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Some properties of Controlled Set Theory

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Abstract: We have defined a new set theory which is called Controlled Set Theory. We have shown that these sets have fundamental properties of mathematics. In this paper, firstly, we defined $(\alpha, \alpha^*)-$Controlled sets which are special type of intuitionistic fuzzy sets and examined the relationship between Controlled Sets and some structures.
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1 Introduction

In order to talk about the opposite of a proposition, we must choose an appropriate universe. If we choose the universe of propositions as $\Omega = \{p\}$ then we can not say anything about the truth value of p. If we can then p' must be itself.

If we consider Zadeh's example in [5] about the length of a person, the membership degree of one whose length 170 cm is almost 1, say 0.8. But, if we choose the universe as

$$U = \{x | \text{the length of the person } \geqslant 170\text{cm}\}$$

then the membership degree of the person 171 cm tall is subject to discussion. In the above universe U, the membership degree of the person 190 cm tall can be almost 1. Can we say that the membership degree of the person 171cm tall is 0.8? In Zadeh's example $\mu(171) = 0.8$, $\mu'(171) = 0.2$. If we study in the universe U then we are in the expectations that the universe should have an element which membership degree is 0.2. Hence, the membership degree of the persons who are 171cm tall is 0.8. This is a contradiction. In this statement, the best solution can be to assign a membership degree of 0.5 to the persons who are 171 cm tall. Also in this case, the membership degree of the persons 180cm tall is 0.9 thus the non-membership degree of the persons 180cm tall is 0.1. This is a contradiction, too.

This problem can be solved by defining a bijective and order-preserving function between the image of fuzzy set on Zadeh's universe, as above, and the image of fuzzy set on subset of Zadeh's same universe. However, in this case, problems associated with the concept appear to be greater. So, in order to find any element in the universe with its membership and non-membership degree, there must also exist an element such that it controls the other. With this idea, if we use Zadeh's universe for the tall people, then the membership degree of the persons who are 171 cm tall is 0.8 and thus the non-membership degree of the persons who are 171 cm tall is 0.2. But, if we use the subuniverse of the Zadeh's universe, then the membership degree of the persons who are 171cm tall is 0.8 and thus the non-membership degree of the persons who are 171cm tall is 0.0. This is so, because there is no element that controls the non-membership degree of the persons who are 171cm tall.

An extension of the fuzzy theory is the Intuitionistic fuzzy theory [1] which has the hesitation degree that is not present the fuzzy theory. But, there are same problems in Intuitionistic fuzzy theory like in Fuzzy theory. This is so, because there is no criterion for non-membership degree of an element. For example, the set $A = \{(x, 0.8, 0.2), (y, 0.4, 0.3)\}$ is an intuitionistic fuzzy set on $U = \{x, y\}$. But, there is no controlling element for x like y.

In addition to the representation of a set, we use the characteristic functions for sets. The characteristic function is a function that is defined for a set A by $\chi_A = U \to \{0, 1\}$

$$\chi_A(a) = \begin{cases} 1, a \in A \\ 0, a \notin A \end{cases}$$

From this definition, A can be represented by

$$\chi_A(A) = \{(a, \chi_A(a), 1 - \chi_A(a)) \mid a \in A\}.$$

For example: let $U = \{a, b, c, d\}$ and $A = \{a, b\}$, then we represent A with

$$A = \{(a, 1, 0), (b, 1, 0), (c, 0, 1), (d, 0, 1)\}.$$

Now, we must ask the fundamental question:

"How can we say the value of $\chi_A(a) = 1$ or $\chi_A(a) = 0$?" $(*)$

The answer is clear for the set A because, in mathematical logic, the opposite of any proposition is a proposition and it is element of the set of propositions, [2].

From this discussion, if we ask the question (*) for the universe set, How can we say that all of $a \in U$ the value of $\chi_A(a) = 1$. Because, we do not have any information that it is not element of set U.

2 $\alpha-$Controlled sets

Definition 1 ([4]). *Let E be an universe, α is a function from E to I then E is called α-set.*

From the definition, if X is a universe and μ is a fuzzy set on X then X is a $\mu-$set.

Definition 2 ([4]). *Let E be an α-set. The set E is called α-controlled set if*

$$\forall x \in E, \exists y \in E \ni 1 - \alpha(x) = \alpha(y).$$

The family of α-controlled set on a universe E is represented by $E \in CS(\alpha)$.

