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Elements of intuitionistic fuzzy logic. Part I

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Abstract

The definition of the notion of intuitionistic fuzzy set is the basis for defining intuitionistic fuzzy logics of different kinds. In this paper, we construct two versions of intuitionistic fuzzy propositional calculus (IFPC) and a version of intuitionistic fuzzy predicate logic (IFPL). © 1998 Elsevier Science B.V.

Keywords: Intuitionistic fuzzy predicate logic; Intuitionistic fuzzy propositional calculus; Predicate logic; Propositional calculus

0. Introduction

In this paper, we discuss logical issues related to the so-called intuitionistic fuzzy sets (cf. [1, 2]). It is the first part of a series of papers devoted to the problem of logical systems with intuitionistic fuzzy semantics. In the sequel, we plan an exposition of systems of modal and temporal logic. Some generalizations of the notion of intuitionistic fuzzy logic will be considered, too.

It should be mentioned that the term was coined with the idea of emphasizing a further liberalization of the notion of membership by the introduction of a "measure" of non-membership in addition to the "measure" of membership, with the provision that the sum of the two measures be less than 1. This restriction expresses a kind of "consistency" of the measures.

Recently there appeared several papers on the so-called bilattice approach [8, 9, 12]. Bilattices, due to Ginsberg, are meant to express the interrelation between missing (incomplete) and conflicting (contradictory) information.

The notion of intuitionistic fuzzy set is a particular case of this bilattice approach: membership is evaluated by pair of elements of the lattice $L_0 = \langle [0, 1], \min, \max \rangle$, restricted with the above condition.

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1. Intuitionistic fuzzying of the validity of propositions

Here we shall introduce the elements of an intuitionistic fuzzy propositional calculus (IFPC), basing our constructions on the definition of the IFSs and [3, 5], and using the notations from the theory of propositional calculus after [14].

To each proposition (in the classical sense) one can assign its truth value: truth – denoted by 1, or falsity – 0. In the case of fuzzy logics this truth value is a real number in the interval [0, 1] and can be called "truth degree" of a particular proposition. Here we add one more value – "falsity degree" – which will be in the interval [0, 1] as well. Thus, one assigns to the proposition p two real numbers $\mu(p)$ and $\gamma(p)$ with the following constraint to hold:

$$\mu(p) + \gamma(p) \leqslant 1.$$

Let this assignment be provided by an evaluation function V defined over a set of propositions S in such a way that

$$V(p) = \langle \mu(p), \gamma(p) \rangle.$$

Hence, the function $V: S \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all propositions in S.

We assume that the evaluation function V assigns to the logical truth T: $V(T) = \langle 1, 0 \rangle$, and to F: $V(F) = \langle 0, 1 \rangle$.

We shall discuss below the truth and falsity degrees of propositions which result from the application of logical operations (unary and binary) over input propositions which have known values according to a given evaluation function.

The evaluation of the negation $\neg p$ of the proposition p will be defined through

$$V(\neg p) = \langle \gamma(p), \mu(p) \rangle.$$

When $\gamma(p) = 1 - \mu(p)$, i.e.,

$$V(p) = \langle \mu(p), 1 - \mu(p) \rangle,$$

for $\neg p$ we get

$$V(\neg p) = \langle 1 - \mu(p), \mu(p) \rangle,$$

which coincides with the result for ordinary fuzzy logic from [15].

When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended also for operations "&", "$\veebar$" through the definition

$$V(p \,\&\, q) = \langle \min(\mu(p), \mu(q)), \max(\gamma(p), \gamma(q)) \rangle,$$

$$V(p \veebar q) = \langle \max(\mu(p), \mu(q)), \min(\gamma(p), \gamma(q)) \rangle.$$

Depending on the way of defining the operation "$\supset$", different variants of IFPC can be obtained.

