Notes on Intuitionistic Fuzzy Sets
Vol. 19, 2013, No. 4, 1–14

A proposed axiomatic system for Atanassov Intuitionistic Fuzzy Logic

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Abstract: In this paper, we continue our studies on Intuitionistic Fuzzy Residuated Lattices (IFRLs) defined in [11]. We investigate more properties of the implication operator of these symmetric residuated lattices. We observe that most axioms of the Basic Fuzzy Logic and Intuitionistic Logic hold in Intuitionistic Fuzzy Residuated Lattices (IFRLs). Accepting these axioms together with the basic properties of operators, we propose an axiomatic system for Atanassov Intuitionistic Fuzzy Logic (A-IFL).
Keywords: Intuitionstic fuzzy residuated lattice, Residuated lattice, Symmetric lattice, Intuitionstic fuzzy logic.
AMS Classification: 03B47, 03G10.

1 Introduction

L. A. Zadeh introduced the concept of fuzzy subsets of a well-defined set in his paper [30] for modeling the vague concept in the real world. After him, Goguen generalized this to L-fuzzy set [13] where L is an appropriate lattice. Fuzzy logic based on the theory of fuzzy sets(TFS) is developed. This is called fuzzy logic in wide sense denoted by FL_w and contains mostly applications of TFS [31]. An other version of fuzzy logic called fuzzy logic in narrow sense denoted by FL_n which studies fuzzy logic as a many valued logic. There are many researchers to work on this sense [7, 15, 27] too that we can not cite them even a few. Ward and Diworth introduced residuated lattices [29] and gave the main properties of these lattices, although they existed before this paper such as a Boolean algebras and Heyting algebras. H. Ono considers residuated lattices as an algebraic structure of substructural logics in [20]. P. Hajek in 1998 [14] introduced the notion of BL-algebra as a residuated lattice with two more conditions, namely divisibility and prelinearity to prove the completeness of Lukasiewicz logic as a many valued logic. He showed that these algebras are the best algebraic counterparts of fuzzy logics generated by continuous t-norms [14]. K. Atanassov [1, 3] introduced the notion of an Intuitionistic Fuzzy set (IFS in short) as a generalization of a Fuzzy set (FS). In fact, from his point of view for each element of the universe there are two degrees one a degree of membership to a vague subset and the other is a degree of non-membership to that given subset. Many researchers have been working on the theory of this subject and developed it in interesting different branches [5, 9, 16, 17, 18, 23, 24]. Many studied and applied it in a broad range of applications [23, 28, 32]. K. Atanassov and S. Stoeva generalized the concept of IFS to Intuitionstic L-fuzzy sets [2] where L is an appropriate lattice. A. Tepavcevic and T. Gerstenkorn give a new definition of lattice valued Intuitionstic fuzzy sets in [26]. Glad Deschrijver, et al. [8, 10] considered the Intuitionistic operators and defined negator, t-norms, t-conorms and implicators on the lattice $L^* = \{(x, y) \in [0, 1]^2 \mid x \leq y - 1\}$. A. Tepavcevic et al. considered the general form of lattice valued IFSs in [25].

E. Eslami introduced the notion of Intuitionistic fuzzy residuated lattices (IFRL) built from some symmetric residuated lattices [11]. He gives some properties of these lattices which are consistent with the properties that come from IFSs. Lattice operators, as well as residuation adjoint pair, are widely discussed. As in any logic the implication operator [4, 22] is the most important operator in this paper. The author emphasizes on it and gives the more important properties.

This paper is organized as follows: in the next section we give the preliminaries including the basic definitions and theorems that are needed in the other parts. In Section 3 by giving some examples we will show that the structure $\tilde{L}$ is not a BL-algebra even though we assume that L is a BL-algebra. Finally, we present some axioms for Intuitionistic Fuzzy logic (IFL).

2 Preliminaries

In this section we give some definitions and theorems that we need in the sequel. Assume that **U** is the universe. A fuzzy set **A** in **U** is characterized by the same symbol **A** as a function $\mathbf{A} : \mathbf{U} \longrightarrow [0, 1]$ where $\mathbf{A}(u) \in [0, 1]$ is the membership degree of the element $u \in \mathbf{U}$ [30].