Definition 3 ([4]). *Let $E \in CS(\alpha)$ and $a \in E$. The following set is called control set of a,*

$$\overline{a} = \{b \in E | 1 - \mu(a) = \mu(b)\}$$

Theorem 1 ([4]). *Let $E \in CS(\alpha)$ and $a, b \in E$. Then the following hold.*

1. $a \in \overline{a} \iff \mu(a) = 1/2$;
2. $a \in \overline{b} \iff b \in \overline{a}$;
3. *If* $\overline{a} \neq \overline{b}$ *then* $x \in \overline{a} \iff x \notin \overline{b}$;
4. *If* $\overline{a} \cap \overline{b} \neq \varnothing$ *then* $\overline{a} = \overline{b}$.

From this theorem we can say easily that there is not any equivalence relation on E such that the control sets of the elements are equivalence class of it. Because, any element may not belong to its class.

Proposition 1 ([4]). *Let E be an α-set and $C = \{X \subset E | X \in CS(\alpha)\}$. Then,*

1. $\cup_{X \epsilon C' \subset C} X \in CS(\alpha)$.
2. The chain of α-controlled sets $X_1 \subset X_2 \subset X_3 \subset \cdots$ is finite.
3. The class C have maximal element property.

Proposition 2 ([4]). *Let G be a group, α is a fuzzy group on G such that $G \in CS(\alpha)$ then*

$$\overline{e}_G = \{c \in G | \mu(c) = 0\}$$

Proof. The proof is clear. Because, if α is a fuzzy group, then $\alpha(e) = 1$. □

Corollary 1 ([4]). *Let G be a group, α is a fuzzy group on G and $G \in CS(\alpha)$. If the set A defined by $A = \{a \in G : \widetilde{a} = \widetilde{e}\}$ then $G - A$ is a group.*

If we use the fuzzy group definition and remember the level subsets of fuzzy groups then we see that the lattice of the subgroups which are get from the level sets is a chain. But from the above proposition then we get another subgroup of G such that the family of them are not chain.

Corollary 2 ([4]). *Let G be a cyclic group, μ is a fuzzy group on G and $G \in CS(\alpha)$ then,*

$$\overline{e} = \{a^n | \exists m \in \mathbb{N} \ni a^{m+1} \neq e, a^m = e, 0 < n < m+1\}$$

Corollary 3 ([4]). *Let G be a group, α is a fuzzy group on G and $G \in CS(\alpha)$ then for every $a \in G - \{e\}$,*

$$\widetilde{a} \neq \widetilde{e}$$

In the same way, we obtain $\alpha_{\theta^{(1)}} \subseteq \alpha_{\theta^{(2)}}$ as well. Therefore $\alpha_{\theta^{(1)}} = \alpha_{\theta^{(2)}}$ i.e. $\theta^{(1)} =_A \theta^{(2)}$.

3 $(\alpha, \alpha^*)-$Controlled sets

The membership degree is very important for an element in any set. But the non-membership degree is very important, too. We can not claim that all sets are controlled set. But, we can introduce controlled set using the membership degrees of elements. So, we can give the following definition

Definition 4. *Let E be an α-set. We define the following mapping on E so that*

$$\alpha^*(x) = \begin{cases} 1-\alpha(x), & x \in E_\alpha \\ \sup\limits_y \alpha(y), & y \in E \ni \alpha(x) < 1-\alpha(y) \\ 0, & \textit{otherwise}. \end{cases}$$

where $E_\alpha = \bigcup\limits_{a \in E} \overline{a}$.

It is clear that α^* is a mapping from E to I. In addition, it can be easily seen that the sum of α and α^* is less or equal than 1. From this properties, we can give the following definition

Definition 5. *Let E be α-set. Then the set $A = \{\langle x, \alpha(x), \alpha^*(x)\rangle \, | x \in E\}$ is called $(\alpha, \alpha^*)-$controlled set.*

From the definition, it can be easily seen that every $(\alpha, \alpha^*)-$controlled set is intuitionistic fuzzy set. But the converse of this is not true generaly.