2. sg-variant of IFPC

One possibility for the evaluation of the compound proposition $p \supset q$ is given by

$$V(p \supset q) = \langle 1 - (1 - \mu(q)) \cdot \mathrm{sg}(\mu(p) - \mu(q)), \gamma(q) \cdot \mathrm{sg}(\mu(p) - \mu(q)) \cdot \mathrm{sg}(\gamma(q) - \gamma(p)) \rangle,$$

Table 1

p	$V(p)$	q	$V(q)$	$V(p \supset q)$
F	$\langle 0,1 \rangle$	F	$\langle 0,1 \rangle$	$\langle 1,0 \rangle$
F	$\langle 0,1 \rangle$	T	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$
T	$\langle 1,0 \rangle$	F	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$
T	$\langle 1,0 \rangle$	T	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$

where

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

While the two clauses concerning conjunction and disjunction are transferred from the classical and ordinary fuzzy cases and coincide entirely with the corresponding definitions there, the definition of "$\supset$" is more complex. Nevertheless, the same is valid for it too when $p, q \in \{F, T\}$ the function V has the values given in Table 1.

By analogy with the operations over IFS, it will be convenient to define for the propositions $p, q \in S$

$$\neg V(p) = V(\neg p),$$

$$V(p) \wedge V(q) = V(p \,\&\, q),$$

$$V(p) \vee V(q) = V(p \veebar q),$$

$$V(p) \rightarrow V(q) = V(p \supset q).$$

A given propositional form A (cf. [14]: each proposition is a propositional form; if A is a propositional form then $\neg A$ is a propositional form; if A and B are propositional forms, then $A \,\&\, B$, $A \veebar B$, $A \supset B$ are propositional forms) will be called a tautology if $V(A) = \langle 1, 0 \rangle$, for all valuation functions V.

Theorem 1. *If A and $A \supset B$ are tautologies, then B is also a tautology.*

Proof. Since A and $A \supset B$ are tautologies then for every V,

$$V(A) = V(A \supset B) = \langle 1, 0 \rangle,$$

i.e.,

$$\mu(A) = 1, \qquad \gamma(A) = 0,$$

$$\mu(A \supset B) = 1 - (1 - \mu(B)) \cdot \mathrm{sg}(\mu(A) - \mu(B)) = 1,$$

$$\gamma(A \supset B) = \gamma(B) \cdot \mathrm{sg}(\mu(A) - \mu(B)) \cdot \mathrm{sg}(\gamma(B) - \gamma(A)) = 0.$$

Hence,

$$1 - \mu(B) = 0 \quad \text{or} \quad \mathrm{sg}(1 - \mu(B)) = 0$$

and at the same time

$$\gamma(B) = 0 \quad \text{or} \quad \mathrm{sg}(\mu(A) - \mu(B)) = 0 \quad \text{or} \quad \mathrm{sg}(\gamma(B) - \gamma(A)) = 0.$$

But

$$1 - \mu(B) = \mathrm{sg}(1 - \mu(B)) = 0$$

exactly then when

$$\mu(B) = 1,$$

from where it follows directly that

$$\gamma(B) = 0,$$

i.e., B is a tautology.

Let us assume everywhere below that

$$V(A) = \langle a, b\rangle, \qquad V(B) = \langle c, d\rangle, \qquad V(C) = \langle e, f\rangle.$$

Theorem 2. *If A, B and C are arbitrary propositional forms then*
(a) $A \supset A$,
(b) $A \supset (B \supset A)$,
(c) $A \,\&\, B \supset A$,
(d) $A \,\&\, B \supset B$,
(e) $A \supset (A \veebar B)$,
(f) $B \supset (A \veebar B)$,
(g) $A \supset (B \supset (A \,\&\, B))$,
(h) $(A \supset C) \supset ((B \supset C) \supset ((A \veebar B) \supset C))$,
(i) $\neg\neg A \supset A$,
(j) $(A \supset (B \supset C)) \supset ((A \supset B) \supset \cdot (A \supset C))$,
are tautologies.

Proof. We shall prove (j); (a)–(i) are proved analogously.