Let $L = (L, \wedge, \vee, 0, 1)$ be a bounded (complete) lattice. By an L-fuzzy set **A** in **U** we mean a function $\mathbf{A} : \mathbf{U} \longrightarrow \mathbf{L}$ [13].

2.1 IFSs and RLs

In this part, we recall original definition of an intuitionistic fuzzy set given by K. Atanassov and the definition of a residuated lattices together with their basic properties:

Definition 2.1. *[1] Let **U** be the universe. By an Intuitionistic Fuzzy set (IFS) in **U** we mean a set of ordered triples* $\mathbf{A} = \{(x, \mu_A(x), \nu_A(x)) | x \in \mathbf{U}\}$, *where* $\mu_A(x)$ *is the membership degree of x to **A** and* $\nu_A(x)$ *is the non-membership degree of x to **A** such that* $\mu : \mathbf{U} \longrightarrow [0, 1]$ *and* $\nu : \mathbf{U} \longrightarrow [0, 1]$ *satisfying* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for all* $x \in \mathbf{U}$. *The complement of an IFS **A** is defined by* $\mathbf{A}^c = \{(x, \nu_A(x), \mu_A(x)) | x \in \mathbf{U}\}$. *We omit the coordinate x when it is clear that where they are coming from.*

We recall from [10] that $\mathbf{L}^* = \{(x, y) \in [0, 1]^2 | 0 \leq x + y \leq 1\}$ is a complete lattice with the order defined by

$$(x_1, x_2) \preceq (y_1, y_2) \; if \; and \; only \; if \; x_1 \leq y_1 \; and \; y_2 \leq x_2 \tag{1}$$

The notion of a residuated lattice defined as follows:

Definition 2.2. *[14] A residuated lattice is an algebra $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:*
(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(ii) $(L, *, 1)$ is a commutative monoid,and
(iii) the operation $*$ and $\rightarrow$ form an adjoint pair, i.e.,

$$x * y \leq z \; if \; and \; only \; if \; x \leq y \rightarrow z \tag{2}$$

for all $x, y, z \in L$.

Definition 2.3. *[14] A BL-algebra is a residuated lattice such that:*
(iv) $x * (x \rightarrow y) = x \wedge y$ (Divisibility), and
(v) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (Prelinearity)
for all $x, y \in L$.

Based on the above definition, the basic properties of residuated lattices are summarized in the following theorem:

Theorem 2.1. *[14] In any residuated lattice $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ the following properties hold for all $x, y, z \in L$:*
(1) $x * (x \rightarrow y) \leq x \wedge y$,
(2) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
(3) $x \leq y \; if \; and \; only \; if \; x \rightarrow y = 1$,
(4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(6) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,and
(7) $1 \rightarrow x = x$.

We may have a negator in a bounded lattice.

Definition 2.4. *Let $L = (L, \wedge, \vee, 0, 1)$ be a bounded lattice. A unary operator $N : L \rightarrow L$ is a negator if it is non-increasing with respect to the usual order on L, $N(0) = 1$ and $N(1) = 0$. Moreover, if N satisfies $N(N(x)) = x$ it is called an involutive negator or a strong negation.*

We know from [11] that if $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice and $\neg : L \rightarrow L$ is defined by $\neg x = x \rightarrow 0$. Then $\neg$ is a negator on L which is not necessarily involutive.

We recall from [14] that if L is an MV-algebra, $\neg$ is involutive. Also, We have the following theorem:

Theorem 2.2. *[14] In any residuated lattice $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ the following properties hold for all $x, y \in L$:*

(1) $x \leq \neg\neg x$,

(2) $\neg 1 = 0, \neg 0 = 1$,

(3) $x \rightarrow y \leq \neg y \rightarrow \neg x$,and

(4) $\neg\neg\neg x = \neg x, x \rightarrow \neg y = y \rightarrow \neg x$.

Definition 2.5. *[6] A residuated lattice is called an ivolutive residuated lattice if the negation $\neg$ is involutive, i.e., when the following equation holds;*

$$\neg\neg x = (x \rightarrow 0) \rightarrow 0 = x. \tag{3}$$

But in the case where $\neg$ is not involutive, it is possible to have another negation on the given residuated lattice which is involutive as:

Definition 2.6. *[6] A residuated lattice is called a symmetric residuated lattice if it is equipped with a unary operation $\sim$ satisfying:*

$$\sim\sim x = x, \tag{4}$$

$$\sim (x \vee y) = \sim x \wedge \sim y, \tag{5}$$

$$\sim (x \wedge y) = \sim x \vee \sim y. \tag{6}$$

It is easily verified that any involutive negator or strong negation (Definition 2.3) satisfies all properties in the above Definition 2.5.

