On intuitionistic fuzzy ideals in $\Gamma$-near-rings

N. Palaniappan$^1$, P. S. Veerappan$^2$ and D. Ezhilmaran$^3$

$^1$ Alagappa University, Karaikudi – 630 003, Tamilnadu, India, e-mail: palaniappan.nallappan@yahoo.com
$^2$ Department of Mathematics, K. S. R. College of Technology Tiruchengode-637215, Tamilnadu, India e-mail: peeyesvee@yahoo.co.in
$^3$ Department of Mathematics, K. S. R. College of Technology Tiruchengode-637215, Tamilnadu, India e-mail: ezhil.devarasan@yahoo.com

Abstract: In this paper, we study some properties of intuitionistic fuzzy ideals of a $\Gamma$-near-ring and prove some results on these.

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1 Introduction

The notion of a fuzzy set was introduced by L. A. Zadeh [10], and since then this concept have been applied to various algebraic structures. The idea of intuitionistic fuzzy set was first published by K. T. Atanassov [1] as a generalization of the notion of fuzzy set. $\Gamma$-near-rings were defined by Bh. Satyanarayana [9] and G. L. Booth [2, 3] studied the ideal theory in $\Gamma$-near-rings. W. Liu [7] introduced fuzzy ideals and it has been studied by several authors. The notion of fuzzy ideals and its properties were applied to semi groups, BCK-algebras and semi rings. Y.B. Jun [5, 6] introduced the notion of fuzzy left (respectively, right) ideals.

In this paper, we introduce the notion of intuitionistic fuzzy ideals in $\Gamma$-near-rings and study some of its properties.

2 Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Definition 2.1. A non-empty set $R$ with two binary operations “$+$” (addition) and “$.$” (multiplication) is called a near-ring if it satisfies the following axioms:

(i) $(R, +)$ is a group,
(ii) $(R, \cdot)$ is a semigroup,
(iii) $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$. It is a right near-ring because it satisfies the right distributive law.

Definition 2.2 A $\Gamma$-near-ring is a triple $(M, +, \Gamma)$ where

(i) $(M, +)$ is a group,
A subset \(A\) of a \(\Gamma\)-near-ring \(M\) is called a left (respectively, right) ideal of \(M\) if
\begin{enumerate}[(i)]  \item \((A, +)\) is a normal divisor of \((M, +),\)
  \item \(ua(x + v) - uav \in A\) (respectively, \(xau \in A\)) for all \(x \in A, \alpha \in \Gamma\) and \(u, v \in M\).
\end{enumerate}

Definition 2.4 A fuzzy set \(\mu\) in a \(\Gamma\)-near-ring \(M\) is called a fuzzy left (respectively, right) ideal of \(M\) if
\begin{enumerate}[(i)]  \item \(\mu(x - y) \geq \min\{\mu(x), \mu(y)\},\)
  \item \(\mu(y + x - y) \geq \mu(x),\) for all \(x, y \in M\).
  \item \(\mu(u(x + v) - uav) \geq \mu(x)\) (respectively, \(\mu(xau) \geq \mu(x)\)) for all \(x, u, v \in M\) and \(\alpha \in \Gamma\).
\end{enumerate}

Definition 2.5 [1] Let \(X\) be a nonempty fixed set. An intuitionistic fuzzy set (IFS) \(A\) in \(X\) is an object having the form \(A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}\), where the functions \(\mu_A : X \rightarrow [0, 1]\) and \(\nu_A : X \rightarrow [0, 1]\) denote the degree of membership and degree of non membership of each element \(x \in X\) to the set \(A\), respectively, and \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\).

Notation. For the sake of simplicity, we shall use the symbol \(A = \langle \mu_A, \nu_A \rangle\) for the IFS \(A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}\).

Definition 2.6 [1]. Let \(X\) be a non-empty set and let \(A = \langle \mu_A, \nu_A \rangle\) and \(B = \langle \mu_B, \nu_B \rangle\) be IFSs in \(X\). Then,
\begin{enumerate}[(1)]  \item \(A \subseteq B\) iff \(\mu_A \leq \mu_B\) and \(\nu_A \geq \nu_B\).
  \item \(A = B\) iff \(A \subseteq B\) and \(B \subseteq A\).
  \item \(A^c = \langle \nu_A, \mu_A \rangle\).
  \item \(A \cap B = \langle \mu_A \land \mu_B, \nu_A \lor \nu_B \rangle\).
  \item \(A \cup B = \langle \mu_A \lor \mu_B, \nu_A \land \nu_B \rangle\).
  \item \(\Delta A = \langle \mu_A, 1 - \mu_A \rangle\) \(\bar{A} = \langle 1 - \nu_A, \nu_A \rangle\).
\end{enumerate}

Definition 2.7. Let \(\mu\) and \(\nu\) be two fuzzy sets in a \(\Gamma\)-near-ring. For \(s, t \in [0, 1]\) the set \(U(\mu, s) = \{x \in \mu(x) \geq s\}\) is called upper level of \(\mu\). The set \(L(\nu, t) = \{x \in \nu(x) \leq t\}\) is called lower level of \(\nu\).

