19th Int. Workshop on IFSs, Burgas, 4–6 June 2015
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 2, 51–62

Solution of intuitionistic fuzzy differential equations by successive approximations method

R. Ettoussi, S. Melliani, M. Elomari and L. S. Chadli

Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University
BP 523, 23000 Beni Mellal, Morocco

Abstract: In this paper, we discuss the existence and uniqueness of a solution of the intuitionistic fuzzy differential equation using the method of successive approximation.
Keywords: Levelwise continuous, Intuitionistic fuzzy solution.
AMS Classification: 03E72.

1 Introduction

The concept of intuitionistic fuzzy is introduced by K. Atanasov (1984) [1, 2]. This concept is a generalization of fuzzy theory introduced by L. Zadeh [3]. Several works made in the study of the Cauchy problem with fuzzy initial condition [4]. By the metric space defined in [5] we have something that makes sense to study this problem in intuitionistic fuzzy theory.

Jong Y. P. and Hho K. H. in [6] gives the demonstrate of the existence and uniqueness of a problem of Cauchy with fuzzy initial condition, using the method of successive approximation and in [7] the authors studied Cauchy problem is level-wise continuous and satisfies the generalized Lipschitz condition. From this work we attempt to give generalization of this existence and uniqueness in the intuitionistic fuzzy case.

In this paper, we prove the existence and uniqueness theorem of a solution to the intuitionistic fuzzy differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where x_0 is an intuitionistic fuzzy quantity and $f : I \times IF_n \to IF_n$ is level-wise continuous and satisfies a generalized Lipschitz condition.

At first, in Section 2, we give some definitions and properties regarding the concept of an intuitionistic fuzzy metric. The main results of this work is discussed in Section 3.

2 Preliminaries

Throughout this paper, $\Big(\mathbb{R}^n, B(\mathbb{R}^n), \mu\Big)$ denotes a complete finite measure space. Let us $P_k(\mathbb{R}^n)$ the set of all nonempty compact convex subsets of $\mathbb{R}^n$ and let $T = [c, d] \subset \mathbb{R}$ be a compact interval. we denote by

$$IF_n = \mathbb{F}(\mathbb{R}^n) = \Big\{ \langle u, v \rangle \ : \ \mathbb{R}^n \to [0,1]^2 \,, |\forall\, x \in \mathbb{R}^n 0 \leq u(x) + v(x) \leq 1 \Big\}$$

An element $\langle u, v \rangle$ of IF_n is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v \rangle$ is normal i.e there exists x_0, $x_1 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp \, \langle u, v \rangle = \mathrm{cl}\{x \in \mathbb{R}^n \ : | \ v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by IF_n

For $\alpha \in [0, 1]$ and $\langle u, v \rangle \in IF^n$, the upper and lower α-cuts of $\langle u, v \rangle$ are defined by

$$\Big[\langle u, v \rangle \Big]^\alpha = \Big\{ x \in \mathbb{R}^n : v(x) \leq 1 - \alpha \Big\}$$

and

$$\Big[\langle u, v \rangle \Big]_\alpha = \Big\{ x \in \mathbb{R}^n : u(x) \geq \alpha \Big\}$$

Remark 2.1. *If $\langle u, v \rangle \in IF_n$, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

We define $0_{(1,0)} \in IF_n$ as

$$0_{(1,0)}(t) = \begin{cases} (1, 0) & t = 0 \\ (0, 1) & t \neq 0 \end{cases}$$

Let $\langle u, v \rangle$,$\langle u', v' \rangle \in IF_n$ and $\lambda \in \mathbb{R}$, we define the following operations by :

$$\Big(\langle u, v \rangle \oplus \langle u', v' \rangle \Big)(z) = \Big(\sup_{z=x+y} \min \Big(u(x), u'(y) \Big), \inf_{z=x+y} \max \Big(v(x), v'(y) \Big) \Big)$$

$$\lambda \, \langle u, v \rangle = \begin{cases} \langle \lambda u, \lambda v \rangle & \text{if } \lambda \neq 0 \\ 0_{(1,0)} & \text{if } \lambda = 0 \end{cases}$$

For $\langle u, v \rangle$, $\langle z, w \rangle \in IF_n$ and $\lambda \in \mathbb{R}$, the addition and scaler-multiplication are defined as follows

$$\Big[\langle u, v \rangle \oplus \langle z, w \rangle \Big]^\alpha = \Big[\langle u, v \rangle \Big]^\alpha + \Big[\langle z, w \rangle \Big]^\alpha, \quad \Big[\lambda \, \langle z, w \rangle \Big]^\alpha = \lambda \Big[\langle z, w \rangle \Big]^\alpha$$

$$\Big[\langle u, v \rangle \oplus \langle z, w \rangle \Big]_\alpha = \Big[\langle u, v \rangle \Big]_\alpha + \Big[\langle z, w \rangle \Big]_\alpha, \quad \Big[\lambda \, \langle z, w \rangle \Big]_\alpha = \lambda \Big[\langle z, w \rangle \Big]_\alpha$$