(j): $V((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$

$$\begin{aligned}
&= (V(A) \to (V(B) \to V(C))) \to ((V(A) \to V(B)) \to (V(A) \to V(C)))\\
&= (\langle a, b\rangle \to \langle 1 - (1-e)\cdot \mathrm{sg}(c-e), f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d)\rangle)\\
&\quad \to (\langle 1 - (1-c)\cdot \mathrm{sg}(a-c), d\cdot \mathrm{sg}(a-c)\cdot \mathrm{sg}(d-b)\rangle\\
&\quad \to \langle 1 - (1-e)\cdot \mathrm{sg}(a-e), f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b)\rangle)\\
&= \langle 1 - (1-e)\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(a - 1 + (1-e)\cdot \mathrm{sg}(c-e)), f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d)\\
&\quad \cdot \mathrm{sg}(a - 1 + (1-e)\cdot \mathrm{sg}(c-e))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d) - b)\rangle\\
&\quad \to \langle 1 - (1-e)\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e) - (1-c)\cdot \mathrm{sg}(a-c)),\\
&\quad f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b)\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e) - (1-c)\cdot \mathrm{sg}(a-c))\\
&\quad \cdot \mathrm{sg}(f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b) - d\cdot \mathrm{sg}(a-c)\cdot \mathrm{sg}(d-b))\rangle\\
&= \langle 1 - (1-e)\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e) - (1-c)\cdot \mathrm{sg}(a-c))\\
&\quad \cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e) - (1-c)\cdot \mathrm{sg}(a-c))\\
&\quad - (1-e)\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(a - 1 + (1-e)\cdot \mathrm{sg}(c-e))), f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b)\\
&\quad \cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e) - (1-c)\cdot \mathrm{sg}(a-c))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b) - d\cdot \mathrm{sg}(a-c)\cdot \mathrm{sg}(d-b))
\end{aligned}$$

$$\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e)-(1-c)\cdot \mathrm{sg}(a-c))$$
$$-(1-e)\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e)))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b)$$
$$\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}(a-e)-(1-c)\cdot \mathrm{sg}(a-c))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(a-e)\cdot \mathrm{sg}(f-b)-d\cdot \mathrm{sg}(a-c)\cdot \mathrm{sg}(d-b))$$
$$-f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d)-b))\rangle$$
$$=\langle 1,0\rangle \quad \text{if } a\leqslant e.$$
$$=\langle 1-(1-e)\cdot \mathrm{sg}((1-e)-(1-c)\cdot \mathrm{sg}(a-c))\cdot \mathrm{sg}((1-e)\cdot(\mathrm{sg}((1-e)$$
$$-(1-c)\cdot \mathrm{sg}(a-c)))-\mathrm{sg}(c-e)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e))), f\cdot \mathrm{sg}(f-b)$$
$$\cdot \mathrm{sg}((1-e)-(1-c)\cdot \mathrm{sg}(a-c))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(f-b)-d\cdot \mathrm{sg}(a-c)\cdot \mathrm{sg}(d-b))$$
$$\cdot \mathrm{sg}((1-e)\cdot \mathrm{sg}((1-e)-(1-c)\cdot \mathrm{sg}(a-c))-(1-e)\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(a-1+(1-e)$$
$$\cdot \mathrm{sg}(c-e)))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(f-b)\cdot \mathrm{sg}((1-e)-(1-c)\cdot \mathrm{sg}(a-c))\cdot \mathrm{sg}(f\cdot \mathrm{sg}(f-b)$$
$$-d\cdot \mathrm{sg}(a-c)\cdot \mathrm{sg}(d-b))-f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e))$$
$$\cdot \mathrm{sg}(f\cdot \mathrm{sg}(c-e)\cdot \mathrm{sg}(f-d)-b))\rangle \quad \text{if } a>e$$

If $a\leqslant c$ (hence $c>e$, and in view of the equality, for $x\geqslant 0$ $x\cdot \mathrm{sg}(x)=x$ we get $\mathrm{sg}((1-e)-\mathrm{sg}(c-e)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e)))=\mathrm{sg}(1-e-\mathrm{sg}(a-e))=\mathrm{sg}(1-e-1)=\mathrm{sg}(-e)=0$)

$$=\langle 1,0\rangle.$$

If $a>c$ (for the same expression we get $\mathrm{sg}((1-e)\cdot \mathrm{sg}((1-e)-(1-c))-\mathrm{sg}(c-e)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e)))=\mathrm{sg}((1-e)\cdot \mathrm{sg}(c-e)-\mathrm{sg}(c-e)\cdot \mathrm{sg}(a-1+(1-e)\cdot \mathrm{sg}(c-e)))$; if $c\leqslant e$ the expression is $\mathrm{sg}(0)=0$; if $c>e$ (because $a>e$) $=\mathrm{sg}((1-e)-\mathrm{sg}(a-1+1-e))=\mathrm{sg}(1-e-1)=\mathrm{sg}(-e)=0$)

$$=\langle 1,0\rangle. \qquad \square$$

In this way we find that some of the basic tautologies in the classical propositional calculus are tautologies in the IFPC. Moreover, as Theorem 1 shows, the set of tautologies is closed under modus ponens. Unfortunately, the classical tautology (see [14])

(k) $(\neg A\supset\neg B)\supset((\neg A\supset B)\supset A)$

is not valid, so there are differences between the two notions – classical $\{0,1\}$ tautology and ours.