Definition 2.8. Let \(A\) be an IFS in a \(\Gamma\)-near-ring \(M\). For each pair \((t, s) \in [0, 1]\) with \(t + s \leq 1\), the set \(A_{(t, s)} = \{x \in X | \mu_A(x) \geq t\) and \(\nu_A(x) \leq s\}\) is called a \((t, s)\)-level subset of \(A\).

Definition 2.9. Let \(A = \langle \mu_A, \nu_A \rangle\) be an intuitionistic fuzzy set in \(M\) and let \(t \in [0, 1]\). Then, the sets \(U(\mu_A ; t) = \{x \in M | \mu_A(x) \geq t\}\) and \(L(\nu_A ; t) = \{x \in M | \nu_A(x) \leq t\}\) are called upper level set and lower level set of \(A\), respectively.

3 Intuitionistic fuzzy ideals

In what follows, let \(M\) denote a \(\Gamma\)-near-ring, unless otherwise specified.

Definition 3.1. An IFS \(A = \langle \mu_A, \nu_A \rangle\) in \(M\) is called an intuitionistic fuzzy left (respectively, right) ideal of a \(\Gamma\)-near-ring \(M\) if
\begin{enumerate}[(i)]  \item \(\mu_A(x - y) \geq \{\mu_A(x) \land \mu_A(y)\},\)
  \item \(\mu_A(x + y - x) \geq \mu_A(x)\)
  \item \(\mu_A(u(x + v) - uav) \geq \mu_A(x)\) (respectively, \(\mu_A(xau) \geq \mu_A(x)\).
\end{enumerate}
(iv) $v_A(x - y) \leq \{v_A(x) \lor v_A(y)\}$,  
(v) $v_A(y + x - y) \leq v_A(x)$,  
(vi) $v_A(u\alpha(x + v) - u\alpha v) \leq v_A(x)$ (respectively, $v_A(x\alpha u) \leq v_A(x)$),

for all $x, y, u, v \in M$ and $\alpha \in \Gamma$.

Example 3.2. Let $R$ be the set of all integers then $R$ is a ring. Take $M = \Gamma = R$. Let $a, b \in M$, $\alpha \in \Gamma$, suppose $a\alpha b$ is the product of $a$, $b \in R$. Then, $M$ is a $\Gamma$-near-ring.

Define an IFS $A = \langle \mu_A, v_A \rangle$ in $R$ as follows.
\[
\begin{align*}
\mu_A(0) & = 1 \\
v_A(0) & = 0, \quad \mu_A(1) = \mu_A(2) = \mu_A(3) = \ldots = t \\
v_A(0) & = 0 \\
v_A(1) & = v_A(2) = v_A(3) = \ldots = s, \quad \text{where } t \in [0, 1], \ s \in [0, 1] \text{ and } t + s \leq 1.
\end{align*}
\]

By routine calculations, clearly $A$ is an intuitionistic fuzzy ideal of a $\Gamma$-near-ring $R$.

Theorem 3.3. If $A$ is an ideal of a $\Gamma$-near-ring $M$, then the IFS $\hat{A} = \langle \chi_A, \tilde{\chi}_A \rangle$ is an intuitionistic fuzzy ideal of $M$.

Proof. Let $x, y \in M$.
If $x, y, u, v \in A$ and $\alpha \in \Gamma$, then $x - y \in A, (y + x - y) \in A$ and $(u\alpha(x + v) - u\alpha v) \in A$, since $A$ is an ideal of $M$.
Hence, $\chi_A(x - y) = 1 \geq \{\chi_A(x) \land \chi_A(y)\}$,  
$\chi_A(x + y - y) = 1 \geq \chi_A(x)$ and  
$\chi_A(u\alpha(x + v) - u\alpha v) = 1 \geq \chi_A(x)$ (respectively, $\chi_A(x\alpha u) \geq \chi_A(x)$) .

Also, we have
\[
\begin{align*}
0 & = 1 - \chi_A(x - y) = \tilde{\chi}_A(x - y) \leq \{\chi_A(x) \lor \chi_A(y)\},  \\
0 & = 1 - \chi_A(y + x - y) = \tilde{\chi}_A(y + x - y) \leq \chi_A(x), \text{ and}  \\
0 & = 1 - \chi_A(u\alpha(x + v) - u\alpha v) = \tilde{\chi}_A(u\alpha(x + v) - u\alpha v) \leq \tilde{\chi}_A(x) \text{ (respectively, } \chi_A(x\alpha u) \leq \tilde{\chi}_A(x)).
\end{align*}
\]

If $x \notin A$ or $y \notin A$, then $\chi_A(x) = 0$ or $\chi_A(y) = 0$. Thus, we have
$\chi_A(x - y) \geq \{\chi_A(x) \land \chi_A(y)\}$,  
$\chi_A(y + x - y) \geq \chi_A(x)$ and  
$\chi_A(u\alpha(x + v) - u\alpha v) \geq \chi_A(x)$ (respectively, $\chi_A(x\alpha u) \geq \chi_A(x)$) for all $\alpha \in \Gamma$.

Also
\[
\begin{align*}
\tilde{\chi}_A(x - y) & \leq \{\chi_A(x) \lor \tilde{\chi}_A(y)\},  \\
= & \{(1 - \chi_A(x)) \lor (1 - \chi_A(y))\} = 1  \\
\tilde{\chi}_A(y + x - y) & \leq \tilde{\chi}_A(x)  \\
= & (1 - \chi_A(x)) = 1  \\
\text{and}  \\
\tilde{\chi}_A(u\alpha(x + v) - u\alpha v) & = 1 - \chi_A(u\alpha(x + v) - u\alpha v) \leq 1 - \chi_A(x) = \tilde{\chi}_A(x).
\end{align*}
\]

This completes the proof.