Definition 2.1. *Let $\langle u,v\rangle$ an element of IF_n and $\alpha\in[0,1]$, we define the following sets :*

$\Big[\langle u,v\rangle\Big]_l^+(\alpha)=\inf\{x\in\mathbb{R}^n\mid u(x)\geq\alpha\},\quad \Big[\langle u,v\rangle\Big]_r^+(\alpha)=\sup\{x\in\mathbb{R}^n\mid u(x)\geq\alpha\}$

$\Big[\langle u,v\rangle\Big]_l^-(\alpha)=\inf\{x\in\mathbb{R}^n\mid v(x)\leq 1-\alpha\},\quad \Big[\langle u,v\rangle\Big]_r^-(\alpha)=\sup\{x\in\mathbb{R}^n\mid v(x)\leq 1-\alpha\}$

Remark 2.2.

$$\Big[\langle u,v\rangle\Big]_\alpha=\bigg[\Big[\langle u,v\rangle\Big]_l^+(\alpha),\Big[\langle u,v\rangle\Big]_r^+(\alpha)\bigg]$$
$$\Big[\langle u,v\rangle\Big]^\alpha=\bigg[\Big[\langle u,v\rangle\Big]_l^-(\alpha),\Big[\langle u,v\rangle\Big]_r^-(\alpha)\bigg]$$

Proposition 2.1. *For all α, $\beta\in[0,1]$ and $\langle u,v\rangle\in IF_n$*

(i) $\Big[\langle u,v\rangle\Big]_\alpha\subset\Big[\langle u,v\rangle\Big]^\alpha$

(ii) $\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}^n$

(iii) if $\alpha\leq\beta$ then $\Big[\langle u,v\rangle\Big]_\beta\subset\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\beta\subset\Big[\langle u,v\rangle\Big]^\alpha$

(iv) If $\alpha_n\nearrow\alpha$ then $\Big[\langle u,v\rangle\Big]_\alpha=\bigcap_n\Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^\alpha=\bigcap_n\Big[\langle u,v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha\in[0,1]$ we denote by

$$M_\alpha=\{x\in\mathbb{R}^n\ :\ u(x)\geq\alpha\}\quad\text{and}\quad M^\alpha=\{x\in\mathbb{R}^n\ :\ v(x)\leq 1-\alpha\}$$

Lemma 2.1. *[6] Let $\Big\{M_\alpha,\ \alpha\in[0,1]\Big\}$ and $\Big\{M^\alpha,\ \alpha\in[0,1]\Big\}$ two families of subsets of $\mathbb{R}^n$ satisfies (i)–(iv) in Proposition 2.1, if u and v define by*

$$u(x)=\begin{cases}0 & \text{if } x\notin M_0\\ \sup\{\alpha\in[0,1]\ :\ x\in M_\alpha\} & \text{if } x\in M_0\end{cases}$$
$$v(x)=\begin{cases}1 & \text{if } x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^\alpha\} & \text{if } x\in M^0\end{cases}$$

Then $\langle u,v\rangle\in IF_n$

Lemma 2.2. *Let I a dense subset of $[0,1]$, if $\Big[\langle u,v\rangle\Big]_\alpha=\Big[\langle u',v'\rangle\Big]_\alpha$ and $\Big[\langle u,v\rangle\Big]^\alpha=\Big[\langle u',v'\rangle\Big]^\alpha$, for all $\alpha\in I$ then $\langle u,v\rangle=\langle u',v'\rangle$*

On the space IF_n we will consider the following metric,

$$\begin{aligned}d_\infty^n\Big(\langle u,v\rangle,\langle z,w\rangle\Big)=&\frac{1}{4}\sup_{0<\alpha\leq1}\Big\|\Big[\langle u,v\rangle\Big]_r^+(\alpha)-\Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big\|\\&+\frac{1}{4}\sup_{0<\alpha\leq1}\Big\|\Big[\langle u,v\rangle\Big]_l^+(\alpha)-\Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big\|\\&+\frac{1}{4}\sup_{0<\alpha\leq1}\Big\|\Big[\langle u,v\rangle\Big]_r^-(\alpha)-\Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big\|\\&+\frac{1}{4}\sup_{0<\alpha\leq1}\Big\|\Big[\langle u,v\rangle\Big]_l^-(\alpha)-\Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big\|\end{aligned}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 2.1. *([6])* d_∞^n *defines a metric on* IF_n.

