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Intuitionistic fuzzy implication $\rightarrow^C$ and negation $\neg^C$

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Abstract: A new type of intuitionistic fuzzy implication $\rightarrow^C$ and negation $\neg^C$ are introduced in intuitionistic fuzzy logic and some of their properties are discussed. These new operations are extensions of the operations $\neg^{\varepsilon,\eta}$ and $\rightarrow^{\varepsilon,\eta}$.
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1 Introduction

In [1, 2] a new type of negations and implications over an Intuitionistic Fuzzy Set (IFS; see [3, 4]) are introduced. Here, we continue to research in this area. We extend the introduced already negations and implications. The used notation for IFS is from [3, 4]. In the second book these operations are described in details.

In [2], the set of IF-negations that has the form

$$\mathcal{N} = \{\neg^{\varepsilon,\eta} \mid 0 \le \varepsilon < 1 \;\&\; 0 \le \eta < 1\},$$

where for each IFS A,

$$\neg^{\varepsilon,\eta} A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle | x \in E\}$$

is constructed and it is proved that the inequality $\varepsilon \le \eta$ is necessity for correctness of $\neg^{\varepsilon,\eta} A$.

There, the implication, generated by the new negation, is constructed, as

$$A \rightarrow^{\varepsilon,\eta} B = \{\langle x, \max(\mu_B(x), \min(1, \nu_A(x) + \varepsilon)),$$

$$\min(\nu_B(x), \max(0, \mu_A(x) - \eta))\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \max(\mu_B(x), \nu_A(x) + \varepsilon)),$$
$$\max(0, \min(\nu_B(x), \mu_A(x) - \eta))\rangle | x \in E\}.$$

Now, we extend these definitions.

2 Main results

Let everywhere below C be an Intuitionistic Fuzzy Topological Set (IFTS), i.e., for every $x \in E$: $\mu_A(x) \geq \nu_A(x)$.

We construct the negation on the basis of C, that has the form for each IFS A,

$$\neg^C A = \{\langle x, \min(1, \nu_A(x) + \nu_C(x)), \max(0, \mu_A(x) - \mu_C(x))\rangle | x \in E\}. \quad (1)$$

Theorem 1. For every IFS A and IFTS C, the set $\neg^C A$ is an IFS.

Proof. Let A be an IFS and C – an IFTS. Then for each $x \in E$, if $\mu_A(x) \leq \mu_C(x)$ then,

$$\min(1, \nu_A(x) + \nu_C(x)) + \max(0, \mu_A(x) - \mu_C(x)) = \min(1, \nu_A(x) + \nu_C(x)) \leq 1;$$

if $\mu_A(x) \geq \mu_C(x)$ then,

$$\min(1, \nu_A(x) + \nu_C(x)) + \max(0, \mu_A(x) - \mu_C(x)) = \min(1, \nu_A(x) + \nu_C(x)) + \mu_A(x) - \mu_C(x)$$
$$\leq \nu_A(x) + \nu_C(x) + \mu_A(x) - \mu_C(x) \leq \nu_A(x) + \mu_A(x) \leq 1.$$

Obviously, in the partial case, when

$$C = \{\langle x, \eta, \varepsilon\rangle | x \in E\}$$

and $\varepsilon + \eta \leq 1$, we obtain the above negation and implication.

In Figure 1, x and $\neg_1 x$ are shown (where, the classical (the first) negation defined over IFSs is marked by $\neg_1$), while in Figures 2 and 3, y and $\neg^C y$ and z and $\neg^C z$ are shown.

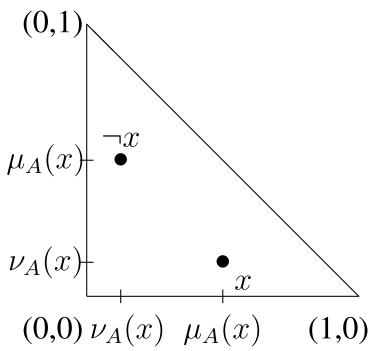

Fig. 1.

Now, by analogy with the above construction, we can construct e new implication, generated by the new negation as

$$A \rightarrow^C B = \{\langle x, \max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))),$$

$$\min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \max(\mu_B(x), \nu_A(x) + \nu_C(x))),$$

$$\max(0, \min(\nu_B(x), \mu_A(x) - \mu_C(x)))\rangle | x \in E\}. \quad (2)$$

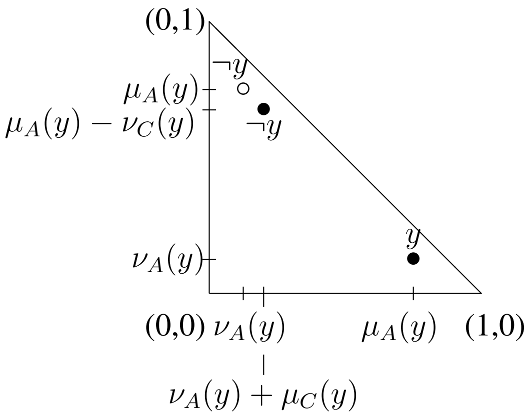

Fig. 2.

