Notes on Intuitionistic Fuzzy Sets
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More on Intuitionistic fuzzy relations

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Abstract: In this paper, we extend the intuitionistic fuzzy relation defined on a crisp set to an intuitionistic fuzzy relation defined on an intuitionistic fuzzy set. Various properties like symmetry, reflexivity, transitivity etc. are studied.
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1 Introduction

A fuzzy binary relation is considered as a fuzzy subset of the set $A \times B$ where A and B are two crisp sets. In [1], a generalization of fuzzy relations was introduced and their properties were studied. Intuitionistic fuzzy sets, defined by K. Atanassov, helps us to model uncertainty with an additional degree. Intuitionistic Fuzzy Relations (IFRs) has already been studied by many researchers. Commonly, IFRs are intuitionistic fuzzy sets in a cartesian product of universes [3]. Here an attempt is made to extend IFRs to a relation between two intuitionistic fuzzy sets.

The notion of generalized IFRs is introduced in section 2. *i.e.*, IFR defined on IFS. Then various binary and unary operations of these relations are defined and symmetry, reflexivity and transitivity are studied in section 3. Throughout this paper, unless otherwise stated, by a relation, we mean an intuitionistic fuzzy binary relation defined on IFSs over the universe U.

Definition 1.1 [4] Let X be an ordinary (non fuzzy) set. An intuitionistic fuzzy set A in X is given by

$$A = \{(x, \mu_A(x), \nu_A(x))/x \in X\}$$

where $\mu_A : X \to [0, 1]$, $\nu_A : X \to [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

2 Relations on intuitionistic fuzzy sets

Let U be any nonempty set and A, B be IFS in U given by the membership function μ_A and μ_B respectively and the nonmembership functions ν_A and ν_B respectively, where

$$\mu_A, \mu_B, \nu_A, \nu_B : U \to [0, 1].$$

$A \times B$ is the IFS in $U \times U$ defined by

$$\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$
$$\nu_{A\times B}(x, y) = \max\{\nu_A(x), \nu_B(y)\}$$

for all $x, y \in U$.

Definition 2.1 Let $R \subseteq A \times B$,

$$\text{i.e.} \quad \mu_R(x, y) \le \mu_{A\times B}(x, y)$$
$$\text{and} \quad \nu_R(x, y) \ge \nu_{A\times B}(x, y)$$

with the condition that

$$0 \le \mu_R(x, y) + \nu_R(x, y) \le 1.$$

Then R is an IFR from A to B.

Definition 2.2 Let R, R_1, R_2 be IFRs from A to B. Then $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 + R_2$, $R_1 \cdot R_2$, $R_1 \bigcup R_2$, $R_1 \bigcap R_2$, $R_1 \odot R_2$, $R_1 \otimes R_2$, $\bar{R}$ and R^{-1} are defined as follows:

1. $\mu_{R_1\cup R_2}(x, y) = \max\{\mu_{R_1}(x, y), \mu_{R_2}(x, y)\}$
   $\nu_{R_1\cup R_2}(x, y) = \min\{\nu_{R_1}(x, y), \nu_{R_2}(x, y)\}$

2. $\mu_{R_1\cap R_2}(x, y) = \min\{\mu_{R_1}(x, y), \mu_{R_2}(x, y)\}$
   $\nu_{R_1\cap R_2}(x, y) = \max\{\nu_{R_1}(x, y), \nu_{R_2}(x, y)\}$

3. $\mu_{R_1+R_2}(x, y) = \mu_{R_1}(x, y) + \mu_{R_2}(x, y) - \mu_{R_1}(x, y)\mu_{R_2}(x, y)$,
   $\nu_{R_1+R_2}(x, y) = \nu_{R_1}(x, y)\nu_{R_2}(x, y)$

4. $\mu_{R_1\cdot R_2}(x, y) = \mu_{R_1}(x, y)\mu_{R_2}(x, y)$
   $\nu_{R_1\cdot R_2}(x, y) = \nu_{R_1}(x, y) + \nu_{R_2}(x, y) - \nu_{R_1}(x, y)\nu_{R_2}(x, y)$