Theorem 2.2. *The metric space* (IF_n, d_∞^n) *is complete.*

Proof. There exists $i_0 \le n$ such that

$$d_\infty^n(<u,v>, <u',v'>) \le \sqrt{n} d_\infty(<u,v>_{i_0}, <u',v'>_{i_0})$$

Since d_∞ defined a complete topology if IF_1, then d_∞^n also is complete. $\square$

Define $D_1, D_2 : IF_n \times IF_n \to \mathbb{R}^+$ by the equations

$$\begin{aligned} D_1\Big(\langle u,v\rangle, \langle u',v'\rangle\Big) &= \sup_{0\le\alpha\le1} d_H\Big(\Big[\langle u,v\rangle\Big]_\alpha, \Big[\langle u',v'\rangle\Big]_\alpha\Big) \\ D_2\Big(\langle u,v\rangle, \langle u',v'\rangle\Big) &= \sup_{0\le\alpha\le1} d_H\Big(\Big[\langle u,v\rangle\Big]^\alpha, \Big[\langle u',v'\rangle\Big]^\alpha\Big) \end{aligned}$$

where d_H is the Hausdorff metric defined in $P_k(\mathbb{R}^n)$ by $d_H([a,b][c,d]) = \max\{||a-c||; ||b-d||\}$.

Remark 2.3.

$$d_\infty^n\Big(\langle u,v\rangle, \langle u',v'\rangle\Big) \le \frac{1}{2}\sup_{0\le\alpha\le1} d_H\Big(\Big[\langle u,v\rangle\Big]_\alpha, \Big[\langle u',v'\rangle\Big]_\alpha\Big) + \frac{1}{2}\sup_{0\le\alpha\le1} d_H\Big(\Big[\langle u,v\rangle\Big]^\alpha, \Big[\langle u',v'\rangle\Big]^\alpha\Big)$$

$$\le \frac{1}{2}D_1\Big(\langle u,v\rangle, \langle u',v'\rangle\Big) + \frac{1}{2}D_2\Big(\langle u,v\rangle, \langle u',v'\rangle\Big)$$

In the sequel we gives some results of measurability, integrability and differentiability for the proof is similar to [6]

$F : T \to IF_n$ is called integrable bounded if there exists an integrable function $h : T \to \mathbb{R}$ such that $\|y\| \le h(t)$ holds for any $y \in supp(F(t))$, $t \in T$.

Definition 2.2. *We say that a mapping* $F : T \to IF_n$ *is strongly measurable if for all* $\alpha \in [0,1]$ *the set-valued mappings* $F_\alpha : T \to P_k(\mathbb{R}^n)$ *defined by* $F_\alpha(t) = [F(t)]_\alpha$ *and* $F^\alpha : T \to P_k(\mathbb{R}^n)$ *defined by* $F^\alpha(t) = [F(t)]^\alpha$ *are (Lebesgue) measurable, when* $P_k(\mathbb{R}^n)$ *is endowed with the topology generated by the Hausdorff metric* d_H.

Lemma 2.3. *Let* $F : T \to IF_n$ *be strongly measurable and denote* $F_\alpha(t) = \Big[\lambda_\alpha(t), \lambda^\alpha(t)\Big]$, $F^\alpha(t) = \Big[\mu_\alpha(t), \mu^\alpha(t)\Big]$ *for* $\alpha \in [0,1]$. *Then* $\lambda_\alpha, \lambda^\alpha, \mu_\alpha, \mu^\alpha$ *are measurable.*

Definition 2.3. *Suppose* $F : T \to IF_n$ *is integrably bounded and strongly measurable for each* $\alpha \in (0,1]$ *write*

$$\left[\int_T F(t)dt\right]_\alpha = \int_T [F(t)]_\alpha\, dt = \left\{\int_T f dt | f : T \to \mathbb{R}^n \text{ is a measurable selection for } F_\alpha\right\}$$

$$\left[\int_T F(t)dt\right]^\alpha = \int_T [F(t)]^\alpha\, dt = \left\{\int_T f dt | f : T \to \mathbb{R}^n \text{ is a measurable selection for } F^\alpha\right\}.$$

if there exists $\langle u,v\rangle \in IF_n$ *such that* $[\langle u,v\rangle]^\alpha = \left[\int_A F(t)dt\right]^\alpha$ *and* $[\langle u,v\rangle]_\alpha = \left[\int_T F(t)dt\right]_\alpha$ $\forall\alpha \in (0,1]$. *Then F is called integrable on T, write* $\langle u,v\rangle = \int_T F(t)dt$.

Remark 2.4.

