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On the operators partially extending the extended intuitionistic fuzzy operators from modal type

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Abstract: Up to now, over the intuitionistic fuzzy sets different operators from modal, topological, level and other types are defined. Here we introduse two new operators that partially extend the extended intuitionistic fuzzy operators from modal type. Some of the basic properties of the new operators are discussed.
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1 On the extended intuitionistic fuzzy operators from modal type

Over Intuitionistic Fuzzy Sets (IFSs, see [1, 2]) different operators from modal, topological, level and other types are defined. Here, we discuss some of their extensions and some of the properties of these extensions.

First, we start with the definitions of the modal types of operators.

Let $\alpha, \beta \in [0, 1]$ and let:

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1$$

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$

$$H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

$$J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\},$$

$$X_{a,b,c,d,e,f}(A) = \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)), \\ d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$$

$$f_{\alpha,\beta}(A) = \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \mu_A(x) + \beta.\pi_A(x)\rangle \mid x \in E\}, \text{ where } \alpha + \beta \leq 1;$$

$$g_{\alpha,\beta}(A) = \{\langle x, \alpha.\nu_A(x), \beta.\mu_A(x)\rangle \mid x \in E\};$$

$$h_{\alpha,\beta}(A) = \{\langle x, \alpha.\nu_A(x), \mu_A(x) + \beta.\pi_A(x)\rangle \mid x \in E\};$$

$$h^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\nu_A(x), \mu_A(x) + \beta.(1 - \alpha.\nu_A(x) - \mu_A(x))\rangle \mid x \in E\};$$

$$j_{\alpha,\beta}(A) = \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \beta.\mu_A(x)\rangle \mid x \in E\};$$

$$j^*_{\alpha,\beta}(A) = \{\langle x, \nu_A(x) + \alpha.(1 - \nu_A(x) - \beta.\mu_A(x)), \beta.\mu_A(x)\rangle \mid x \in E\};$$

$$x_{a,b,c,d,e,f}(A) = \{\langle x, a.\nu_A(x) + b.(1 - \nu_A(x) - c.\mu_A(x)), \\ d.\mu_A(x) + e.(1 - f.\nu_A(x) - \mu_A(x))\rangle | x \in E\}$$

where $a, b, c, d, e, f \in [0, 1]$ and

$$a + e - e.f \leq 1, \tag{1}$$

$$b + d - b.c \leq 1. \tag{2}$$

For every two IFSs A and B a lot of relations and operations are defined (see, e.g. [1, 2]), the most important of which are:

$$A \subseteq B \text{ iff } (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x));$$

$$A \supseteq B \text{ iff } B \subseteq A;$$

$$A = B \text{ iff } (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x));$$

$$\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\},$$

$$A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle | x \in E\};$$

$$A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle | x \in E\};$$

$$A@B = \{\langle x, \tfrac{\mu_A(x)+\mu_B(x)}{2}, \tfrac{\nu_A(x)+\nu_B(x)}{2}\rangle | x \in E\}.$$

In [2], the following extension operators of the above ones are introduced:

Let A and B be two IFSs. In [2], we generalized the above operators to the forms,

$$F_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle | \, x \in E\},$$

$$G_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},$$

$$H_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},$$

$$H^*_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).(1 - \mu_B(x),$$

$$J_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},$$

$$J^*_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).(1 - \mu_A(x) - \nu_B(x).\nu_A(x)), \nu_B(x).\nu_A(x)\rangle | \, x \in E\},$$

$$f_B(A) = \{\langle x, \nu_A(x) + \mu_B(x).\pi_A(x), \mu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\};$$

$$g_B(A) = \{\langle x, \mu_B(x).\nu_A(x), \nu_B(x).\mu_A(x)\rangle \mid x \in E\};$$

$$h_B(A) = \{\langle x, \mu_B(x).\nu_A(x), \mu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},$$

$$h^*_B(A) = \{\langle x, \mu_B(x).\nu_A(x), \mu_A(x) + \nu_B(x).(1 - \mu_B(x).\nu_A(x) - \nu_A(x))\rangle | \, x \in E\},$$

$$j_B(A) = \{\langle x, \nu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\mu_A(x)\rangle \mid x \in E\},$$

$$j^*_B(A) = \{\langle x, \nu_A(x) + \mu_B(x).(1 - \nu_A(x) - \nu_B(x).\nu_A(x)), \nu_B(x).\mu_A(x)\rangle | \, x \in E\}.$$

2 Operators $X_{A,B,C}$ and $x_{A,B,C}$

Here, we will introduce two new operators, that extend the previous ones.

Let A, B, C and D are four IFSs over universe E.