3. (max–min) version of IFPC

Using the above definitions for "&" and "$\vee$" here, another version of IFPC is constructed by giving the following definition for "$\supset$":

$$V(p\supset q)=\langle \max(\gamma(p),\mu(q)),\min(\mu(p),\gamma(q))\rangle$$
$$V(p)\rightarrow V(q)=V(p\supset q).$$

For the needs of the discussion below, we shall define the notion of intuitionistic fuzzy tautology (IFT) through

"A is an IFT" iff "if $V(A)=\langle a,b\rangle$, then $a\geqslant b$".

Theorem 3. *If A, B and C are propositional forms, then* (a)–(j) *from Theorem* 2 *and the classical tautology* (k) *are IFTs.*

Proof. We shall prove (h) and (k). The other assertions are proved analogously.

(h): $V((A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)))$

$$= \langle \max(b,e), \min(a,f)\rangle \to (\langle \max(d,e), \min(c,f)\rangle \to (\langle \max(a,c), \min(b,d)\rangle \to \langle e,f\rangle))$$

$$= \langle \max(b,e), \min(a,f)\rangle \to (\langle \max(d,e), \min(c,f)\rangle \to (\max(e, \min(b,d)), \min(f, \max(a,c))\rangle)$$

$$= \langle \max(b,e), \min(a,f)\rangle \to \langle \max(\min(c,f), e, \min(b,d)), \min(\max(d,e), f, \max(a,c))\rangle$$

$$= \langle \max(\min(a,f), \min(c,f), e, \min(b,d)), \min(\max(b,e), \max(d,e), f, \max(a,c))\rangle$$

and

$$\max(\min(a,f), \min(c,f), e, \min(b,d)) \geqslant \max(\min(a,f), \min(c,f)) = \min(f, \max(a,c))$$
$$\geqslant \min(\max(b,e), \max(d,e), f, \max(a,c))\rangle.$$

(k) $V((\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A))$

$$= (\langle b,a\rangle \to \langle d,c\rangle) \to (\langle \max(a,c), \min(b,d)\rangle \to \langle a,b\rangle)$$

$$= \langle \max(a,d), \min(b,c)\rangle \to \langle \max(a, \min(b,d)), \min(b, \max(a,c))\rangle$$

$$= \langle \max(\min(b,c), a, \min(b,d)), \min(b, \max(a,c), \max(a,d))\rangle$$

and

$$\max(\min(b,c), a, \min(b,d)) \geqslant \max(a, \min(b,c))$$
$$\geqslant \min(b, \max(a,c)) \geqslant \min(b, \max(a,c), \max(a,d))\rangle. \qquad \square$$

With this choice of operations, notion of tautology, and evaluation, it turns out that the modus ponens is not valid. On the other hand, a well-known fact from classical logic is also valid here:

$$\langle a,b\rangle \to \langle 0,1\rangle = \langle b,a\rangle.$$

Some approximations (in different respects) of modus ponens are valid for this notion of tautology:

Theorem 4. (a) *If A and* $(A \,\&\, B)$ *are IFTs, then B is an IFT.*
(b) *If A and* $\neg(A \supset B)$ *are IFTs, then* $\neg B$ *is an IFT.*
(c) $(A \,\&\, (A \supset B)) \supset B$ *is an IFT.*

Proof. (a) Let us assume that $c < d$ and by the above conditions

$$a \geqslant b, \qquad \min(a,c) \geqslant \max(b,d).$$

are valid. Then

$$d > c \geqslant \min(a,c) \geqslant \max(b,d) \geqslant d,$$

which is a contradiction, i.e. $c \geqslant d$. Hence B is an IFT.