Definition 3.4[3]. An intuitionistic fuzzy left (respectively, right) ideal $A = \langle \mu_A, v_A \rangle$ of a $\Gamma$-near-ring $M$ is said to be normal if $\mu_A(0) = 1$ and $v_A(0) = 0$.

Theorem 3.5. Let $A = \langle \mu_A, v_A \rangle$ be an intuitionistic fuzzy left (respectively, right) ideal of a $\Gamma$-near-ring $M$ and let $(x) = \mu_A(x) + 1 - \mu_A(0)$, $(x) = v_A(x) - v_A(0)$. If $(x) + (x) \leq 1$ for all $x \in M$, then $A' = \langle \cdot \cdot \rangle$ is a normal intuitionistic fuzzy left (respectively, right) ideal of $M$.

Proof. We first observe that $\mu_A^+(0) = 1$, $v_A^+(0) = 0$ and $\mu_A^+(x), v_A^+(x) \in [0,1]$ for every $x \in M$.

Hence, $A' = \langle \mu_A^+, v_A^+ \rangle$ is a normal intuitionistic fuzzy set. To prove that it is an intuitionistic fuzzy left (respectively, right) ideal, let $x, y \in M$ and $\alpha \in \Gamma$. Then,
$\mu_A^+(x - y) = \mu_A(x - y) + 1 - \mu_A(0)$
\[ \geq \{ \mu_A(x) \land \mu_A(y) \} + 1 - \mu_A(0) \]
\[ = \{ \mu_A(x) + 1 - \mu_A(0) \} \land \{ \mu_A(y) + 1 - \mu_A(0) \} \]
\[ = \mu_A^\ast(x) \land \mu_A^\ast(y) \]
\[ v_A^\ast(x - y) = v_A(x - y) - v_A(0) \]
\[ \leq \{ v_A(x) \lor v_A(y) \} - v_A(0) \]
\[ = \{ v_A(x) - v_A(0) \} \lor \{ v_A(y) - v_A(0) \} \]
\[ = v_A^\ast(x) \lor v_A^\ast(y) \]
\[ \mu_A^\ast(y + x - y) = \mu_A(y + x - y) + 1 - \mu_A(0) \]
\[ \geq \{ \mu_A(x) + 1 - \mu_A(0) \} \]
\[ = \mu_A^\ast(x) \]
\[ v_A^\ast(y + x - y) = v_A(y + x - y) - v_A(0) \]
\[ \leq \{ v_A(x) - v_A(0) \} \]
\[ = v_A^\ast(x) \]

and
\[ \mu_A^\ast(u\alpha(x + v) - u\alpha v) = \mu_A(u\alpha(x + v) - u\alpha v) + 1 - \mu_A(0) \]
\[ \geq \mu_A(x) + 1 - \mu_A(0) = \mu_A^\ast(x) \]
\[ v_A^\ast(u\alpha(x + v) - u\alpha v) = v_A(u\alpha(x + v) - u\alpha v) - v_A(0) \]
\[ \leq v_A(x) - v_A(0) = v_A^\ast(x) \]

This shows that \( \Lambda^\ast \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M \). So, \( \Lambda^\ast \) is a normal intuitionistic fuzzy left (respectively, right) ideal of \( M \).

**Definition 3.6.** Let \( I \) be an ideal of a \( \Gamma \)-near-ring \( M \). If for each \( a+I, b+I \) in the factor group \( M/I \) and each \( \alpha \in \Gamma \), we define \( (a+I)\alpha(b+I) = ab+I \), then \( M/I \) is a \( \Gamma \)-near-ring which we shall call the \( \Gamma \)-residue class ring of \( M \) with respect to \( I \).

**Theorem 3.7.** Let \( I \) be an ideal of a \( \Gamma \)-near-ring \( M \). If \( A \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M \), then the IFS \( \tilde{A} \) of \( M/I \) defined by \( \mu_{\tilde{A}}(a+I) = \bigvee_{x \in I} \mu_A(a+x) \) and \( v_{\tilde{A}}(a+I) = \bigwedge_{x \in I} v_A(a+x) \) is an intuitionistic fuzzy left (respectively, right) ideal of the \( \Gamma \)-residue class ring \( M/I \) of \( M \) with respect to \( I \).

**Proof.** Let \( a, b \in M \) be such that \( a + I = b + I \).

Then \( b = a + y \) for some \( y \in I \) and so
\[ \mu_{\tilde{A}}(b+I) = \bigvee_{x \in I} \mu_A(b+x) = \bigvee_{x \in I} \mu_A(a+y+x) = \bigvee_{x+y \in z+I} \mu_A(a+z) = \mu_{\tilde{A}}(a+I), \]
\[ v_{\tilde{A}}(b+I) = \bigwedge_{x \in I} v_A(b+x) = \bigwedge_{x \in I} v_A(a+y+x) = \bigwedge_{x+y \in z+I} v_A(a+z) = v_{\tilde{A}}(a+I). \]

Hence, \( \tilde{A} \) is well defined.