An extension of operators $F_A, G_A, H_A, H^*_A, J_A, J^*_A$ and $X_{a,b,c,d,e,f}$ has the form:

$$X_{A,B,C}(D) = \{\langle x, \mu_A(x).\mu_D(x) + \mu_B(x).(1 - \mu_D(x) - \mu_C(x).\nu_D(x)),$$

$$\nu_B(x).\nu_D(x) + \nu_A(x).(1 - \nu_C(x).\mu_D(x) - \nu_D(x))\rangle | x \in E\}$$

and the extension of operators $f_{A,B}, g_{A,B}, h_{A,B}, h^*_{A,B}, j_{A,B}, j^*_{A,B}$ and $x_{a,b,c,d,e,f}$ has the form:

$$x_{A,B,C}(D) = \{\langle x, \mu_A(x).\nu_D(x) + \mu_B(x).(1 - \nu_D(x) - \mu_C(x).\mu_D(x)),$$

$$\nu_B(x).\mu_D(x) + \nu_A(x).(1 - \nu_C(x).\nu_D(x) - \mu_D(x))\rangle | x \in E\}.$$

As discussed in [2], operators F_A and f_A totally extend operators $F_{\alpha,\beta}$ and $f_{\alpha,\beta}$, respectively, while operators $G_A, H_A, ..., J^*_A$ and $g_A, h_A, ..., j^*_A$ only partially extend operators $G_{\alpha,\beta}, H_{\alpha,\beta}, ...,$ $J^*_{\alpha,\beta}$ and $g_{\alpha,\beta}, h_{\alpha,\beta}, ..., j^*_{\alpha,\beta}$, respectively.

Now we see that operator F_B cannot be represented by operator $X_{A,B,C}$, because, if for some IFSs A, B, C, D:

$$F_B(D) = X_{A,B,C}(D),$$

then the following equalities hold: $\mu_A(x) = 1, \mu_C(x) = 1, \nu_A(x) = 1, \nu_C(x) = 1$, that is impossible. Analogously, operator f_B cannot be represented by operator $x_{A,B,C}$.

Let

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

$$E^* = \{\langle x, 1, 0\rangle | x \in E\},$$

$$U^* = \{\langle x, 0, 0\rangle | x \in E\}.$$

Having in mind (see [2]) that

$$P_B(A) = \{\langle x, \max(\mu_B(x), \mu_A(x)), \min(\nu_B(x), \nu_A(x))\rangle \mid x \in E\},$$

$$Q_B(A) = \{\langle x, \min(\mu_B(x), \mu_A(x)), \max(\nu_B(x), \nu_A(x))\rangle \mid x \in E\},$$

then

$$P_B(A) = B \cup A = A \cup B = P_A(B),$$

$$Q_B(A) = B \cap A = A \cap B = Q_A(B).$$

Therefore,

$$\{\langle x, \mu_A(x), 0\rangle \mid x \in E\} = P_{U^*}(A)$$

and

$$\{\langle x, 0, \nu_A(x)\rangle \mid x \in E\} = Q_{U^*}(A).$$

Theorem 1: For every two IFSs A and D over universe E,

(a) $G_A(D) = X_{P_{U^*}(A), Q_{U^*}(A), C}(D)$, where C is an arbitrary IFS over E,

(b) $H_A(A) = X_{A,O^*,O^*}(D)$,

(c) $J_A(A) = X_{E^*,A,E^*}(D)$,

(d) $H^*_A(A) = X_{A,O^*,\overline{A}}(D)$,

(e) $J_A(A) = X_{E^*,A,\overline{A}}(D)$,

(f) $g_A(D) = x_{P_{U^*}(A), Q_{U^*}(A), C}(D)$, where C is an arbitrary IFS over E

(g) $h_A(A) = x_{A,O^*,O^*}(D)$,

(h) $j_A(A) = x_{E^*,A,E^*}(D)$,

(i) $h^*_A(A) = x_{A,O^*,\overline{A}}(D)$,

(j) $j_A(A) = x_{E^*,A,\overline{A}}(D)$.

Proof. Let the IFSs A, C and D over universe E be given. Then

$$X_{P_{U^*}(A), Q_{U^*}(A), C}(D) = X_{\{\langle x, \mu_A(x), 0\rangle \mid x \in E\},\ \{\langle x, 0, \nu_A(x)\rangle \mid x \in E\}, C}(D)$$

$$= \{\langle x, \mu_A(x).\mu_D(x) + 0.(1 - \mu_D(x) - \mu_C(x).\nu_D(x)),$$

$$\nu_A(x).\nu_D(x) + 0.(1 - \nu_C(x).\mu_D(x) - \nu_D(x))\rangle | x \in E\}$$

$$= \{\langle x, \mu_A(x).\mu_D(x), \nu_A(x).\nu_D(x)\rangle | x \in E\} = G_A(D).$$

The assertions (b) - (j) are proved analogously.