5. $\mu_{R_1\bigcup R_2}(x, y) = \min\{1, \mu_{R_1}(x, y) + \mu_{R_2}(x, y)\}$
   $\nu_{R_1\bigcup R_2}(x, y) = \max\{0, \nu_{R_1}(x, y) + \nu_{R_2}(x, y) - 1\}$

6. $\mu_{R_1\bigcap R_2}(x, y) = \max\{0, \mu_{R_1}(x, y) + \mu_{R_2}(x, y) - 1\}$
   $\nu_{R_1\bigcap R_2}(x, y) = \min\{1, \nu_{R_1}(x, y) + \nu_{R_2}(x, y)\}$

7. $\mu_{R_1\odot R_2}(x, y) = \frac{\mu_{R_1}(x,y)+\mu_{R_2}(x,y)}{2}$
   $\nu_{R_1\odot R_2}(x, y) = \frac{\nu_{R_1}(x,y)+\nu_{R_2}(x,y)}{2}$

8. $\mu_{R_1 \otimes R_2}(x,y) = \sqrt{\mu_{R_1}(x,y)\mu_{R_2}(x,y)}$
   $\nu_{R_1 \otimes R_2}(x,y) = \sqrt{\nu_{R_1}(x,y)\nu_{R_2}(x,y)}$

9. $\mu_{\bar{R}}(x,y) = \min\{1-\mu_R(x,y), \mu_{A\times B}(x,y)\}$
   $$\nu_{\bar{R}}(x,y) = \begin{cases} \max\{1-\nu_R(x,y), \nu_{A\times B}(x,y)\} = C(x,y), \\ \qquad\qquad\qquad \text{if } 0 \le \mu_{\bar{R}}(x,y) + C(x,y) \le 1 \\ \mu_R(x,y) \qquad \text{if } \mu_{\bar{R}}(x,y) + C(x,y) > 1 \end{cases}$$

10. $\mu_{R^{-1}}(x,y) = \mu_R(y,x)$
    $\nu_{R^{-1}}(x,y) = \nu_R(y,x)$ for all $x, y \in U$.

Note 1. All the above definitions are intuitionistic fuzzy relations on intuitionistic fuzzy sets.
Note 2. If A and B are ordinary subsets of U, then

$$\mu_{\bar{R}}(x,y) = \begin{cases} 1-\mu_R(x,y), & \text{if } (x,y) \in A \times B \\ 0, & \text{if } (x,y) \notin A \times B \end{cases}$$

$$\nu_{\bar{R}}(x,y) = \begin{cases} 1-\nu_R(x,y), & \text{if } (x,y) \in A \times B \\ 1, & \text{if } (x,y) \notin A \times B \end{cases}$$

Notation

We use the following matrix representation for IFS in $U \times U$. If the universal set $U = \{a_1, a_2, \dots, a_n\}$ and if G is an IFS in $U \times U$ with membership function μ_G and nonmembership function ν_G, then G is represented as

$$G : \begin{pmatrix} (\mu_G(a_1,a_1), \nu_G(a_1,a_1)) & (\mu_G(a_2,a_1), \nu_G(a_2,a_1)) & \cdots (\mu_G(a_n,a_1), \nu_G(a_n,a_1)) \\ (\mu_G(a_1,a_2), \nu_G(a_1,a_2)) & (\mu_G(a_2,a_2), \nu_G(a_2,a_2)) & \cdots (\mu_G(a_n,a_2), \nu_G(a_n,a_2)) \\ \cdots & \cdots & \cdots \\ (\mu_G(a_1,a_n), \nu_G(a_1,a_n)) & (\mu_G(a_2,a_n), \nu_G(a_2,a_n)) & \cdots (\mu_G(a_n,a_n), \nu_G(a_n,a_n)) \end{pmatrix}$$