For any \( x + I, y + I \in M/I \) and \( \alpha \in \Gamma \), we have
\[ \mu_{\tilde{A}}((x+I) - (y+I)) = \mu_{\tilde{A}}((x-y)+I) \]
\[ = \bigvee_{z \in I} \mu_A((x-y)+z) \]
\[ = \bigvee_{z \in I} \mu_A((x-y) + (u-v)) \]
\[ = \bigvee_{z \in I} \mu_A((x+u) - (y+v)) \]
\[ \geq \bigvee_{z \in I} (\mu_A(x+u) \land \mu_A(y+v)) \]
\[ = (\bigvee_{z \in I} \mu_A(x+u)) \land (\bigvee_{z \in I} \mu_A(y+v)) \]
\[ = \mu_{\tilde{A}}(x+I) \land \mu_{\tilde{A}}(y+I). \]
\[ \begin{align*}
\nu_\bar{A}(x + I) - (y + I) &= \nu_\bar{A}(x-y + I) \\
&= \bigwedge_{z \in I} \nu_\bar{A}(x-y + z) \\
&= \bigwedge_{z = u-v \epsilon I} \nu_\bar{A}(x-y + (u-v)) \\
&= \bigwedge_{u \epsilon I} \nu_\bar{A}(x+u) - (y+v) \\
&\leq \bigwedge_{u \epsilon I} (\nu_\bar{A}(x+u) \vee \nu_\bar{A}(y+v)) \\
&= \nu_\bar{A}(x + I) \vee \nu_\bar{A}(y + I),
\end{align*} \]

\[ \begin{align*}
\mu_\bar{A}(y + I) + ((x + I) - (y + I)) &= \mu_\bar{A}(y + x - y + I) \\
&= \bigwedge_{z \in I} \mu_\bar{A}(y + x - y + z) \\
&= \bigwedge_{z = v + (u-v) \epsilon I} \mu_\bar{A}(y + x - y + (v + (u-v))) \\
&= \bigwedge_{u \epsilon I} \mu_\bar{A}(y + v) + ((x + u) - (y + v)) \\
&\geq \bigwedge_{u \epsilon I} (\mu_\bar{A}(x+u)) \\
&= \mu_\bar{A}(x + I).
\end{align*} \]

\[ \begin{align*}
\nu_\bar{A}(y + I) + ((x + I) - (y + I)) &= \nu_\bar{A}(y + x - y + I) \\
&= \bigwedge_{z \in I} \nu_\bar{A}(y + x - y + z) \\
&= \bigwedge_{z = u-v \epsilon I} \nu_\bar{A}(y + x - y + (v + (u-v))) \\
&= \bigwedge_{u \epsilon I} \nu_\bar{A}(y + v) + ((x + u) - (y + v)) \\
&\leq (\bigwedge_{u \epsilon I} \nu_\bar{A}(x+u)) \\
&= \nu_\bar{A}(x + I),
\end{align*} \]

\[ \begin{align*}
\mu_\bar{A}((a + I)\alpha((x + I) + (b + I)) - ((a + I)\alpha(b + I))) &= \mu_\bar{A}((a\alpha(x + b) - a\alpha b + I)) \\
&= \bigwedge_{z \in I} \mu_\bar{A}((a\alpha(x + b) - a\alpha b + z)) \\
&\geq \bigwedge_{z \in I} \mu_\bar{A}(a\alpha x + a\alpha z) \text{ because } a\alpha z \in I \\
&= \bigwedge_{z \in I} \mu_\bar{A}(a\alpha(x + z)) \geq \bigwedge_{z \in I} \mu_\bar{A}(x + z) = \mu_\bar{A}(x + I),
\end{align*} \]

\[ \begin{align*}
\nu_\bar{A}((a + I)\alpha((x + I) + (b + I)) - ((a + I)\alpha(b + I))) &= \nu_\bar{A}((a\alpha(x + b) - a\alpha b + I)) \\
&= \bigwedge_{z \in I} \nu_\bar{A}((a\alpha(x + b) - a\alpha b + z)) \\
&\leq \bigwedge_{z \in I} \nu_\bar{A}(a\alpha x + a\alpha z) \text{ because } a\alpha z \in I
\end{align*} \]
\[ \bigwedge_{zd} v_A(\alpha x + z) \leq \bigwedge_{zd} v_A(x + z) = v_A(x + I). \]

Similarly,
\[ \mu_A((x + I)\alpha(a + I)) \geq \mu_A(x + I) \text{ and } v_A((x + I)\alpha(a + I)) \leq v_A(x + I). \]

Hence, \( \hat{A} \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M/I \).

**Theorem 3.8.** If the IFS \( A = \langle \mu_A, v_A \rangle \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M \), then the set \( M_A = \{ x \in M \mid \mu_A(x) = \mu_A(0) \text{ and } v_A(x) = v_A(0) \} \) is an ideal of \( M \).