Theorem 2: For every four IFSs A, B, C and D over universe E,

(a) $X_{A,B,C}(D)$ and $x_{A,B,C}(D)$ are IFSs,

(b) $\overline{X_{A,B,C}(\overline{D})} = X_{\overline{B},\overline{A},\overline{C}}(D)$,

(c) $\overline{x_{A,B,C}(\overline{D})} = x_{\overline{B},\overline{A},\overline{C}}(D)$.

Theorem 3: For every four IFSs A, B, C and D over universe E,
(a) $X_{A,B,C}(x_{A,B,C}(D)) = x_{A,B,C}(X_{\overline{B},\overline{A},\overline{C}}(D))$,

(b) $x_{A,B,C}(X_{A,B,C}(D)) = X_{A,B,C}(x_{\overline{B},\overline{A},\overline{C}}(D))$.

More generally, the previous assertion has the form

Theorem 4: For every seven IFSs A, B, C, U, V, W and D over universe E,

$$X_{A,B,C}(x_{U,V,W}(D)) = x_{A,B,C}(X_{\overline{V},\overline{U},\overline{W}}(D)).$$

Proof. Let the seven IFSs A, B, C, U, V, W, D over universe E be given. Then

$$X_{A,B,C}(x_{U,V,W}(D))$$

$$= X_{A,B,C}(\{\langle x, \mu_U(x).\nu_D(x) + \mu_V(x).(1 - \nu_D(x) - \mu_W(x).\mu_D(x)),$$

$$\nu_V(x).\mu_D(x) + \nu_U(x).(1 - \nu_W(x).\nu_D(x) - \mu_D(x))\rangle | x \in E\})$$

$$= \{\langle x, \mu_A(x).(\mu_U(x).\nu_D(x) + \mu_V(x).(1 - \nu_D(x) - \mu_W(x).\mu_D(x)))$$

$$+\mu_B(x).(1 - \mu_U(x).\nu_D(x) - \mu_V(x).(1 - \nu_D(x) - \mu_W(x).\mu_D(x))$$

$$-\mu_C(x).(\nu_V(x).\mu_D(x) + \nu_U(x).(1 - \nu_W(x).\nu_D(x) - \mu_D(x))),$$

$$\nu_B(x).(\nu_V(x).\mu_D(x) + \nu_U(x).(1 - \nu_W(x).\nu_D(x) - \mu_D(x)))$$

$$+\nu_A(x).(1 - \nu_C(x).(\mu_U(x).\nu_D(x) + \mu_V(x).(1 - \nu_D(x) - \mu_W(x).\mu_D(x)))$$

$$-\nu_V(x).\mu_D(x) + \nu_U(x).(1 - \nu_W(x).\nu_D(x) - \mu_D(x))\rangle | x \in E\})$$

$$= \{\langle x, \mu_A(x).(\mu_U(x).\nu_D(x) + \mu_V(x).(1 - \mu_W(x).\mu_D(x) - \nu_D(x)))$$

$$+\mu_B(x).(1 - \mu_U(x).\nu_D(x) - \mu_V(x).(1 - \nu_D(x) - \mu_W(x).\mu_D(x))$$

$$-\mu_C(x).(\nu_V(x).\mu_D(x) + \nu_U(x).(1 - \nu_W(x).\nu_D(x) - \mu_D(x))),$$

$$= x_{A,B,C}(\{\langle x, \nu_V(x).\mu_D(x) + \nu_U(x).(1 - \mu_D(x) - \nu_W(x).\nu_D(x)),$$

$$\mu_U(x).\nu_D(x) + \mu_V(x).(1 - \mu_W(x).\mu_D(x) - \nu_D(x))\rangle | x \in E\})$$

$$= x_{A,B,C}(X_{\overline{V},\overline{U},\overline{W}}(D)).$$

Theorem 5: For every five IFSs A, B, C, P, Q over universe E,
(a) $X_{A,B,C}(P \cup Q) \supseteq X_{A,B,C}(P) \cup X_{A,B,C}(Q)$,
(b) $X_{A,B,C}(P \cap Q) \subseteq X_{A,B,C}(P) \cap X_{A,B,C}(Q)$,