- *If $F(t) = \langle u_t, v_t \rangle$ is integrable, then $\int \langle u_t, v_t \rangle = \left\langle \int u_t, \int v_t \right\rangle$*
- *If $F : T \to IF_n$ is integrable then in view of Lemma (2.3) $\int F$ is obtained by integrating the α-level curves, that is*

$$\left[\int F\right]_\alpha = \left[\int \lambda_\alpha, \int \lambda^\alpha\right] \text{ and } \left[\int F\right]^\alpha = \left[\int \mu_\alpha, \int \mu^\alpha\right], \alpha \in [0,1]$$

$F_\alpha(t) = [F(t)]_\alpha = [\lambda_\alpha(t), \lambda^\alpha(t)], F^\alpha(t) = [F(t)]^\alpha = [\mu_\alpha(t), \mu^\alpha(t)]$ *for* $\alpha \in [0,1]$.

Theorem 2.3. *If $F : T \to IF_n$ is strongly measurable and integrably bounded, then F is integrable.*

Proof: If we denote $\mathcal{M}_\alpha = \int F_\alpha$ and $\mathcal{M}^\alpha = \int F^\alpha$, then properties (i)–(iii) of Lemma (2.1) are checked .
Since $F_\alpha \subset F^\alpha \Rightarrow \int F_\alpha \subset \int F^\alpha$ for all $\alpha \in [0,1]$, by Lemma (2.1), There exists unique $\langle u, v \rangle \in IF_n$ such that $[\langle u, v \rangle]^\alpha = \int F^\alpha$ et $[\langle u, v \rangle]_\alpha = \int F_\alpha$, which completes the proof. □

Definition 2.4. *A mapping $F : T \to IF_n$ is called level-wise continuous at $t_0 \in T$ if the set-valued mappings $F_\alpha(t) = [F(t)]_\alpha$ and $F^\alpha(t) = [F(t)]^\alpha$ are continuous at $t = t_0$ with respect to the Hausdorff metric d_H for all $\alpha \in [0,1]$*

Proposition 2.2. *If $F : T \to IF_n$ is level-wise continuous then it is strongly measurable.*

Proof: By the level-wise continuity of F; F_α and F^α are continuous with respect to the Hausdorff metric d_H for all $\alpha \in [0,1]$. Therefore $F_\alpha^{-1}(U)$ and $F^\alpha(U)^{-1}$ are open, and hence its are measurable for each open $U \in P_k(\mathbb{R}^n)$. □

Proposition 2.3. *If $F : T \to IF_n$ is level-wise continuous, then it is integrable.*

Proof: By **Proposition 2.2** F is strongly measurable. Since F^0 is continuous, $F^0(t) \in P_k(\mathbb{R}^n)$ for all $t \in T$ and T is compact, then $\cup_{t \in T} F^0(t)$ is compact. So F is integrably bounded. □

Definition 2.5. *A mapping $F : T \to IF_n$ is said to be differentiable at t_0 if there exist $F'(t_0) \in IF_n$ such that limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t} \text{ and } \lim_{\Delta t \to 0^+} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t}$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0) \rangle$.

Here the limit is taken in the metric space (IF_n, d_∞^n). At the end points of T we consider only the one-sided derivatives.

Remark 2.5. *From the definition it directly follows that if F is differentiable then the multivalued mappings F_α and F^α are Hukuhara differentiable for all $\alpha \in [0,1]$ and*

$$DF_\alpha(t) = [F'(t)]_\alpha \tag{2}$$

Here DF_α denotes the Hukuhara derivative of F_α.

$$DF^\alpha(t) = [F'(t)]^\alpha \tag{3}$$

Here DF^α denotes the Hukuhara derivative of F^α.

If $F : T \to IF_n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the intuitionistic fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.4. *Let $F : T \to IF_n$ be differentiable. Denote $F_\alpha(t) = [F(t)]_\alpha = [\lambda_\alpha(t), \lambda^\alpha(t)]$, $F^\alpha(t) = [F(t)]^\alpha = [\mu_\alpha(t), \mu^\alpha(t)]$. Then $\lambda_\alpha(t), \lambda^\alpha(t), \mu_\alpha(t)$ and $\mu^\alpha(t)$ are differentiable and*

$$[F(t)']_\alpha = [\lambda'_\alpha(t), \lambda^{\alpha\prime}(t)]$$

$$[F(t)']^\alpha = [\mu'_\alpha(t), \mu^{\alpha\prime}(t)]$$

Theorem 2.5. *If $F : T \to IF_n$ is differentiable; then it is levelwise continuous.*