2.2 IFRLs

Now, we recall the definition of an Intuitionistic Fuzzy Residuated lattice (IFRL) given by E. Eslami together with related concepts and some properties that he proves (see [11] for details).

Definition 2.7. *[11] Let $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ be a symmetric residuated lattice. Let $\tilde{L} = \{(x, y) \in L^2 | x \leq \sim y\}$. Define*

$$(x_1, y_1) \preceq (x_2, y_2)\ if\ and\ only\ if\ x_1 \leq x_2\ and\ y_2 \leq y_1 \tag{7}$$

it is easily verified that the above relation on $\tilde{L}$ is a partially order. This order induces the following lattice operators on $\tilde{L}$:

$$(x_1, y_1) \bigwedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2) \tag{8}$$

$$(x_1, y_1) \bigvee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2) \tag{9}$$

It follows from above that if L is a complete lattice, then $\tilde{L}$ is a complete lattice (see [10] for proof). In this paper we assume that L is complete.

The notions of t-norm and t-conorms on arbitrary lattices are defined as follows:

Definition 2.8. *[11] Let $L = (L, \wedge, \vee, 0, 1)$ be a bounded lattice. (a) A lattice traingular norm (Lt-norm for short) is a binary operator $T : L^2 \to L$ which is commutative,associative,isotone and $T(1, x) = x$ for all $x \in L$. (b) A lattice traingular conorm (Lt-conorm for short) is a binary operator $S : L^2 \to L$ which is commutative,isotone and $S(0, x) = x$ for all $x \in L$.*

Remark 2.1. *If the operator $\sim$ is an involutive negator on a bounded lattice L and T is an Lt-norm on L, then S defined by*

$$S(x, y) =\sim T(\sim x, \sim y) \tag{10}$$

for all $x, y \in L$ is an Lt-conorm. Conversely, if S is an Lt-conorm on L, then T defined by

$$T(x, y) =\sim S(\sim x, \sim y) \tag{11}$$

for all $x, y \in L$ is an Lt-norm.

The notion of an implicator in lattices is also defined by:

Definition 2.9. *[11] Let $L = (L, \wedge, \vee, 0, 1)$ be a bounded lattice. A lattice implicator (L-implicator) on L is a binary operator $I : L^2 \to L$ satisfying:*

(i) $I(0, 0) = I(0, 1) = (1, 1) = 1$ and $I(1, 0) = 0$ (Boundary conditions), and

(ii)I is decreasing w.r.t. first component and increasing w.r.t. second component (Monotonic conditions).

In the next lemma, an Lt-norm on $\tilde{L}$ is introduced.

Lemma 2.1. *[11] Let L and $\tilde{L}$ be as in definition 2.6. Then T defined by*

$$T((x, y), (u, v)) = (x * u, S(y, v)) \tag{12}$$

for all $(x, y), (u, v) \in \tilde{L}$and $S(a, b) =\sim (\sim a* \sim b)$ for all $a, b \in L$, is an Lt-norm on $\tilde{L}$.

We recall that this Lt-norm is an extension of IF t-norms that are defined on $\mathbf{L}^*$ in [8] and call them t-representable.

Lemma 2.2. *[11] Let L and $\tilde{L}$ be as in Definition 2.6. Define I by*

$$I((x, y), (u, v)) = ((x \to u) \wedge (\sim y \to\sim v), \sim (\sim y \to\sim v)) \tag{13}$$

for all $(x, y), (u, v) \in \tilde{L}$. Then I is an L-implicator on $\tilde{L}$.

Two of the basic properties of the above L-implicator I is given in the following Theorem:

Theorem 2.3. *[11] Let $I : \tilde{L}^2 \to \tilde{L}$ be the L-implicator defined in Lemma 2.11. Then I has the following properties:*

(a) $I(\tilde{1}, (u, v)) = (u, v)$ for all $(u, v) \in \tilde{L}$,

(b) $X \preceq Y$ if and only if $I(X, Y) = \tilde{1}$.

when $\tilde{1} = (1, 0)$, for all $X, Y \in \tilde{L}$.