Definition 6. *Let E be α-set. If $\alpha^*(a) = 0$ then a is called universal element of E.*

Definition 7. *Let E be $(\alpha, \alpha^*)-$controlled set and $t \in I$. The following set is called $t-$cut of α,*

$$\alpha_t = \{x \in E | \alpha(x) \geqslant t \vee \alpha^*(x) \leqslant 1-t\}$$

Example 1. *Let E={a,b,c,d,e} and $\alpha(a) = 0.7, \alpha(b) = 0.6, \alpha(c) = 1.0, \alpha(d) = 0.3, \alpha(e) = 0.5$.*
Then we get the following value from the definition of α^*.
$\alpha^*(a) = 0.3, \alpha^*(b) = 0.3, \alpha^*(c) = 0.0, \alpha^*(d) = 0.7,$
$\alpha^*(e) = 0.5$. *If we consider α as a fuzzy set then some level sets are as*

$$\alpha_{0.7} = \{a, c\}, \alpha_{0.6} = \{a, b, c\}, \alpha_{0.5} = \{a, b, c, e\}$$

but if we consider as $(\alpha, \alpha^*)-$controlled set then the same value cut set are as

$$\alpha_{0.7} = \{a, c\}, \alpha_{0.6} = \{a, b, c, e\}, \alpha_{0.5} = \{a, b, c, e\}.$$

Remark 1. *In fuzzy set theory, if t,s*$\in Im\,\alpha \ni t \neq s$ *then* $\alpha_t \neq \alpha_s$. *From the above example, in general, this is not true in* $(\alpha, \alpha^*)-$*controlled sets.*

Definition 8. *Let G be a group and* $(\alpha, \alpha^*)-$*controlled set. Then* α *is called* $(\alpha, \alpha^*)-$*controlled group if and only if*

1. $\alpha(e) = 1.0$
2. $\alpha(a.b) \geqslant \alpha(a) \wedge \alpha(b)$
3. $\alpha^*(a.b) \leqslant \alpha^*(a) \vee \alpha^*(b)$
4. $\alpha(a) \geqslant \alpha(a^{-1})$ and $\alpha^*(a) \geqslant \alpha^*(a^{-1})$

Theorem 2. *Let* α *is* $(\alpha, \alpha^*)-$*controlled group on G. Then* α *is a fuzzy group on G.*

Proof. Proof is clear from definition. □

The opposite of the theorem is generally not true.

Example 2. *Let* $G = \{e, a, b, ab\}$ *known group and* $\alpha(a) = 0.4, \alpha(b) = 0.4, \alpha(a.b) = 0.7, \alpha(e) = 1.0$.

Then α *is a fuzzy group on G, but it is not* $(\alpha, \alpha^*)-$*controlled group on G.*

Lemma 1. *Let* α *be* $(\alpha, \alpha^*)-$*controlled group on G and* $\theta_\alpha : G \to I$ *such that* $\theta_\alpha(a) = 1 - \alpha^*(a)$ *then* θ_α *is a fuzzy group on G.*

Proof. Let $a, b \in G$ and $a.b \in G_\alpha$ then

$$\begin{aligned} &\theta_\alpha(ab) = 1 - \alpha^*(ab) = 1 - (1 - \alpha(ab)) = \alpha(ab) \\ &\geqslant \alpha(a) \wedge \alpha(b) = 1 - (1 - \alpha(a)) \wedge 1 - (1 - \alpha(b)) \\ &= \theta_\alpha(a) \wedge \theta_\alpha(b). \end{aligned}$$

If $a.b \notin G_\alpha$ then

$$\begin{aligned} &\theta_\alpha(ab) = \\ &1 - \alpha^*(ab) = 1 - \sup_{\alpha(ab) < 1 - \alpha(y)} \alpha(y) = \inf_{\alpha(ab) < 1 - \alpha(y)} (1 - \alpha(y)) \geq \alpha(ab) \\ &\geqslant \alpha(a) \wedge \alpha(b) \\ &\geqslant 1 - (1 - \inf_{\alpha(a) < 1 - \alpha(y)} (1 - \alpha(y))) \wedge 1 - (1 - \inf_{\alpha(b) < 1 - \alpha(z)} (1 - \alpha(z))) \\ &= \theta_\alpha(a) \wedge \theta_\alpha(b). \end{aligned}$$

□

Theorem 3. *Let* α *be* $(\alpha, \alpha^*)-$*controlled group on G and* $\theta_\alpha : G \to I$ *such that* $\theta_\alpha(a) = 1 - \alpha^*(a)$ *then* $\alpha \cup \theta_\alpha$ *and* $\alpha \cap \theta_\alpha$ *are fuzzy groups on G*

Proof. From the above Lemma, the proof is clear. □

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