{T^A_\beta(g_2,q) \mid g_2 \in G, f(g_2) = h_2\}) \\ &= C(f(T^A_\beta)(h_1,q), f(T^A_\beta)(h_2,q)), \end{split}$$

then $f(T_{\beta}^{A})(h_{1}h_{2},q) \leq C(f(T_{\beta}^{A})(h_{1},q), f(T_{\beta}^{A})(h_{2},q)).$

(4)

$$f(T_{\beta}^{A})(h_{1}^{-1},q) = \inf\{T_{\beta}^{A}(g_{1}^{-1},q) \mid g_{1} \in G, f(g_{1}^{-1}) = h_{1}^{-1}\}$$

$$\leq \inf\{T_{\beta}^{A}(g_{1},q) \mid g_{1} \in G, f(g_{1},q) = h_{1}\} = f(T_{\beta}^{A})(h_{1},q),$$

then $f(T_{\beta}^{A})(h_{1}^{-1},q) \leq f(T_{\beta}^{A})(h_{1},q).$

Therefore (1)–(4) give us that

$$f(T^{A}_{(\alpha,\beta)}) = (f(T^{A}_{\alpha}, f(T^{A}_{\beta})) \in QIFSN(H).$$

Proposition 3.19. Let f be a homomorphism from group G into group H. If $B = (\mu_B, \nu_B) \in$ QIFSN(H) and $T^B_{(\alpha,\beta)} = (T^B_{\alpha}, T^B_{\beta})$ is a translation of B, then $f^{-1}(T^B_{(\alpha,\beta)}) \in QIFSN(G)$.

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$f^{-1}(T^B_{\alpha})(g_1g_2,q) = T^B_{\alpha}(f(g_1g_2),q)$$

= $T^B_{\alpha}(f(g_1)f(g_2),q)$
 $\geq T(T^B_{\alpha}(f(g_1),q),T^B_{\alpha}(f(g_2),q))$
= $T(f^{-1}(T^B_{\alpha})(g_1,q),f^{-1}(T^B_{\alpha})(g_2,q)),$

then $f^{-1}(T^B_{\alpha})(g_1g_2,q) \ge T(f^{-1}(T^B_{\alpha})(g_1,q), f^{-1}(T^B_{\alpha})(g_2,q)).$

$$\begin{aligned} f^{-1}(T^B_\beta)(g_1g_2,q) &= T^B_\beta(f(g_1g_2),q) \\ &= T^B_\beta(f(g_1)f(g_2),q) \\ &\leq C(T^B_\beta(f(g_1),q),T^B_\beta(f(g_2),q)) \\ &= C(f^{-1}(T^B_\beta)(g_1,q),f^{-1}(T^B_\beta)(g_2,q)), \end{aligned}$$

then $f^{-1}(T^B_\beta)(g_1g_2,q) \leq C(f^{-1}(T^B_\beta)(g_1,q), f^{-1}(T^B_\beta)(g_2,q)).$

Let $g \in G$ and $q \in Q$. Then

$$f^{-1}(T^B_{\alpha})(g^{-1},q) = T^B_{\alpha}(f(g^{-1}),q) = T^B_{\alpha}(f(g)^{-1},q) \ge T^B_{\alpha}(f(g),q) = f^{-1}(T^B_{\alpha})(g,q)$$

and

$$f^{-1}(T^B_{\beta})(g^{-1},q) = T^B_{\beta}(f(g^{-1}),q) = T^B_{\beta}(f(g)^{-1},q) \le T^B_{\beta}(f(g),q) = f^{-1}(T^B_{\beta})(g,q).$$

Then $f^{-1}(T^B_{(\alpha,\beta)}) = (f^{-1}(T^B_{\alpha}), f^{-1}(T^B_{\beta})) \in QIFSN(G).$

Proposition 3.20. Let f be an anti homomorphism from group G into group H. If $B = (\mu_B, \nu_B) \in$ QIFSN(H) and $T^B_{(\alpha,\beta)} = (T^B_{\alpha}, T^B_{\beta})$ is a translation of B, then $f^{-1}(T^B_{(\alpha,\beta)}) \in QIFSN(G)$.