We also recall that:

Lemma 2.3. *[11] Let L, $\tilde{L}$, T and I be as above. Then I is an R-implicator generated by T, i.e.,*

$$I(X,Y) = \sup\{Z \in \tilde{L} | T(X,Z) \preceq Y\} \tag{14}$$

for all $(X,Y) \in \tilde{L}$.

Which gives:

Theorem 2.4. *[11] Let L, $\tilde{L}$, T and I be as above. Then $\tilde{L} = (\tilde{L}, \bigwedge, \bigvee, T, I, \tilde{0}, \tilde{1})$ is a residuated lattice.*

The main definition of an Intuitionistic fuzzy lattice is given as:

Definition 2.10. *[11] An Intuitionistic Fuzzy Residuated lattice (IFRL) is a residuated lattice*

$$\tilde{L} = (\tilde{L}, \wedge, \vee, T, I, \tilde{0}, \tilde{1}) \tag{15}$$

such that its universe

$$\tilde{L} = \{(x,y) \in L^2 | x \leq \sim y\} \tag{16}$$

where

$$L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1) \tag{17}$$

is a symmetric residuated lattice and

$$T((x,y),(u,v)) = (x * u, S(y,v)) \tag{18}$$

and

$$I((x,y),(u,v)) = ((x \rightarrow u) \wedge (\sim y \rightarrow \sim v), \sim (\sim y \rightarrow \sim v)). \tag{19}$$

The lattice $\tilde{L}$ is called the intuitionistic fuzzy residuated lattice corresponding to L.

Example 2.1. *[11] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ is a symmetric residuated lattice with the operators defined by the following tables:*

x	$\sim x$
0	1
a	b
b	a
1	0

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	0	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Table 1: Operators $\sim$, $*$ and $\rightarrow$

Now $\tilde{L} = \{(0,0), (0,a), (0,b), (0,1), (a,b), (a,a), (a,0), (b,a), (b,0), (1,0)\}$ with the order $\preceq$ shown by the following diagram in Figure 1 and corresponding T and I is a Intuitionistic Fuzzy residuated lattice. We denote this special IFRL by $\tilde{L}_0$ for future use.

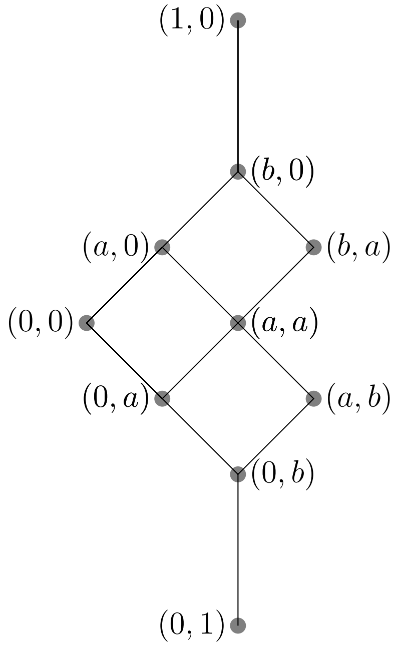

Figure 1: Lattice structure of $\tilde{L}$.

3 Propositional language of IFL

We introduce a propositional language consisting of the propositional symbols A, B, C ..., connectives $\barwedge, \veebar, \&, \supset, ', N$ and constants $\tilde{0}, \tilde{1}$. Our propositional forms [19] are defined as:

(i) Each propositional symbol is a propositional form,

(ii) $\tilde{0}, \tilde{1}$ are propositional forms,

(iii) If A is a propositional form, then A' and NA are propositional forms,

(iv) If A and B are propositional forms, then $A \barwedge B$, $A \veebar B$, $A\&B$ and $A \supset B$ are propositional forms.

Let us fix some notations and terminologies trough out this section for some conveniences. We assume that the lattice $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ is a symmetric residuated lattice and $\neg x = x \rightarrow 0$, $\tilde{L} = (\tilde{L}, \wedge, \vee, T, I, N, \tilde{0}, \tilde{1})$ is the IFRL corresponding to L and $(x, y)' = I((x, y), \tilde{0})$, $N(x, y) = (y, x)$. We assume elements A, B and C as typical elements of $\tilde{L}$ have ordered pair forms as $(a, b), (c, d)$ and (e, f) respectively.