**Proof.** Let \( x, y \in M_A \). Then \( \mu_A(x) = \mu_A(y) = \mu_A(0) \) and \( v_A(x) = v_A(y) = v_A(0) \). Since \( A \) is an intuitionistic fuzzy ideal of \( M \), it follows that
\[ \mu_A(x - y) \geq \{ \mu_A(x) \wedge \mu_A(y) \} = \{ \mu_A(0) \wedge \mu_A(0) \} = \mu_A(0), \]
\[ v_A(x - y) \leq \{ v_A(x) \vee v_A(y) \} = \{ v_A(0) \vee v_A(0) \} = v_A(0). \]

Hence, \( \mu_A(x - y) = \mu_A(0) \) and \( v_A(x - y) = v_A(0) \). So, \( x - y \in M_A \).

Let \( x \in M, \alpha \in \Gamma \) and \( y \in M_A \). Therefore, \( \mu_A(\alpha x + z) = \mu_A(x) = \mu_A(0) \) (respectively, \( \mu_A(y \alpha x) = \mu_A(x) = \mu_A(0) \)) and \( v_A(\alpha x + z) = v_A(0) \) (respectively, \( v_A(y \alpha x) = v_A(x) = v_A(0) \)). Hence, \( \mu_A(\alpha x + z) = \mu_A(0) \) and \( v_A(\alpha x + z) = v_A(0) \). So, \( (x \alpha(y + z) - xaz) \in M_A \). Hence, \( M_A \) is an intuitionistic fuzzy ideal of \( M \).

**Theorem 3.9.** Let \( A \) be an intuitionistic fuzzy left (respectively, right) ideal of a \( \Gamma \)-near-ring \( M \). For each pair \( (t, s) \in [0, 1] \), the level set \( A_{(t, s)} \) is an ideal of \( M \).

**Proof.** Let \( x, y \in A_{(t, s)} \). Then \( \mu_A(x) \geq t, \mu_A(y) \geq t \) and \( v_A(x) \leq s, v_A(y) \leq s \). Since \( A \) is an intuitionistic fuzzy left ideal (respectively, right ideal), we have
\[ \mu_A(x - y) \geq \{ \mu_A(x) \wedge \mu_A(y) \} \geq t \text{ and } v_A(x - y) \leq \{ v_A(x) \vee v_A(y) \} \leq s. \]

So \( x - y \in A_{(t, s)} \).

Since \( \mu_A(x + y - z) = \mu_A(x) + \mu_A(y) \geq t \) and \( v_A(x + y - z) = v_A(0) \leq s \).

So \( x + y - z \in A_{(t, s)} \).

Let \( x \in M, y \in A_{(t, s)} \) and \( \alpha \in \Gamma \). Then \( \mu_A(\alpha x + z) \geq \mu_A(y) \geq t \) and \( v_A(\alpha x + z) \leq v_A(0) \leq s \).

Hence, \( A_{(t, s)} \) is an ideal of \( M \).

**Definition 3.10.** Let \( A \) and \( B \) be two intuitionistic fuzzy subsets of a \( \Gamma \)-near-ring \( M \) and \( \alpha \in \Gamma \). The product \( A \Gamma B \) is defined by
\[
\mu_{A \Gamma B}(x) = \begin{cases} \bigvee_{x=(u,v)+(w,v)} \left( \mu_A(u) \wedge \mu_B(v) \right) & \text{for } u, v \in M, \ \gamma \in \Gamma \\ 0 & \text{otherwise,} \end{cases}
\]
\[
v_{A \Gamma B}(x) = \begin{cases} \bigwedge_{x=(u,v)+(w,v)} \left( v_A(u) \vee v_B(v) \right) & \text{for } u, v \in M, \ \gamma \in \Gamma \\ 1 & \text{otherwise.} \end{cases}
\]

**Definition 3.11** Let \( A = \langle \mu_A, v_A \rangle \) and \( B = \langle \mu_B, v_B \rangle \) be two IFSs in a \( \Gamma \)-near-ring \( M \). Then, the composition of \( A \) and \( B \) is defined to be the intuitionistic fuzzy set \( A \circ B = \langle \mu_A \circ_B, v_A \circ_B \rangle \) in \( M \) given by
$$\mu_{A \odot B}(x) = \begin{cases} \bigwedge_{i \in S_k}\left(\mu_A(u_i) \land \mu_B(v_i)\right) & : x = \sum_{i=1}^{k} (u_i \alpha_i(v_i + w_i) - u_i \alpha_i w_i), u_i, v_i \in M, \gamma_i \in \Gamma, k \in \mathbb{N} \\ 0 & \text{Otherwise} \end{cases}$$

$$\nu_{A \odot B}(x) = \begin{cases} \bigvee_{i \in S_k}\left(\nu_A(u_i) \lor \nu_B(v_i)\right) & : x = \sum_{i=1}^{k} (u_i \alpha_i(v_i + w_i) - u_i \alpha_i w_i), u_i, v_i \in M, \gamma_i \in \Gamma, k \in \mathbb{N} \\ 1 & \text{Otherwise} \end{cases}$$

**Theorem 3.12.** If $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ are intuitionistic fuzzy ideals in a $\Gamma$-near-ring $M$, then $A \odot B$ is an intuitionistic fuzzy ideal in $M$.