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$f^{-1}(T^B_{\alpha})(g_1g_2, q) = T^B_{\alpha}(f(g_1g_2), q)$$

= $T^B_{\alpha}(f(g_2)f(g_1), q)$
 $\geq T(T^B_{\alpha}(f(g_2), q), T^B_{\alpha}(f(g_1), q))$
= $T(T^B_{\alpha}(f(g_1), q), T^B_{\alpha}(f(g_2), q))$
= $T(f^{-1}(T^B_{\alpha})(g_1, q), f^{-1}(T^B_{\alpha})(g_2, q)),$

then $f^{-1}(T^B_{\alpha})(g_1g_2,q) \ge T(f^{-1}(T^B_{\alpha})(g_1,q), f^{-1}(T^B_{\alpha})(g_2,q)).$

Also

$$\begin{aligned} f^{-1}(T^B_\beta)(g_1g_2,q) &= T^B_\beta(f(g_1g_2),q) \\ &= T^B_\beta(f(g_2)f(g_1),q) \\ &\leq C(T^B_\beta(f(g_2),q),T^B_\beta(f(g_1),q)) \\ &= C(T^B_\beta(f(g_1),q),T^B_\beta(f(g_2),q)) \\ &= C(f^{-1}(T^B_\beta)(g_1,q),f^{-1}(T^B_\beta)(g_2,q)), \end{aligned}$$

then $f^{-1}(T^B_\beta)(g_1g_2,q) \leq C(f^{-1}(T^B_\beta)(g_1,q), f^{-1}(T^B_\beta)(g_2,q)).$ Let $g \in G$ and $q \in Q$. Then

$$f^{-1}(T^B_{\alpha})(g^{-1},q) = T^B_{\alpha}(f(g^{-1}),q) = T^B_{\alpha}(f(g)^{-1},q) \ge T^B_{\alpha}(f(g),q) = f^{-1}(T^B_{\alpha})(g,q)$$

and

$$f^{-1}(T^B_{\beta})(g^{-1},q) = T^B_{\beta}(f(g^{-1}),q) = T^B_{\beta}(f(g)^{-1},q) \le T^B_{\beta}(f(g),q) = f^{-1}(T^B_{\beta})(g,q).$$

Then $f^{-1}(T^B_{(\alpha,\beta)}) = (f^{-1}(T^B_{\alpha}), f^{-1}(T^B_{\beta})) \in QIFSN(G).$

Proposition 3.21. Let f be an epimorphism from group G into group H. If $A = (\mu_A, \nu_A) \in NQIFSN(G)$ and $T^A_{(\alpha,\beta)}$ is a translation of A, then $f(T^A_{(\alpha,\beta)}) \in NQIFSN(H)$.

Proof. As Proposition 3.18 we have $f(T^A_{(\alpha,\beta)}) \in QIFSN(H)$. Let $x, y \in H$ and $q \in Q$. Since f is a surjection, f(u) = x for some $u \in G$, then

$$f(T^{A}_{\alpha})(xyx^{-1},q) = \sup\{T^{A}_{\alpha}(w,q) \mid w \in G, f(w) = xyx^{-1}\}$$