By a valuation we mean a function v from propositional symbols to $\tilde{L}$. It is obvious that v assigns to each propositional symbol p an element $< a, b >$ of $\tilde{L}$, where a and b are degrees of truth and falsity of p, respectively. We extend v to V on propositional forms as follows:

$$V(A \barwedge B) = V(A) \wedge V(B) \tag{20}$$

$$V(A \veebar B) = V(A) \vee V(B) \tag{21}$$

$$V(A\&B) = T(V(A), V(B)) \tag{22}$$

$$V(A \supset B) = I(V(A), V(B)) \tag{23}$$

$$V(A') = I(V(A), \tilde{0}) \tag{24}$$

$$V(NA) = N\big(V(A)\big) \tag{25}$$

where A and B are propositional forms.

For the needs of the discussion below we adopt the notion of the (standard) tautology [1, 2] in $\tilde{L}$ as:

$$\text{A is a (standard) tautology if and only if } V(A) = \langle 1, 0\rangle, \tag{26}$$

while Intuitionistic Fuzzy Tautology (IFT) is defined by:

$$\text{A propositional form } A \text{ with } V(A) = \langle a, b\rangle \in \tilde{L} \text{ is an IFT if and only if } b \leq a. \tag{27}$$

It is obvious that if A is a (standard) Tautology, then it is Intuitionistic Fuzzy Tautology (IFT).

3.1 On the relation between IFL and BL

In the following we show that the IFRL satisfies some BL axioms. (c.f. [14])

Theorem 3.1. *If A,B and C are propositional forms, then:*
(BL_1) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
(BL_2) $(A\&B) \supset A$
(BL_3) $(A\&B) \supset (B\&A)$
(BL_4) $(A \supset (B \supset C)) \supset ((A\&B) \supset C)$
(BL_5) $((A\&B) \supset C) \supset (A \supset (B \supset C))$
(BL_6) $\tilde{0} \supset A$
are $\tilde{L}$-tautologies.

Proof. We show that (BL_4) is a tautology in $\tilde{L}$. The validities of the other propositional forms can be checked similarly. Let A, B and C be propositional forms whose truth values are:

$$V(A) = \langle a, b\rangle$$

$$V(B) = \langle c, d\rangle$$

$$V(C) = \langle e, f\rangle$$

we have

$$V\big((A \supset (B \supset C)\big) = I\big[\langle a, b\rangle, I\big(\langle c, d\rangle, \langle e, f\rangle\big)\big]$$

$$= \Big[\Big(a \to \big((c \to e) \wedge (\sim d \to \sim f)\big)\Big) \wedge \big(\sim b \to (\sim d \to \sim f)\big), \sim \big(\sim b \to (\sim d \to \sim f)\big)\Big]$$

$$= \Big[\big(a \to (c \to e)\big) \wedge \big(a \to (\sim d \to \sim f)\big) \wedge \big(\sim b \to (\sim d \to \sim f)\big), \sim \big(\sim b \to (\sim d \to \sim f)\big)\Big]$$

$$= \Big[\big((a * c) \to e\big) \wedge \Big((\sim b * \sim d) \to \sim f\Big), \sim \Big((\sim b * \sim d) \to \sim f\Big)\Big]$$

$$= I\Big[T\big(\langle a, b\rangle, \langle c, d\rangle\big), \langle e, f\rangle\Big]$$

$$= V\big((A\&B) \supset C\big)$$

So,

$$V\Big(\big(A \supset (B \supset C)\big) \supset \big((A\&B) \supset C\big)\Big) = \langle 1, 0\rangle = \tilde{1}$$

□

It is worth to note that $BL_1 - BL_6$ are the same as $A_1, A_2, A_3, A_{5a}, A_{5b}, A_7$ in [14].

Now we show that the two axioms of BL (A_4 and A_6 from the list of $A_1 - A_7$ in [14]) are not satisfied in IFRL, generally.