**Proof.** For any $x, y \in M$, we have

$$\mu_{A \circ B}(x - y) = \vee\left\{ \bigwedge_{i \in S_m}\mu_A(u_i) \land \bigvee_{i \in S_m}\mu_B(v_i) : x - y = \sum_{i=1}^{m} (u_i \alpha_i(v_i + u_i') - u_i \alpha_i u_i'), u_i, v_i, u_i' \in M, \alpha_i \in \Gamma \text{ and } k \in \mathbb{N} \right\}$$

$$\geq \vee\left\{ \bigwedge_{i \in S_m}\mu_A(a_i) \land \bigvee_{i \in S_m}\mu_B(b_i) : x = \sum_{i=1}^{m} (a_i \alpha_i(b_i + a_i') - a_i \alpha_i a_i'), a_i, b_i \in M, \alpha_i \in \Gamma \text{ and } m \in \mathbb{N} \right\}$$

$$-y = \sum_{i=1}^{m} (c_i \alpha_i(d_i + c_i') - c_i \alpha_i c_i'), a_i, b_i, c_i, d_i, a_i', c_i' \in M, \alpha_i \in \Gamma \text{ and } m, n \in \mathbb{N}$$

$$= \vee\left\{ \bigwedge_{i \in S_m}\mu_A(a_i) \land \bigvee_{i \in S_m}\mu_B(b_i) : x = \sum_{i=1}^{m} (a_i \alpha_i(b_i + a_i') - a_i \alpha_i a_i'), a_i, b_i, a_i' \in M, \alpha_i \in \Gamma \text{ and } m \in \mathbb{N} \right\}$$

$$\land \vee\left\{ \bigwedge_{i \in S_m}\mu_A(c_i) \land \bigvee_{i \in S_m}\mu_B(d_i) : y = \sum_{i=1}^{n} (c_i \alpha_i(d_i + c_i') - c_i \alpha_i c_i'), c_i, d_i, c_i' \in M, \alpha_i \in \Gamma \text{ and } n \in \mathbb{N} \right\}$$

$$= \mu_{A \circ B}(x) \land \mu_{A \odot B}(y)$$

$$\nu_{A \circ B}(x - y) = \bigvee\left\{ \bigwedge_{i \in S_m}\nu_A(u_i) \lor \bigvee_{i \in S_m}\nu_B(v_i) : x - y = \sum_{i=1}^{k} (u_i \alpha_i(v_i + u_i') - u_i \alpha_i u_i'), u_i, v_i, u_i' \in M, \alpha_i \in \Gamma \text{ and } k \in \mathbb{N} \right\}$$

$$\leq \bigwedge\left\{ \bigvee_{i \in S_m}\nu_A(u_i) \lor \bigvee_{i \in S_m}\nu_B(v_i) : x = \sum_{i=1}^{k} (u_i \alpha_i(v_i + u_i') - u_i \alpha_i u_i'), u_i, v_i, u_i' \in M, \alpha_i \in \Gamma \text{ and } k \in \mathbb{N} \right\}$$

$$y = \sum_{i=1}^{k} (c_i \alpha_i(d_i + c_i') - c_i \alpha_i c_i'), a_i, b_i, c_i, d_i, a_i', c_i' \in M, \alpha_i \in \Gamma \text{ and } m, n \in \mathbb{N}$$

$$= \bigwedge\left\{ \bigvee_{i \in S_m}\nu_A(u_i) \lor \bigvee_{i \in S_m}\nu_B(v_i) : x = \sum_{i=1}^{k} (u_i \alpha_i(v_i + u_i') - u_i \alpha_i u_i'), u_i, v_i, u_i' \in M, \alpha_i \in \Gamma \text{ and } k \in \mathbb{N} \right\}$$

$$\land \bigvee\left\{ \bigwedge_{i \in S_m}\nu_A(c_i) \lor \bigvee_{i \in S_m}\nu_B(d_i) : y = \sum_{i=1}^{n} (c_i \alpha_i(d_i + c_i') - c_i \alpha_i c_i'), c_i, d_i, c_i' \in M, \alpha_i \in \Gamma \text{ and } n \in \mathbb{N} \right\}$$

$$= \nu_{A \circ B}(x) \lor \nu_{A \odot B}(y)$$

$$\mu_{A \circ B}(y + x - y) \geq \bigwedge\left\{ \bigvee_{i \in S_m}\mu_A(u_i) : x = \sum_{i=1}^{k} (u_i \alpha_i(v_i + u_i') - u_i \alpha_i u_i'), u_i, v_i, u_i' \in M, \alpha_i \in \Gamma \text{ and } k \in \mathbb{N} \right\}$$
= \bigvee \{ \bigwedge_{i \in \mathcal{I}} \mu_A(a_i) \land \mu_B(b_i) : x = \sum_{i=1}^{m} (a_i \alpha(b_i + a') - a_i \alpha a'), a_i, b_i, a' \in M, \alpha \in \Gamma \text{ and } m \in N \}
\mu_{A \circ B}(x)

v_{A \circ B}(y + x - y) \leq \bigwedge \{ \bigvee_{i \in \mathcal{I}} v_A(u_i) : x = \sum_{i=1}^{k} (u_i \alpha(v_i + u') - u_i \alpha u'), u_i, v_i, u' \in M, \alpha \in \Gamma \text{ and } k \in N \}
= \bigwedge \{ \bigvee_{i \in \mathcal{I}} v_A(a_i) \lor v_B(b_i) : x = \sum_{i=1}^{m} (a_i \alpha(b_i + a') - a_i \alpha a'), a_i, b_i, a' \in M, \alpha \in \Gamma \text{ and } m \in N \}
= v_{A \circ B}(x).