Example 2.1 Let $U = \{a, b\}$ and A, B be given by

$$A = \{(a, 0.1, 0.3), (b, 0.6, 0.2)\}$$
$$B = \{(a, 0.8, 0.1), (b, 0.3, 0.7)\}$$

Then

$$A \times B : \begin{pmatrix} (0.1, 0.3) & (0.6, 0.2) \\ (0.1, 0.7) & (0.3, 0.7) \end{pmatrix}$$

Let R_1, R_2 be two relations from A to B defined by

$$R_1 : \begin{pmatrix} (0.1, 0.4) & (0.5, 0.3) \\ (0.01, 0.8) & (0.2, 0.7) \end{pmatrix} \qquad R_2 : \begin{pmatrix} (0.05, 0.5) & (0.5, 0.3) \\ (0.1, 0.8) & (0.2, 0.7) \end{pmatrix}.$$

Then $R_1 \cap R_2$ is a relation from A to B defined by

$$R_1 \cap R_2 : \begin{pmatrix} (0.05, 0.5) & (0.5, 0.3) \\ (0.01, 0.8) & (0.2, 0.7) \end{pmatrix}$$

and $\bar{R}_1$ is a relation from A to B defined by

$$\bar{R}_1 : \begin{pmatrix} (0.1, 0.6) & (0.5, 0.5) \\ (0.1, 0.7) & (0.3, 0.7) \end{pmatrix}$$

Definition 2.3 The composition $\circ$ of two IFRs, R_1 and R_2 is defined by

$$\mu_{R_1 \circ R_2}(x, y) = \max_{z \in U}[\min(\mu_{R_1}(x, z), \mu_{R_2}(z, y))] \text{ and}$$
$$\nu_{R_1 \circ R_2}(x, y) = \min_{z \in U}[\max(\nu_{R_1}(x, z), \nu_{R_2}(z, y))]$$

where R_1 is a relation from A to B and R_2 is a relation from B to C.

Theorem 2.1 *Let R_1 be a relation from A to B and R_2 a relation from B to C, then $R_1 \circ R_2$ is a relation from A to C [8].*

3 Symmetry, reflexivity and transitivity

Definition 3.1 An IFR R on IFS A is symmetric if

$$\mu_R(x, y) = \mu_R(y, x) \quad \text{and} \quad \nu_R(x, y) = \nu_R(y, x) \text{ for all } x, y \in U.$$

Theorem 3.1 *If R is symmetric, then so is R^{-1}.*

Proof 3.1

$$\mu_{R^{-1}}(x, y) = \mu_R(y, x) = \mu_R(x, y) = \mu_{R^{-1}}(y, x)$$
$$\nu_{R^{-1}}(x, y) = \nu_R(y, x) = \nu_R(x, y) = \nu_{R^{-1}}(y, x)$$

for all $x, y \in U$.

Theorem 3.2 *R is symmetric if and only if $R = R^{-1}$.*

Proof 3.2 *Let R be symmetric. Then*

$$\mu_{R^{-1}}(x, y) = \mu_R(y, x) = \mu_R(x, y)$$
$$\nu_{R^{-1}}(x, y) = \nu_R(y, x) = \nu_R(x, y) \quad \textit{for all } x, y \in U.$$

So, $R^{-1} = R$.
Conversely, let $R^{-1} = R$

$$\mu_R(x, y) = \mu_{R^{-1}}(x, y) = \mu_R(y, x)$$
$$\nu_R(x, y) = \nu_{R^{-1}}(x, y) = \nu_R(y, x)$$

Theorem 3.3 *If R_1 and R_2 are symmetric IFRs on an IFS A, then $R_1 * R_2$ is also symmetric on A.*

Proof follows immediately from the definitions.
* could be anyone of $\cup, \cap, +, \cdot, \bigcup, \bigcap \odot, \otimes$
Note. $R_1 \circ R_2$ is not in general symmetric as is obvious from the definition. The following theorem gives the condition for it being symmetric. The proof is analogous to that in [1].