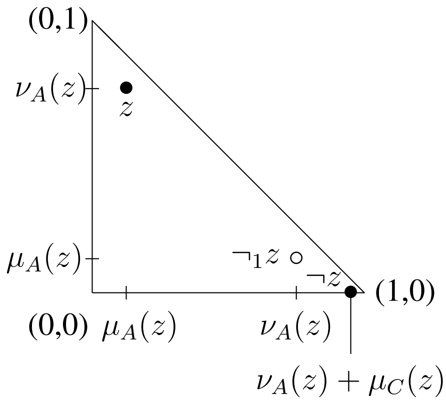

Fig. 3.

Theorem 2. For every two IFSs A, B and IFTS C, the set $A \to^C B$ is an IFS.
Proof. Let A and B be IFSs and C – an IFTS. Let for each $x \in E$, $\mu_A(x) - \mu_C(x) \geq \nu_B(x)$. Then,

$$\nu_A(x) + \nu_C(x) \leq 1 - \mu_A(x) + \mu_C(x) \leq 1 - \nu_B(x)$$

and

$$\mu_B(x) \leq 1 - \nu_B(x).$$

Hence

$$\max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) + \min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))$$

$$\leq \max(\mu_B(x), \min(1, 1 - \nu_B(x))) + \min(\nu_B(x), \mu_A(x) - \mu_C(x))$$

$$\leq \max(\mu_B(x), 1 - \nu_B(x)) + \nu_B(x) = 1.$$

Let for each $x \in E$, $\mu_A(x) - \mu_C(x) < \nu_B(x)$. If $\mu_A(x) \leq \mu_C(x)$, then

$$\max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) + \min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))$$

$$\leq \max(\mu_B(x), 1) + \min(\nu_B(x), 0) = 1 + 0 = 1.$$

If $\mu_A(x) > \mu_C(x)$, then, as above

$$\nu_A(x) + \nu_C(x) \leq 1 - \mu_A(x) + \mu_C(x)$$

and

$$1 - \mu_B(x) \geq \nu_B(x) > \mu_A(x) - \mu_C(x),$$

i.e.,

$$\mu_B(x) \leq 1 - \mu_A(x) + \mu_C(x).$$

Hence

$$\max(\mu_B(x), \min(1, \nu_A(x) + \nu_C(x))) + \min(\nu_B(x), \max(0, \mu_A(x) - \mu_C(x)))$$

$$\leq \max(\mu_B(x), \min(1, 1 - \mu_A(x) + \mu_C(x))) + \mu_A(x) - \mu_C(x)$$

$$= \max(\mu_B(x), 1 - \mu_A(x) + \mu_C(x)) + \mu_A(x) - \mu_C(x)$$

$$1 - \mu_A(x) + \mu_C(x)) + \mu_A(x) - \mu_C(x) = 1.$$

Therefore, $A \rightarrow^C B$ is an IFS.

In [5], George Klir and Bo Yuan introduced the following axioms for implications and negations.

Axiom 1 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z))$.
Axiom 2 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(z, x) \leq I(z, y))$.
Axiom 3 $(\forall y)(I(0, y) = 1)$.
Axiom 4 $(\forall y)(I(1, y) = y)$.
Axiom 5 $(\forall x)(I(x, x) = 1)$.
Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.
Axiom 7 $(\forall x, y)(I(x, y) = 1$ iff $x \leq y)$.
Axiom 8 $(\forall x, y)(I(x, y) = I(N(y), N(x)))$, where N is an operation for a negation.
Axiom 9 I is a continuous function.
Theorem 3. Implication $\rightarrow^C$ and negation $\neg^C$:
(a) satisfy Axioms 1, 2, 3, 6 and 9;
(b) satisfy Axioms 4 and 5 as IFTs, but not as tautologies;
(c) satisfy Axiom 8 in the form
Axiom 8': $(\forall x, y)(I(x, y) \leq I(N(y), N(x)))$.

Theorem 4.: For each IFS A:
(a) $A \cup \neg^C A$ is an IFTS, but not always equal to E^*;
(b) $\neg^C \neg^C A \cup \neg^C A$ is an IFTS, but not always equal to E^*.