= $\sup\{T^{A}_{\alpha}(u^{-1}wu,q) \mid w \in G, f(u^{-1}wu) = y\}$
= $\sup\{T^{A}_{\alpha}(w,q) \mid w \in G, f(w) = y\}$
= $f(T^{A}_{\alpha})(y,q).$

$$\begin{split} \text{Then } f(T^A_{\alpha})(xyx^{-1},q) &= f(T^A_{\alpha})(y,q). \\ \text{Also} \\ f(T^A_{\beta})(xyx^{-1},q) &= \inf\{T^A_{\beta}(w,q) \mid w \in G, f(w) = xyx^{-1}\} \\ &= \inf\{T^A_{\beta}(u^{-1}wu,q) \mid w \in G, f(u^{-1}wu) = y\} \\ &= \inf\{T^A_{\beta}(w,q) \mid w \in G, f(w) = y\} \\ &= f(T^A_{\beta})(y,q), \end{split}$$

thus $f(T^A_\beta)(xyx^{-1},q)=f(T^A_\beta)(y,q).$ Therefore

$$f(T^{A}_{(\alpha,\beta)})(xyx^{-1},q) = (f(T^{A}_{\alpha})(xyx^{-1},q), f(T^{A}_{\beta})(xyx^{-1},q))$$

= $(f(T^{A}_{\alpha})(y,q), f(T^{A}_{\beta})(y,q))$
= $f(T^{A}_{(\alpha,\beta)})(y,q)$

and then $f(T^A_{(\alpha,\beta)}) \in NQIFSN(H)$.

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Proposition 3.22. Let f be a homomorphism from group G into group H. If $B = (\mu_B, \nu_B) \in NQIFSN(H)$ and $T^B_{(\alpha,\beta)} = (T^B_\alpha, T^B_\beta)$ is a translation of B, then $f^{-1}(T^B_{(\alpha,\beta)}) \in NQIFSN(G)$.

Proof. Using Proposition 3.19 implies that $f^{-1}(T^B_{(\alpha,\beta)}) \in QIFSN(G)$. Now for any $x, y \in G$ and $q \in Q$ we obtain

$$\begin{aligned} f^{-1}(T^B_{\alpha})(xyx^{-1},q) &= T^B_{\alpha}(f(xyx^{-1}),q) \\ &= T^B_{\alpha}(f(x)f(y)f(x^{-1}),q) \\ &= T^B_{\alpha}(f(x)f(y)f^{-1}(x),q) \\ &= T^B_{\alpha}(f(y),q) \\ &= f^{-1}(T^B_{\alpha})(y,q) \end{aligned}$$

thus $f^{-1}(T^B_{\alpha})(xyx^{-1},q) = f^{-1}(T^B_{\alpha})(y,q).$ Also

$$\begin{split} f^{-1}(T^B_\beta)(xyx^{-1},q) &= T^B_\beta(f(xyx^{-1}),q) \\ &= T^B_\beta(f(x)f(y)f(x^{-1}),q) \\ &= T^B_\beta(f(x)f(y)f^{-1}(x),q) \\ &= T^B_\beta(f(y),q) \\ &= f^{-1}(T^B_\beta)(y,q) \end{split}$$

so $f^{-1}(T^B_\beta)(xyx^{-1},q) = f^{-1}(T^B_\beta)(y,q)$. Now

$$\begin{split} f^{-1}(T^B_{(\alpha,\beta)})(xyx^{-1},q) &= (f^{-1}(T^B_{\alpha})(xyx^{-1},q), f^{-1}(T^B_{\beta})(xyx^{-1},q)) \\ &= (f^{-1}(T^B_{\alpha})(y,q), f^{-1}(T^B_{\beta})(y,q)) \\ &= f^{-1}(T^B_{(\alpha,\beta)})(y,q) \end{split}$$

thus $f^{-1}(T^B_{(\alpha,\beta)}) \in NQIFSN(G)$.

4 Conclusion and an open problem

In this study, the idea of normality and translation of Q-intuitionistic fuzzy subgroups with respect to norms (t-norm T and t-conorm C) are introduced and given some interesting results of them. Now one can investigate Q-intuitionistic fuzzy subrings with respect to norms (t-norm T and t-conorm C) and obtain some results about them as we did for Q-intuitionistic fuzzy subgroups and this can be an open problem.

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