Also
\mu_{A \circ B}(x) = \bigvee \{ \bigwedge_{i \in \mathcal{I}} \mu_A(a_i) \land \mu_B(b_i) : x = \sum_{i=1}^{m} (a_i \alpha(b_i + a') - a_i \alpha a'), a_i, b_i, a' \in M, \alpha \in \Gamma \text{ and } m \in N \}
\leq \bigvee \{ \bigwedge_{i \in \mathcal{I}} \mu_A(a_i) \land \mu_B(b \alpha y) : (x \alpha(y + z) - x \alpha z) = \sum_{i=1}^{m} ((a_i \alpha(b_i + a') - a_i \alpha a') \alpha y), a_i, b \alpha y \in M, \alpha \in \Gamma \text{ and } m \in N \}
= \bigwedge \{ \bigvee_{i \in \mathcal{I}} v_A(a_i) \lor v_B(v_i) : (x \alpha(y + z) - x \alpha z) = \sum_{i=1}^{m} (u_i \alpha(v_i + u') - u_i \alpha u'), u_i, v_i, u' \in M, \alpha \in \Gamma \text{ and } m \in N \}
= \mu_{A \circ B}(x \alpha(y + z) - x \alpha z)

v_{A \circ B}(x) = \bigwedge \{ \bigvee_{i \in \mathcal{I}} v_A(a_i) \lor v_B(b_i) : x = \sum_{i=1}^{m} (a_i \alpha(b_i + a') - a_i \alpha a'), a_i, b_i \in M, \alpha \in \Gamma \text{ and } m \in N \}
\geq \bigvee \{ \bigwedge_{i \in \mathcal{I}} v_A(a_i) \lor v_B(b \alpha y) : (x \alpha(y + z) - x \alpha z) = \sum_{i=1}^{m} ((a_i \alpha(b_i + a') - a_i \alpha a') \alpha y), a_i, b \alpha y \in M, \alpha \in \Gamma \text{ and } m \in N \}
= \bigwedge \{ \bigvee_{i \in \mathcal{I}} v_A(u_i) \lor v_B(v_i) : (x \alpha(y + z) - x \alpha z) = \sum_{i=1}^{m} (u_i \alpha(v_i + u') - u_i \alpha u'), u_i, v_i, u' \in M, \alpha \in \Gamma \text{ and } m \in N \}
= v_{A \circ B}(x \alpha(y + z) - x \alpha z).

That is, \mu_{A \circ B}(x \alpha(y + z) - x \alpha z) \geq \mu_{A \circ B}(x) \text{ and } v_{A \circ B}(x \alpha(y + z) - x \alpha z) \leq v_{A \circ B}(x).

Similarly, we get \mu_{A \circ B}(y \alpha x) \geq \mu_{A \circ B}(x) \text{ and } v_{A \circ B}(y \alpha x) \leq v_{A \circ B}(x).

Hence, A \circ B is an intuitionistic fuzzy ideal of M.

**Definition 3.13.** A function f : M \rightarrow N, where M and N are \Gamma-near-rings, is said to be a \Gamma-homomorphism if f(a + b) = f(a) + f(b), f(ab) = f(a)\alpha f(b), for all a, b \in M and \alpha \in \Gamma.

**Definition 3.14.** A function f : M \rightarrow N, where f is a \Gamma-homomorphism and M and N are \Gamma-near-rings, is said to be a \Gamma-endomorphism if N \subseteq M.

**Definition 3.15.** Let f : X \rightarrow Y be a mapping of \Gamma-near-rings and A be an intuitionistic fuzzy set of Y. Then, the map f^{-1}(A) is the pre-image of A under f, if \mu_{f^{-1}(A)}(x) = \mu_A(f(x)) and v_{f^{-1}(A)}(x) = v_A(f(x)), for all x \in X.

**Theorem 3.16.** Let f be a \Gamma-homomorphism of M. If the IFS A = \langle \mu_A, v_A \rangle is an intuitionistic fuzzy left (respectively, right) ideal of M, then B = \langle \mu_{f^{-1}(A)}, v_{f^{-1}(A)} \rangle is an intuitionistic fuzzy left (respectively, right) ideal of M.

**Proof.** For any x, y \in M, \alpha \in \Gamma, we have
\mu_{f^{-1}(A)}(x - y) = \mu_A(f(x) - f(y)) = \mu_A(f(x) - f(y))
\geq \{ \mu_A(f(x)) \land \mu_A(f(y)) \}
= \{ \mu_{f^{-1}(A)}(x) \land \mu_{f^{-1}(A)}(y) \},
\mu_{f^{-1}(A)}(y + x - y) = \mu_A(f(y) + f(x) - f(y)) = \mu_A(f(y) + f(x) - f(y))

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Thus, \( t \geq x \) and

\[
\mu_{t}^{-1}(\langle x, y + z \rangle - xaz) = \mu_{A}(f(xaz)) = \mu_{t}(f(x)\alpha f(y)) \geq \mu_{t}^{-1}(\langle x, y \rangle).
\]

Similarly,

\[
v_{t}^{-1}(\langle x, y \rangle - y) = v_{A}(f(x) - y) = v_{A}(f(x) - f(y)) \leq \{ v_{A}(f(x)) \cup v_{A}(f(y)) \} = \{ v_{t}^{-1}(\langle x, y \rangle) \quad \text{and} \quad v_{t}^{-1}(\langle y + x \rangle - y) = v_{A}(f(y + x - y)) = v_{A}(f(y) + f(x) - f(y)) \leq v_{A}(f(x)) \leq v_{t}^{-1}(\langle x, y \rangle) \}
\]

\[
v_{t}^{-1}(\langle (x, y) + z \rangle - xaz) = v_{A}(f(x\alpha y)) = v_{A}(f(x)\alpha f(y)) \leq v_{t}^{-1}(\langle x, y \rangle).
\]

Hence, \( B \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M \).