Theorem 3.4 *If R_1 and R_2 are symmetric relations on A, then $R_1 \circ R_2$ is symmetric on A if, and only if, $R_1 \circ R_2 = R_2 \circ R_1$.*

Corollary. R^n is symmetric for all positive integer n if R is symmetric. (R^n is $R \circ R \cdots \circ R$ n times)

Definition 3.2 An IFR R on A is reflexive of order (α, β) if $\mu_R(x, x) = \alpha$ and $\nu_R(x, x) = \beta$ for all $x \in U$ such that $\mu_A(x) \neq 0$ and $\nu_A(x) \neq 1$.

Note.

1. Clearly $0 \leq \alpha + \beta \leq 1$

2.
$$\mu_{R^{-1}}(x, x) = \mu_R(x, x) = \alpha,$$
$$\nu_{R^{-1}}(x, x) = \nu_R(x, x) = \beta.$$

   So R^{-1} is reflexive of order (α, β)

3. If $\alpha = 1$, $\beta = 0$, IFS A reduces to an ordinary set.

Theorem 3.5 *If R_1 and R_2 are reflexive IFRs on IFS A of orders (α, γ) and (β, δ) respectively, then R_1^{-1}, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 + R_2$, $R_1 \cdot R_2$, $R_1 \bigcup R_2$, $R_1 \bigcap R_2$, $R_1 \odot R_2$, $R_1 \otimes R_2$, are reflexive of orders (α, γ), $(\max[\alpha, \beta], \min[\gamma, \delta])$, $(\min[\alpha, \beta], \max[\gamma, \delta])$, $(\alpha + \beta - \alpha\beta, \gamma\delta)$, $(\alpha\beta, \gamma + \delta - \gamma\delta)$, $(\min[1, \alpha+\beta], \max[0, \gamma+\delta-1])$, $(\max[0, \alpha+\beta-1], \min[1, \gamma+\delta])$, $(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2})$ and $(\sqrt{\alpha\beta}, \sqrt{\gamma\delta})$, respectively*

Proof follows from the respective definitions.
Note. $R_1 \circ R_2$ and $\bar{R}_1$ are not reflexive. See [1].

Definition 3.3 Let R be an IFR on IFS A. Then R is transitive if $R \circ R \subseteq R$

Theorem 3.6 *If R is a transitive relation, then so is R^{-1}.*

Proof 3.3 *$\mu_{R^{-1}}(x,y) \geq \mu_{R^{-1}\circ R^{-1}}(x,y)$ as in [1].*

$$\begin{aligned}
\nu_{R^{-1}}(x,y) &= \nu_R(y,x) \\
&\leq \nu_{R\circ R}(y,x) \\
&= \min_{Z\in U}[\max(\nu_R(y,z),\nu_R(z,x))] \\
&= \min_{Z\in U}[\max(\nu_{R^{-1}}(x,z),\nu_{R^{-1}}(z,y))] \\
&= \nu_{R^{-1}\circ R^{-1}}(x,y)
\end{aligned}$$

So, $R^{-1}\circ R^{-1} \subseteq R^{-1}$.

Hence the theorem.

Lemma 3.1 *If $\Phi$and Ψ are mappings from U to $[0,1]$, then*

$$\min_{Z\in U}\big\{\max[\Psi(z),\Phi(z)]\big\} \geq \max\{\min_{Z\in U}\Psi(z),\min_{Z\in U}\phi(z)\}.$$

Proof 3.4 *For one particular z,*
$\min_{Z\in U}\Psi(z) \leq \Psi(z)$
$\min_{Z\in U}\Phi(z) \leq \Phi(z)$
$\max\{\min_{Z\in U}\Psi(z),\min_{Z\in U}\Phi(z)\} \leq \max\{\Psi(z),\Phi(z)\}$
$\max\{\Psi(z),\Phi(z)\} \geq \max\{\min_{Z\in U}\Psi(z),\min_{Z\in U}\Phi(z)\}$
R.H.S. is a fixed quantity. So,

$$\min_{Z\in U}\{\max[\Psi(z),\Phi(z)]\} \geq \max\{\min_{Z\in U}\Psi(z),\min_{Z\in U}\Phi(z)\}$$

Hence the lemma.