Example 3.1. *We show that the following axiom (A_6 of [14]) does not hold in the model of Example 2.1.*

$$\big((A \supset B) \supset C\big) \supset \Big(\big((B \supset A) \supset C\big) \supset C\Big)$$

Proof. It is sufficient to show that

$$V\Big((A \supset B) \supset C\Big) \not\leq V\Big(\big(((B \supset A) \supset C\big) \supset C)\Big)$$

In other words,

$$I\Big[I\big(V(A), V(B)\big), V(C)\Big] \not\leq I\Big[I\Big(I\big(V(B), V(A)\big), V(C)\Big), V(C)\Big]$$

Let A, B and C be propositional forms whose truth values are $V(A) = \langle 0, a\rangle$, $V(B) = \langle a, b\rangle$ and $V(C) = \langle b, 0\rangle$ in $\tilde{L_0}$ of Example 2.1. Based on these, we get

$$V\Big((A \supset B) \supset C\Big) = I\Big[I\big(V(A), V(B)\big), V(C)\Big] = \langle 1, 0\rangle,$$

and

$$V\Big(\big(((B \supset A) \supset C\big) \supset C)\Big) = I\Big[I\Big(I\big(V(B), V(A)\big), V(C)\Big), V(C)\Big] = \langle b, 0\rangle$$

We see that $\langle 1, 0\rangle \not\leq \langle b, 0\rangle$. Then by Theorem 2.3 (b), our result follows. □

Example 3.2. *Let $L = ([0, 1], \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ be the symmetric Lukasiewicz structure, then the following axiom (A_4 of [14]) does not hold.*

$$\big(A\&(A \supset B)\big) \supset \big(B\&(B \supset A)\big)$$

Proof. It is sufficient to show that

$$T\Big(V(A), I\big(V(A), V(B)\big)\Big) \neq T\Big(V(B), I\big(V(B), V(A)\big)\Big)$$

Put $V(A) = \langle 0.2, 0.5\rangle$, $V(B) = \langle 0.3, 0.6\rangle$; we have
$T\Big(V(A), I\big(V(A), V(B)\big)\Big) = \langle 0.1, 0.6\rangle$,
$T\Big(V(B), I\big(V(B), V(A)\big)\Big) = \langle 0.2, 0.6\rangle$ so

$$\langle 0.1, 0.6\rangle \neq \langle 0.2, 0.6\rangle.$$

□

Based on the above examples, we conclude that:

Corollary 3.1. *The IFRL* $\tilde{L} = (\tilde{L}, \bigwedge, \bigvee, T, I, \tilde{0}, \tilde{1})$ *is not a BL-algebra generally evev if* $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ *is a BL-Symmetric residuated lattice.*

Proof. Note that prelinearity fails by Example 3.1. and divisibility fails by Example 3.2. □

We know that BL-algebras are algebraic counterpart of Fuzzy Logics [14]. Assuming IFRLs are algebraic counterparts of IFL, then based on the above we see that the logics IFL and FL are essentially different.

3.2 On the relation between IFL and IL

The next and more important question is which of the IL axioms is true in Intuitionistic Fuzzy Residuated Lattice (IFRL).

In the following we show that the IFRL satisfies some IL axioms. (c.f. [21])

Theorem 3.2. *If* A, B *and* C *are propositional forms, then:*

(IL_1) $\big(A \supset (B \supset A)\big)$

(IL_2) $\big(A \supset (A \veebar B))\big)$

(IL_3) $\big(B \supset (A \veebar B))\big)$

(IL_4) $\big((A \barwedge B) \supset A\big)$

(IL_5) $\big((A \barwedge B) \supset B\big)$

(IL_6) $(A \supset B') \supset (B \supset A')$

(IL_7) $(A \supset A)' \supset B$

(IL_8) $\big((A \supset C)\big) \supset \Big((B \supset C) \supset \big((A \veebar B) \supset C\big)\Big)$

(IL_9) $\big((A \supset B)\big) \supset \Big((A \supset C) \supset \big((A \supset (B \barwedge C)\big)\Big)$

are $\tilde{L}$*-tautologies.*

Proof. We show that (IL_7) *is a tautology in* $\tilde{L}$*. The validities of the other propositional forms can be checked similarly. Let* A *and* B *be propositional forms whose truth values are:*

$$V(A) = \langle a, b\rangle,$$

$$V(B) = \langle c, d\rangle.$$

We have,

$$\begin{aligned} & V\big((A \supset A)' \supset B\big) \\ & = I\Big[I\Big(I\big\langle\langle a, b\rangle, \langle a, b\rangle\big\rangle, \langle 0, 1\rangle\Big), \langle c, d\rangle\Big] \\ & = I\Big[I\Big(\langle 1, 0\rangle, \langle 0, 1\rangle\Big), \langle c, d\rangle\Big] \\ & = I\Big[\langle 0, 1\rangle, \langle c, d\rangle\Big] \\ & = \langle 1, 0\rangle = \tilde{1}. \end{aligned}$$

□

It is worth to note that $IL_1 - IL_9$ are the same as $A_1, A_3 - A_{10}$ in [21].