(c) $X_{A,B,C}(P@Q) = X_{A,B,C}(P)@X_{A,B,C}(Q)$,

(d) $x_{A,B,C}(P \cup Q) \supseteq x_{A,B,C}(P) \cup x_{A,B,C}(Q)$,

(e) $x_{A,B,C}(P \cap Q) \subseteq x_{A,B,C}(P) \cap x_{A,B,C}(Q)$,

(f) $x_{A,B,C}(P@Q) = x_{A,B,C}(P)@x_{A,B,C}(Q)$,

Proof. (c) Let the five IFSs A, B, C, P, Q over universe E are given. Then

$$X_{A,B,C}(P@Q)$$

$$= X_{A,B,C}(\{\langle x, \frac{\mu_P(x)+\mu_Q(x)}{2}, \frac{\nu_P(x)+\nu_Q(x)}{2}\rangle | x \in E\})$$

$$= \{\langle x, \mu_A(x).\frac{\mu_P(x)+\mu_Q(x)}{2} + \mu_B(x).(1 - \frac{\mu_P(x)+\mu_Q(x)}{2} - \mu_C(x).\frac{\nu_P(x)+\nu_Q(x)}{2}),$$

$$\nu_B(x).\frac{\nu_P(x)+\nu_Q(x)}{2} + \nu_A(x).(1 - \nu_C(x).\frac{\mu_P(x)+\mu_Q(x)}{2} - \frac{\nu_P(x)+\nu_Q(x)}{2}))\rangle | x \in E\})$$

$$= \{\langle x, \frac{1}{2}.((\mu_A(x).\mu_P(x) + \mu_B(x).(1 - \mu_P(x) - \mu_C(x).\nu_P(x))$$

$$+(\mu_A(x).\mu_Q(x) + \mu_B(x).(1 - \mu_Q(x) - \mu_C(x).\nu_Q(x)),$$

$$\frac{1}{2}.((\nu_B(x).\nu_P(x) + \nu_A(x).(1 - \nu_C(x).\mu_P(x) - \nu_P(x))$$

$$+(\nu_B(x).\nu_Q(x) + \nu_A(x).(1 - \nu_C(x).\mu_Q(x) - \nu_Q(x))\rangle | x \in E\})$$

$$\{\langle x, \mu_A(x).\mu_P(x) + \mu_B(x).(1 - \mu_P(x) - \mu_C(x).\nu_P(x)),$$

$$\nu_B(x).\nu_P(x) + \nu_A(x).(1 - \nu_C(x).\mu_P(x) - \nu_P(x))\rangle | x \in E\}$$

$$@\{\langle x, \mu_A(x).\mu_Q(x) + \mu_B(x).(1 - \mu_Q(x) - \mu_C(x).\nu_Q(x)),$$

$$\nu_B(x).\nu_Q(x) + \nu_A(x).(1 - \nu_C(x).\mu_Q(x) - \nu_Q(x))\rangle | x \in E\}$$

$$= X_{A,B,C}(P)@X_{A,B,C}(Q).$$

The rest assertions are proved analogously.

3 Some open problems

We finish with some open problems.

It was proved (see, e.g., [1, 2]) that for every IFS A and for every α, β, γ, $\delta \in [0, 1]$ such that, $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$, we have

$$F_{\alpha,\beta}(F_{\gamma,\delta}(A)) = F_{\alpha+\gamma-\alpha.\gamma-\alpha.\delta,\beta+\delta-\beta.\gamma-\beta.\delta}(A).$$

Similarly, if $\alpha, \beta, \gamma, \delta \in$ [0,1], then $G_{\alpha,\beta}(G_{\gamma,\delta}(A)) = G_{\alpha.\gamma,\beta.\delta}(A)$, but the same equalities are not valid for the rest operators from the same type. The same is the situation for expressions $H_B(H_C(A)), H^*_B(H^*_C(A)), J_B(J_C(A)), J^*_B(J^*_C(A))$. The following open problems can be formulated and are interesting to solve

Open Problem 1: Can the six operators be modified to the form $F_{A,B}(C), G_{A,B}(C)$, $H_{A,B}(C)$, $H^*_{A,B}(C)$, $J_{A,B}(C)$, $J^*_{A,B}(C)$? Which properties will these new operators have?

Open Problem 2: Will operators $G_{A,B}(C), H_{A,B}(C)$, $H^*_{A,B}(C)$, $J_{A,B}(C)$, $J^*_{A,B}(C)$ be total (not partial!) extensions of operators $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, $J^*_{\alpha,\beta}$?

Open Problem 3: What form should the X-operator have in order to totally (not partially!) extend operator $X_{a,b,c,d,e,f}$?

These open problems can be formulated for the operators $f, g, ..., j^*$ and x, too.

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