Proof: Let $t, t+h \in T$ with $h > 0$. Then for any $\alpha \in [0,1]$ by the definition of metric d^n_∞ we have

$$\begin{aligned}
\frac{1}{4}d_H\Big([F(t+h)]_\alpha, [F(t)]_\alpha)\Big) + \frac{1}{4}d_H\Big([F(t+h)]^\alpha, [F(t)]^\alpha)\Big) \\
&\le d^n_\infty\Big(F(t+h), F(t)\Big) \\
&\le d^n_\infty\Big(F(t+h) \ominus F(t), 0_{(1,0)}\Big) \\
&\le hd^n_\infty\Big(\frac{1}{h}\Big(F(t+h) \ominus F(t)\Big), F'(t)\Big) + hd^n_\infty\Big(F'(t), 0_{(1,0)}\Big)
\end{aligned}$$

where h is so small that the H-difference $F(t+h) \ominus F(t)$ exists. By the differentiability we know that the right-hand side goes to zero as $h \to 0^+$ and hence F^α and F_α are right continuous then, F is right continuous level-wise, the left continuity levelwise is proved similarly. □

Theorem 2.6. *Let $F : T \to IF_n$ be level-wise continuous; Then for every $t \in T$ the integral $G(t) = \int_a^t F(s)ds$ is differentiable and $G'(t) = F(t)$.*

Proof: Let $\alpha \in [0,1]$ be fixed. Since F is level-wise continuous, then for arbitrary $\varepsilon > 0$, t, $t+h \in T$ and $h > 0$ there exists an $\delta(\varepsilon, \alpha)$ such that

$$d_H\Big([F(t+h)]_\alpha, [F(t)]_\alpha\Big) < \varepsilon \quad \text{and} \quad d_H\Big([F(t+h)]^\alpha, [F(t)]^\alpha\Big) < \varepsilon$$

whenever $0 < h < \delta(\varepsilon, \alpha)$. According to Proposition 2.3. F is integrable, and [Theorem 4.2 in [7]] gives

$$G(t+h) \ominus G(t) = \int_t^{t+h} F$$

Then

$$
\begin{aligned}
d_\infty^n\Big(\frac{1}{h}(G(t+h)\ominus G(t)), F(t)\Big) &= \frac{1}{h}d_\infty^n\Big(\int_t^{t+h} F(s)ds, hF(t)\Big)\\
&= \frac{1}{h}d_\infty^n\Big(\int_t^{t+h} F(s)ds, \int_t^{t+h} F(t)ds\Big)\\
&\leq \frac{1}{h}\int_t^{t+h} d_\infty^n\Big(F(s), F(t)\Big)ds
\end{aligned}
$$

Consequently, by Remark 2.3 we have,

$$
\begin{aligned}
d_\infty^n\Big(\frac{1}{h}(G(t+h)\ominus G(t)), F(t)\Big) &\leq \frac{1}{2h}\Big[\int_t^{t+h} \sup_{0\leq\alpha\leq 1} d_H\Big([F(s)]_\alpha, [F(t)]_\alpha\Big)ds\\
&\quad + \int_t^{t+h} \sup_{0\leq\alpha\leq 1} d_H\Big([F(s)]^\alpha, [F(t)]^\alpha\Big)ds\Big]\\
&\leq \frac{1}{2h}\Big[h\varepsilon + h\varepsilon\Big] \leq \varepsilon
\end{aligned}
$$

this implies, $\lim_{h\to 0}\frac{1}{h}(G(t+h)\ominus G(t)) = F(t)$,
and similarly $\lim_{h\to 0}\frac{1}{h}(G(t)\ominus G(t-h)) = F(t)$, which proves the theorem. □

Theorem 2.7. *Let $F: T \to IF_n$ be differentiable and assume that the derivative F' is integrable over T Then, for each $s \in T$, we have*

$$
F(s) = F(a) \oplus \int_a^s F'(t)dt. \tag{4}
$$

Proof: We shall prove that

$$
F_\alpha(s) = F_\alpha(a) + \int_a^s DF_\alpha(t)dt \tag{5}
$$

$$
F^\alpha(s) = F^\alpha(a) + \int_a^s DF^\alpha(t)dt \tag{6}
$$

for all $\alpha \in [0,1] \setminus A$ with A being negligible.

So, for $\alpha \in [0,1] \setminus A$ be fixed. We will show that $F_\alpha(s) = F_\alpha(a) + \int_a^s DF_\alpha(t)dt$, where DF_α is the Hukuhara derivative of F_α. Recall that the supporting functional $\delta(.,K): \mathbb{R}^n \to \mathbb{R}$ of $K \in P_k(\mathbb{R}^n)$ is defined by

$$
\delta(a,K) = \sup\Big\{a.k \mid k \in K\Big\} \tag{7}
$$

where $a.k$ denotes the usual scalar product of a and k. If $K_1, K_2 \in P_k(\mathbb{R}^n)$ then Theorem $II-18$ in [8] gives us the equation

$$
d_H(K_1,K_2) = \sup_{\|a\|=1} |\delta(a,K_1) - \delta(a,K_2)| \tag{8}
$$

Now let $t, t+h \in T$ with $h > 0$ so small that the H-difference $F(t+h) - F(t)$ exists. Then by Theorem $II-17$ in [8] we have

$$
\delta\Big(x, F_\alpha(t+h) - F_\alpha(t)\Big) = \delta\Big(x, F_\alpha(t+h)\Big) - \delta\Big(x, F_\alpha(t)\Big) \tag{9}
$$

for all $x \in \mathbb{R}^n$, $\alpha \in [0,1] \setminus A$ and consequently

$$\delta\Big(x, (F_\alpha(t+h) - F_\alpha(t))/_h\Big) = \delta\Big(x, F_\alpha(t+h)\Big) - \delta\Big(x, F_\alpha(t)\Big)/_h \tag{10}$$