The axiom A_2 of IL from the List of axioms $A_1 - A_{10}$ in [21] reads as follows

$$\big(A \supset (B \supset C)\big) \supset \big((A \supset B) \supset (A \supset C)\big)$$

We can easily (with long computation) show that the converse of this axiom is a (standard) tautology in $\tilde{L}$.

The above observation show that in spite of satisfaction of most axioms of IL, the logics IFL and IL are actually differnt.

We leave the investigation of exact relations for a new future paper.

Now we are ready to propose a logical axiomatic system for Intuitionistic Fuzzy Logic.

3.3 Proposed axioms for IFL

In this section we introduce the logical axioms for Intuitionistic Fuzzy Logic (IFL).

Definition 3.1. *The following propositional forms are our proposed axioms for IFL.*

(IFL_1) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

(IFL_2) $(A \barwedge B) \supset A$

(IFL_3) $A \barwedge B = B \barwedge A$, $A \veebar B = B \veebar A$

(IFL_4) $A \barwedge (B \barwedge C) = (A \barwedge B) \barwedge C$, $A \veebar (B \veebar C) = (A \veebar B) \veebar C$

(IFL_5) $A \barwedge (B \veebar C) = (A \barwedge B) \veebar (A \barwedge C)$, $A \veebar (B \barwedge C) = (A \veebar B) \barwedge (A \veebar C)$

(IFL_6) $(A\&B) \supset (B\&A)$

(IFL_7) $(A \supset (B \supset C)) \supset ((A\&B) \supset C)$

(IFL_8) $((A\&B) \supset C) \supset (A \supset (B \supset C))$

(IFL_9) $\tilde{0} \supset A$

(IFL_{10}) $N(NA) = A$

(IFL_{11}) $(A \supset B) \equiv (NA \supset NB)$

(IFL_{12}) $(NA\&NB) \supset N(A \veebar B)$

(IFL_{13}) $N\big(N(A \supset B)\big) \equiv \big(A \supset N(NB)\big)$

where A, B and C are propositional forms, and by $A \equiv B$, we mean $A \supset B$ and $B \supset A$.

Also we consider modus ponens as the deduction rule of **IFL**. Given this, the notions of a proof and of a provable formula in IFL are defined in the obvious way [21] or [14].

Theorem 3.3. *IFL proves the following propositional forms:*

(1) $A \supset (A')'$,

(2) $(A \supset B) \supset (B' \supset A')$,

(3) $\big((A')'\big)' \supset A'$,

(4) $A' \supset \big((A')'\big)'$,

(5) $(A \veebar B)' \supset (A' \wedge B')$,

(6) $(A' \wedge B') \supset (A \veebar B)'$,

(7) $(A' \veebar B') \supset (A \wedge B)'$.

Proof. We present only a proof for (2). The others are similar but might be lengthy. It is sufficient to put $\tilde{0}$ in (IFL_1) instead of C.

$$(A \supset B) \supset ((B \supset \tilde{0}) \supset (A \supset \tilde{0}))$$

so we have

$$(A \supset B) \supset (B' \supset A').$$

□

Note that our axioms are a combination of residuated logic axioms and properties of the added Atanassov negation N. Theorem 3.3 shows that the properties of the intuitionistic negation $'$ are satisfied.

4 Conclusion

We explored more properties of the Intuitionistic Fuzzy Residuated Lattices (IFRLs). Satisfaction of the axioms of Basic Fuzzy Logic and Intuitionistic Logic are investigated. These facts together with the properties of the operators lead us to think of Intuitionistic Fuzzy Residuated Lattice (IFRL) as an algebraic counterpart of the Atanassov Intuitionistic Fuzzy Logic (IFL). We proposed a set of axioms for this logic upon which one may prove the validity of more formulas in Intuitionistic Fuzzy Logic (IFL).

Based on the proved theorems, we can consider some algebraic substructures of IFRLs such as Boolean, BL– and Heyting centers and their relations in near future works.

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