Theorem 3.7 *If R_1 and R_2 are transitive on A, then so is $R_1\cap R_2$.*

Proof 3.5 *$\mu_{R_1\cap R_2}(x,y) \geq \mu_{(R_1\cap R_2)\circ(R_1\cap R_2)}(x,y)$, see [1].*

$$\begin{aligned}
\nu_{R_1\cap R_2}(x,y) &= \max\{\nu_{R_1}(x,y),\nu_{R_2}(x,y)\} \\
&\leq \max\{\nu_{{R_1}^2}(x,y),\nu_{R_2^2}(x,y)\} \\
&= \max\{\min_{Z\in U}[\max(\nu_{R_1}(x,z),\nu_{R_1}(z,y))],\min_{Z\in U}[\max(\nu_{R_2}(x,z),\nu_{R_2}(z,y))]\} \\
&\leq \min_{Z\in U}[\max\{\max(\nu_{R_1}(x,z),\nu_{R_1}(z,y)),\max(\nu_{R_2}(x,z),\nu_{R_2}(z,y))\}] \\
&\qquad\qquad \textit{by lemma 3.1} \\
&= \min_{Z\in U}[\max\{\max(\nu_{R_1}(x,z),\nu_{R_2}(x,z)),\max(\nu_{R_1}(z,y),\nu_{R_2}(z,y))\}] \\
&= \min_{Z\in U}[\max\{\nu_{R_1\cap R_2}(x,z),\nu_{R_1\cap R_2}(z,y)\}] \\
&= \nu_{(R_1\cap R_2)\circ(R_1\cap R_2)}(x,y)
\end{aligned}$$

Therefore, $R_1\cap R_2$ is transitive.

Note. $R_1 \cup R_2$, $R_1 + R_2$, $R_1 \cdot R_2$, $R_1 \bigcup R_2$, $R_1 \bigcap R_2$ are not transitive in general even if R_1 and R_2 are transitive. [1, 2]

Theorem 3.8 *If R_1 and R_2 are transitive relations on an IFS A, then $R_1 \odot R_2$ and $R_1 \otimes R_2$ are not necessarily transitive.*

Proof 3.6 *This will be proved by an example.*
Let $U = \{a, b, c\}$ and A be given by

$$A = \{(a, 0.9, 0.1), (b, 0.9, 0.05), (c, 0, 1)\}.$$

Then

$$A \times A : \begin{pmatrix} (0.9, 0.1) & (0.9, 0.1) & (0, 1) \\ (0.9, 0.1) & (0.9, 0.05) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) \end{pmatrix}$$

Let R_1, R_2 be relations on A defined by

$$R_1 : \begin{pmatrix} (0.4, 0.2) & (0.3, 0.3) & (0, 1) \\ (0.8, 0.1) & (0.4, 0.1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) \end{pmatrix},$$

$$R_2 : \begin{pmatrix} (0.4, 0.2) & (0.8, 0.2) & (0, 1) \\ (0.3, 0.3) & (0.4, 0.1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) \end{pmatrix}$$

One can check that $R_1 \odot R_2$ and $R_1 \otimes R_2$ are not transitive.

Theorem 3.9 *If R is transitive, so is R^2.*

Proof 3.7 $\mu_{R \circ R}(x, y) \geq \mu_{R^2 \circ R^2}(x, y)$ *as in [1].*

$$\begin{aligned} \nu_{R \circ R}(x, y) &= \min_{Z \in U}[\max(\nu_R(x, z), \nu_R(z, y))] \\ &\leq \min_{Z \in U}[\max(\nu_{R \circ R}(x, z), \nu_{R \circ R}(z, y))] \\ &= \nu_{R^2 \circ R^2}(x, y) \\ R^2 \circ R^2 &\subseteq R^2. \end{aligned}$$

Hence the theorem.

4 Conclusion

The IFRs presented in this paper are extensions of generalized fuzzy relations defined in [1].

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