Then by the differentiability of F_α and Eqs. (8) and (10) we obtain $\delta\Big(x, F_\alpha(t)\Big)$ that is right differentiable and the right derivative equals to $\delta(x, DF_\alpha(t))$ where x is an arbitrary element of the surface of the unit ball S in $\mathbb{R}^n$. Applying a similar reasoning for $h < 0$, we conclude that for all $x \in S$, $\delta\Big(x, F_\alpha(t)\Big)$ is differentiable on T and

$$\frac{d}{dt}\delta\Big(x, F_\alpha(t)\Big) = \delta\Big(x, DF_\alpha(t)\Big)$$

Since $DF_\alpha(t)$ is compact and convex it can be expressed as an intersection of all closed half spaces containing it, i.e.

$$DF_\alpha(t) = \bigcap_{x \in S} H_x$$

where $H_x = \{z \in \mathbb{R}^n \mid x.z \leq \delta(x, DF_\alpha(t))\}$. Thus $DF_\alpha(t)$ equals to the derivative of the set-valued mapping F_α defined by Bradley and Datko [4]. The equality (5) now follows from [[4], Theorem 3.5]. The same technique is applied to show The equality (6).

Consequently, according to Lemma 2.2 the result is obtained. □

Definition 2.6. *A mapping $f : T \times IF_n \to IF_n$ is called level-wise continuous at point $(t_0, x_0) \in T \times IF_n$ provided for any fixed $\alpha \in [0,1]$ and arbitrary $\varepsilon > 0$, there exists an $\delta(\varepsilon, \alpha)$ such that*

$$d_H\Big([f(t,x)]_\alpha, [f(t,x_0)]_\alpha\Big) < \varepsilon, \qquad d_H\Big([f(t,x)]^\alpha, [f(t,x_0)]^\alpha\Big) < \varepsilon$$

whenever $|t - t_0| < \delta(\varepsilon, \alpha)$ and for all $t \in T$, $x \in IF_n$

$$d_H\Big([x]_\alpha, [x_0]_\alpha\Big) < \delta(\varepsilon, \alpha), \qquad d_H\Big([x]^\alpha, [x_0]^\alpha\Big) < \delta(\varepsilon, \alpha)$$

3 Main result

Assume that $f : I \times IF_n \to IF_n$ is levelwise continuous, where the interval

$$I = \{t : |t - t_0| \leq \delta \leq a\}.$$

Consider the intuitionistic fuzzy differential equation (1) where $x_0 \in IF_n$.

We denote $J_0 = I \times B(x_0, b)$ where $a > 0,\ b > 0$, $x_0 \in IF_n$

$$B(x_0, b) = \Big\{x \in IF_n | d_\infty^n(x, x_0) \leq b\Big\}$$

Definition 3.1. *A mapping $x : I \to IF_n$ is a solution to the problem (1) if it is level-wise continuous and satisfies the integral equation*

$$x(t) = x_0 \oplus \int_{t_0}^{t} f(s, x(s))ds. \; for \; all \; t \in I$$

According to the method of successive approximation, let us consider the sequence $\{x_n(t)\}$ such that

$$x_n(t) = x_0 \oplus \int_{t_0}^{t} f(s, x_{n-1}(s))ds, \; n = 1, 2, \ldots \tag{11}$$

where $x_0(t) \equiv x_0$, $t \in T$

Theorem 3.1. *Assume that*

1. *A mapping $f : J_0 \to IF_n$ is level-wise continuous,*
2. *for any pair $(t, x), (t, y) \in J_0$, we have*

$$d_\infty^n\Big(f(t, x), f(t, y)\Big) \le L d_\infty^n\Big(x, y\Big) \tag{12}$$

where $L > 0$ is a given constant.
Then there exists a unique solution $x = x(t)$ of (1) defined on the interval

$$|t - t_0| \le \delta = \min\{a, \frac{b}{M}\}, \tag{13}$$

where $M = \frac{1}{2}\Big(D_1\Big(f(t, x), 0_{(1,0)}\Big) + D_2\Big(f(t, x), 0_{(1,0)}\Big)\Big)$ and for any $(t, x) \in J_0$.
Moreover, there exists an intuitionistic fuzzy set-valued mapping $x : I \to IF_n$ such that

$$d_\infty^n\Big(x_n(t), x(t)\Big) \to 0 \;\; for \;\; |t - t_0| \le \delta \;\; as \;\; n \to \infty$$