Usually, in set theory the De Morgan's Laws have the forms:

$$\neg A \cap \neg B = \neg(A \cup B), \tag{3}$$

$$\neg A \cup \neg B = \neg(A \cap B). \tag{4}$$

or

$$\neg(\neg A \cap \neg B) = A \cup B, \tag{5}$$

$$\neg(\neg A \cup \neg B) = A \cap B. \tag{6}$$

but, as we discussed in [4], they can also have the forms:

$$\neg(\neg A \cap \neg B) = \neg\neg A \cup \neg\neg B, \tag{7}$$

$$\neg(\neg A \cup \neg B) = \neg\neg A \cap \neg\neg B. \tag{8}$$

Theorem 5.: For every two IFSs A and B:

(a) the IFSs from (3) and (4) with negation $\neg^C$ are IFTSs, but not always equal to E^*;

(b) the IFSs from (5) – (8) with negation $\neg^C$ are not always IFTSs or not always equal to E^*.

Theorem 6.: For every IFS A:

$$\neg^C \Box A \supset \Box \neg^C A,$$

$$\neg^C \Diamond A \subset \Diamond \neg^C A.$$

Let us prove, for example, the second inclusion. The rest of the assertions can be proved analogously. Let C be an IFTS. Then,

$$\neg^C \Diamond A = \neg^C \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \nu_A(x) + \nu_C(x)), \max(0, 1 - \nu_A(x) - \mu_C(x))\rangle | x \in E\}.$$

$$\Diamond \neg^C A = \Diamond \{\langle x, \min(1, \nu_A(x) + \nu_C(x)), \max(0, \mu_A(x) - \mu_C(x))\rangle | x \in E\}$$

$$= \{\langle x, 1 - \max(0, \mu_A(x) - \mu_C(x)), \max(0, \mu_A(x) - \mu_C(x))\rangle | x \in E\}.$$

Let

$$X \equiv 1 - \max(0, \mu_A(x) - \mu_C(x)) - \min(1, \nu_A(x) + \nu_C(x)).$$

If $\nu_A(x) + \nu_C(x) \geq 1$, then

$$\mu_A(x) - \mu_C(x) \leq 1 - \nu_A(x) - \mu_C(x) \leq \nu_C(x) - \mu_C(x) \leq 0$$

and

$$X = 1 - 1 - 0 = 0.$$

If $\nu_A(x) + \nu_C(x) \leq 1$, then there are two subcases. If $\mu_A(x) - \mu_C(x) \leq 0$, then

$$X = 1 - (\nu_A(x) + \nu_C(x)) - 0 \geq 0$$

and if $\mu_A(x) - \mu_C(x) \geq 0$, then

$$X = 1 - (\nu_A(x) + \nu_C(x)) - \mu_A(x) + \mu_C(x) = 1 - \mu_A(x) - \mu_A(x) + \mu_C(x) - \nu_C(x) \geq 0.$$

Therefore, the first component of the second term is higher than the first component of the first term, while the inequality

$$\max(0, 1 - \nu_A(x) - \mu_C(x)) - \max(0, \mu_A(x) - \mu_C(x)) \geq 0$$

is obvious. Therefore, the inclusion is valid.

Theorem 7. For every IFS A and IFTS C, for every two real numbers α, β, so that $0 \leq \alpha, \beta \leq 1$

(a) $\neg^C G_{\alpha,\beta}(A) \supseteq G_{\beta,\alpha}(\neg^C A)$,

(b) $\neg^C H_{\alpha,\beta}(A) \supseteq H_{\beta,\alpha}(\neg^C A)$,

(c) $\neg^C J_{\alpha,\beta}(A) \subseteq J_{\beta,\alpha}(\neg^C A)$,

(d) $\neg^C H^*_{\alpha,\beta}(A) \supseteq H^*_{\beta,\alpha}(\neg^C A)$,

(e) $\neg^C J^*_{\alpha,\beta}(A) \subseteq J^*_{\beta,\alpha}(\neg^C A)$,

(f) $\neg^C P_{\alpha,\beta}(A) \subseteq P_{\alpha,\beta}(\neg^C A)$,

(g) $\neg^C Q_{\alpha,\beta}(A) \supseteq Q_{\alpha,\beta}(\neg^C A)$.

Theorem 8. For every IFSs A, B and for every IFTS C

(a) $\neg^C G_B(A) \supseteq G_{\neg B}(\neg^C A)$,

(b) $\neg^C H_B(A) \supseteq H_{\neg B}(\neg^C A)$,

(c) $\neg^C J_B(A) \subseteq J_{\neg B}(\neg^C A)$,

(d) $\neg^C H^*_B(A) \supseteq H^*_{\neg B}(\neg^C A)$,

(e) $\neg^C J^*_B(A) \subseteq J^*_{\neg B}(\neg^C A)$,

(f) $\neg^C P_B(A) \subseteq P_B(\neg^C A)$,

(g) $\neg^C Q_B(A) \supseteq Q_B(\neg^C A)$.

3 Conclusion

In the paper, two new operations: a negation and an implication, were introduced. The implication is based on the new negation, but in general, has the classical form. In a next author's research some non-classical forms of the implication $\rightarrow^C$ will be discussed.

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