**Theorem 3.17.** If \( A = \langle \mu_{A}, v_{A} \rangle \) is an intuitionistic fuzzy set in \( M \) such that the non-empty sets \( U(\mu_{A}; t) \) and \( L(v_{A}; t) \) are ideals of \( M \) for all \( t \in [0, 1] \), then \( A \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M \).

**Proof.** Suppose that there exists \( x_{0}, y_{0} \in M \) such that \( \mu_{A}(x_{0} - y_{0}) < \{ \mu_{A}(x_{0}) \wedge \mu_{A}(y_{0}) \} \). Let \( t_{0} = \frac{1}{2} \{ \mu_{A}(x_{0} - y_{0}) + (\mu_{A}(x_{0}) \wedge \mu_{A}(y_{0})) \} \). Then, \( (\mu_{A}(x_{0}) \wedge \mu_{A}(y_{0})) \geq t_{0} > \mu_{A}(x_{0} - y_{0}) \). It follows that \( x_{0}, y_{0} \in U(\mu_{A}; t_{0}) \) and \( x_{0} - y_{0} \notin U(\mu_{A}; t_{0}) \). This is a contradiction.

Hence, \( \mu_{A}(x - y) \geq \{ \mu_{A}(x) \wedge \mu_{A}(y) \} \), for all \( x, y \in M \).

Suppose that there exists \( x_{0}, y_{0} \in M \) such that \( \mu_{A}(x_{0} - y_{0}) < \mu_{A}(x_{0}) \). Let \( t_{0} = \frac{1}{2} \{ \mu_{A}(x_{0} - y_{0}) + \mu_{A}(x_{0}) \} \). Then \( \mu_{A}(x_{0}) \geq t_{0} > \mu_{A}(x_{0} - y_{0}) \). It follows that \( x_{0}, y_{0} \in U(\mu_{A}; t_{0}) \) and \( y_{0} + x_{0} - y_{0} \notin U(\mu_{A}; t_{0}) \). This is a contradiction.

Hence, \( \mu_{A}(y - x) \geq \mu_{A}(x) \), for all \( x, y \in M \).

Now let \( x_{0}, y_{0} \in M \) and \( \alpha \in \Gamma \) such that \( \mu_{A}(x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0})) < \mu_{A}(x_{0}) \). Let \( t_{0} = \frac{1}{2} \{ \mu_{A}(x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0})) + \mu_{A}(x_{0}) \} \).

Then we get \( \mu_{A}(x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0})) < t_{0} < \mu_{A}(x_{0}) \). It follows that \( y_{0} \in U(\mu_{A}; t_{0}) \) and \( x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0}) \notin U(\mu_{A}; t_{0}) \). This is a contradiction.

Hence, \( \mu_{A}(x - y) \geq \mu_{A}(x) \), for all \( x, y \in M \).

Suppose that there exists \( x_{0}, y_{0} \in M \) such that \( v_{A}(x_{0} - y_{0}) < v_{A}(x_{0}) \). Let \( t_{0} = \frac{1}{2} \{ v_{A}(x_{0} - y_{0}) + v_{A}(x_{0}) \} \). Then, \( v_{A}(x_{0}) > t_{0} > v_{A}(x_{0} - y_{0}) \).

It follows that \( x_{0}, y_{0} \in U(\mu_{A}; t_{0}) \) and \( y_{0} + x_{0} - y_{0} \notin U(\mu_{A}; t_{0}) \). This is a contradiction.

Hence, \( v_{A}(x + y - x) \geq v_{A}(x) \), for all \( x, y \in M \).

Now let \( x_{0}, y_{0} \in M \) and \( \alpha \in \Gamma \) such that \( v_{A}(x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0})) > v_{A}(x_{0}) \). Let \( t_{0} = \frac{1}{2} \{ v_{A}(x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0})) + v_{A}(x_{0}) \} \).

Then we get \( v_{A}(x_{0} \alpha((y_{0} + z_{0}) - x_{0}az_{0})) > t_{0} > v_{A}(x_{0}) \). It follows that \( y_{0} \in L(\mu_{A}; t_{0}) \) and \( x_{0} \alpha y_{0} \notin L(v_{A}; t_{0}) \). This is a contradiction. Thus, \( v_{A}(x\alpha(y + z) - xaz) \leq v_{A}(x_{0}) \) (respectively, \( v_{A}(y\alpha x) \leq v_{A}(x_{0}) \)). Hence, \( A \) is an intuitionistic fuzzy left (respectively, right) ideal of \( M \).
References