Proof: Let $t \in T$ by (11), it follows that, for $n = 1$

$$x_1(t) = x_0 \oplus \int_{t_0}^{t} f(s, x_0(s))ds \tag{14}$$

which proves that $x_1(t)$ is level-wise continuous on $|t - t_0| \le a$ and, hence on $|t - t_0| \le \delta$. Moreover, for any $\alpha \in [0, 1]$, we have

$$d_H\Big([x_1(t)]_\alpha, [x_0]_\alpha\Big) = d_H\Big(\Big[\int_{t_0}^{t} f(s, x_0)ds\Big]_\alpha, 0\Big) \le \int_{t_0}^{t} d_H\Big(\Big[f(s, x_0)\Big]_\alpha, 0\Big)ds \tag{15}$$

$$d_H\Big([x_1(t)]^\alpha, [x_0]^\alpha\Big) = d_H\Big(\Big[\int_{t_0}^{t} f(s, x_0)ds\Big]^\alpha, 0\Big) \le \int_{t_0}^{t} d_H\Big(\Big[f(s, x_0)\Big]^\alpha, 0\Big)ds \tag{16}$$

By Remark 2.3., we get

$$d_\infty^n\Big(x_1(t), x_0\Big) \le M|t - t_0| \le M\delta = b \tag{17}$$

if $|t - t_0| \le \delta$, where $M = \frac{M_1 + M_2}{2}$, $M_1 = D_1\Big(f(t, x), 0_{(1,0)}\Big)$ and $M_2 = D_2\Big(f(t, x), 0_{(1,0)}\Big)$ for any $(t, x_1) \in J_0$.

Now, assume that $x_{n-1}(t)$ is level-wise continuous on $|t - t_0| \le \delta$ and that

$$d_\infty^n\Big(x_{n-1}(t), x_0\Big) \le M|t - t_0| \le M\delta = b \tag{18}$$

if $|t-t_0| \le \delta$,where $M = \frac{M_1+M_2}{2}$ and for any $(t, x_{n-1}) \in J_0$.

From (11), we deduce that $x_n(t)$ is level-wise continuous on $|t-t_0| \le \delta$ and that

$$d_\infty^n\Big(x_n(t), x_0\Big) \le M|t-t_0| \le M\delta = b \tag{19}$$

if $|t-t_0| \le \delta$,where $M = \frac{M_1+M_2}{2}$ and for any $(t, x_n) \in J_0$.

Consequently, we conclude that $\{x_n(t)\}$ consists of level-wise continuous mappings on $|t-t_0| \le \delta$ and that

$$(t, x_n(t)) \in J_0,\ |t-t_0| \le \delta,\ n = 1, 2, \dots \tag{20}$$

In addition we will show that there exists an intuitionistic fuzzy set-valued mapping $x : I \to IF_n$ such that

$$d_\infty^n\Big(x_n(t), x(t)\Big) \to 0 \text{ uniformly for } |t-t_0| \le \delta \text{ as } n \to \infty$$

For $n = 2$ from (11)

$$x_2(t) = x_1 \oplus \int_{t_0}^{t} f\Big(s, x_1(s)\Big)ds \tag{21}$$

from (14) ,(21) we have

$$d_\infty^n\Big(x_2(t), x_1(t)\Big) = d_\infty^n\Big(\int_{t_0}^{t} f(s, x_1(s))ds, \int_{t_0}^{t} f(s, x_0)ds\Big) \le \int_{t_0}^{t} d_\infty^n\Big(f(s, x_1(s)), f(s, x_0)\Big)ds \tag{22}$$

According to the condition (12), we obtain

$$d_\infty^n\Big(x_2(t), x_1(t)\Big) \le L\int_{t_0}^{t} d_\infty^n\Big(x_1(s), x_0\Big)ds \tag{23}$$

Now, we can apply the first inequality (17) in the right-hand side of (23) to get

$$d_\infty^n\Big(x_2(t), x_1(t)\Big) \le ML\frac{|t-t_0|^2}{2!} \le ML\frac{\delta^2}{2!} \tag{24}$$

Starting from (17) and (24), assume that

$$d_\infty^n\Big(x_n(t), x_{n-1}(t)\Big) \le ML^{n-1}\frac{|t-t_0|^n}{n!} \le ML^{n-1}\frac{\delta^n}{n!} \tag{25}$$

and let us prove that such an inequality holds for $d_\infty^n\Big(x_{n+1}(t), x_n(t)\Big)$.

In fact, from (11) and condition (12), it follows that

$$\begin{aligned} d_\infty^n\Big(x_{n+1}(t), x_n(t)\Big) &= d_\infty^n\Big(\int_{t_0}^{t} f(s, x_n(s))ds, \int_{t_0}^{t} f(s, x_{n-1}(s))ds\Big) && (26)\\ &\le \int_{t_0}^{t} d_\infty^n\Big(f(s, x_n(s)), f(s, x_{n-1}(s))\Big)ds && (27)\\ &\le \int_{t_0}^{t} Ld_\infty^n\Big(x_n(s), x_{n-1}(s)\Big)ds && (28)\end{aligned}$$

According to (25), we get

$$d_\infty^n\Big(x_{n+1}(t), x_n(t)\Big) \le ML^n\int_{t_0}^{t}\frac{|s-t_0|^n}{n!}ds = ML^n\frac{|t-t_0|^{n+1}}{(n+1)!} \le ML^n\frac{\delta^{n+1}}{(n+1)!} \tag{29}$$

Thus, inequality (25) holds for $n = 1, 2, \ldots$ We can also write

$$d_\infty^n\Big(x_n(t), x_{n-1}(t)\Big) \leq \frac{M}{L}\frac{(L\delta)^n}{(n)!} \tag{30}$$

for $n = 1, 2, \ldots$, and $|t - t_0| \leq \delta$. Furthermore, we can write $x_n(t)$ as follows

$$x_n(t) = x_0 + [x_1(t) - x_0] + \cdots + [x_n(t) - x_{n-1}(t)], \tag{31}$$

which implies that the sequence $\{x_n(t)\}$ and the series

$$x_0 + \sum_{n=1}^{\infty}[x_n(t) - x_{n-1}(t)], \tag{32}$$

have the same convergence properties.

From (29), according to the convergence criterion of Weierstrass, it follows that the series having the general term $x_n(t) \ominus x_{n-1}(t)$, so $d_\infty^n\Big(x_n(t), x_{n-1}(t)\Big) \to 0$ uniformly on $|t - t_0| \leq \delta$ as $n \to \infty$.

Hence, there exists an intuitionistic fuzzy set-valued mapping $x : I \to IF_n$ such that $d_\infty^n\Big(x_n(t), x(t)\Big) \to 0$ uniformly on $|t - t_0| \leq \delta$ as $n \to \infty$.

From (12) we get

$$d_\infty^n\Big(f(t, x_n(t)), f(t, x(t))\Big) \leq L d_\infty^n\Big(x_n(t), x(t)\Big) \to 0 \tag{33}$$

uniformly on $|t - t_0| \leq \delta$ as $n \to \infty$.

Taking (33) into account, from (11), we obtain, for $n \to \infty$,

$$x(t) = x_0 \oplus \int_{t_0}^{t} f(s, x(s))ds \tag{34}$$

Consequently, there is at least one level-wise continuous solution of (1).

It remains to show this solution is unique.

Let

$$y(t) = x_0 \oplus \int_{t_0}^{t} f(s, y(s))ds \tag{35}$$

on $|t - t_0| \leq \delta$, from (11) and (35), for $n = 1, 2, \ldots$ we obtain

$$\begin{aligned} d_\infty^n\Big(y(t), x_n(t)\Big) &= d_\infty^n\Big(\int_{t_0}^{t} f(s, y(s))ds, \int_{t_0}^{t} f(s, x_{n-1}(s))ds\Big) && (36)\\ &\leq \int_{t_0}^{t} d_\infty^n\Big(f(s, y(s)), f(s, x_{n-1}(s))\Big)ds, \\ &\leq L\int_{t_0}^{t} d_\infty^n\Big(y(s), x_{n-1}(s)\Big)ds && (37) \end{aligned}$$

But $d_\infty^n\Big(y(t), x_0\Big) \leq b$ on $|t - t_0| \leq \delta$ being a solution of (35). It follows from (36) that

$$d_\infty^n\Big(y(t), x_1(t)\Big) \leq bL|t - t_0| \tag{38}$$

Now, assume that

$$d_\infty^n\Big(y(t), x_n(t)\Big) \le bL^n \frac{|t-t_0|^n}{n!} \tag{39}$$

so

$$\begin{aligned} d_\infty^n\Big(y(t), x_{n+1}(t)\Big) &\le L\int_{t_0}^{t} d_\infty^n\Big(y(s), x_n(s)\Big)ds \\ &\le bL^{n+1}\frac{|t-t_0|^{n+1}}{(n+1)!} \end{aligned}$$

Consequently, (39) holds for any n, which leads to the conclusion

$$d_\infty^n\Big(y(t), x_n(t)\Big) = d_\infty^n\Big(x(t), x_n(t)\Big) \to 0$$

on the interval $|t-t_0| \le \delta$ as $n \to \infty$. □

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