(b) Let us assume that $c > d$ and by the above conditions

$$a \geqslant b, \qquad \min(a,d) \geqslant \max(b,c).$$

Then

$$d \geqslant \min(a,d) \geqslant \max(b,c) \geqslant c > d,$$

which is a contradiction, i.e. $c \leqslant d$. Hence $\neg B$ is an IFT.

$$\begin{aligned}\text{(c): } V((A \,\&\, (A \supset B)) \supset B) &= (\langle a,b\rangle \wedge (\langle a,b\rangle \to \langle c,d\rangle)) \to \langle c,d\rangle \\ &= (\langle a,b\rangle \wedge \langle \max(b,c), \min(a,d)\rangle) \to \langle c,d\rangle \\ &= \langle \min(a,\max(b,c)), \max(b,\min(a,d))\rangle \to \langle c,d\rangle \\ &= \langle \max(b,c,\min(a,d)), \min(a,d,\max(b,c))\rangle.\end{aligned}$$

From

$$\max(b,c,\min(a,d)) \geqslant \min(a,d) \geqslant \min(a,d,\max(b,c))$$

it follows that $(A \,\&\, (A \supset B)) \supset B$ is an IFT. □

Theorem 5. *$A \supset (\neg A \supset B)$ is an IFT.*

Proof.

$$V(A \supset (\neg A \supset B)) = \langle a,b\rangle \to \langle \max(a,c), \min(b,d)\rangle = \langle \max(a,b,c), \min(a,b,d)\rangle.$$

The validity of the assertion follows from the inequalities: $\max(a,b,c) \geqslant a \geqslant \min(a,b,d)$.

4. A characterization of the set of propositional IFTs

The definition of IFT is a generalization of the corresponding notion in ordinary fuzzy logic, where a propositional form A is assumed to be a fuzzy tautology, if for any evaluation, $V(A) \geqslant 0.5$. A fuzzy tautology is also a classical two-valued tautology. Moreover, we have the well-known fact: "if A is a classical two-valued tautology, then it is a fuzzy tautology" [15].

The case of intuitionistic fuzzy tautological propositional forms is more complicated. Again we have that an IFT is at the same time a (0, 1)-tautology, but the converse implication is not true, as can be shown by an example.

It can be established that if A and B are IFTs, then $A \,\&\, B$ is not necessarily an IFT, e.g. we have $p \veebar \neg p$ and $q \veebar \neg q$, for different propositions p and q are IFT's. Indeed,

$$V(p \veebar \neg p) = \langle \max(\mu(p), \mu(\neg p)), \min(\gamma(p), \gamma(\neg p))\rangle = \langle \max(\mu(p), \gamma(p)), \min(\gamma(p), \mu(\neg p))\rangle$$

and obviously, for any a, b $\max(a,b) \geqslant \min(a,b)$.

Nevertheless the form $A = (p \veebar \neg p) \,\&\, (q \veebar \cdot \neg p)$ is not an IFT. Take for e.g. $V(p) = \langle 0.4, 0.4\rangle$, $V(q) = \langle 0.2, 0.2\rangle$. Then $V(p \veebar \neg p) = \langle 0.4, 0.4\rangle$, $V(q \veebar \neg q) = \langle 0.2, 0.2\rangle$, but $V(A) = \langle 0.2, 0.4\rangle$.

Let us call two propositional forms A and B equivalent ($A \Leftrightarrow B$), if for all evaluation functions $V : V(A) = V(B)$, i.e. $V(A) \geqslant V(B)$ and $V(B) \geqslant V(A)$. Any propositional form A generates a function f_A which maps a subset of $([0,1] \times [0,1])^n \xrightarrow[\text{into}]{} ([0,1] \times [0,1])$, where n is the number of propositional variables of f_A.

Thus equivalent forms represent one and the same function, and the relation "$\Leftrightarrow$" is indeed, an equivalence relation.

Easily provable facts about equivalence of forms show that the set of equivalence classes, with the operations "&", "$\veebar$", "$\neg$" defined in the usual way, is a de Morgan lattice, i.e., modulo, the equivalence, the laws of idempotency, commutativity, associativity, distributivity (for & and $\veebar$) hold as well as de Morgan laws: $\neg(A \& B) \Leftrightarrow \neg A \veebar \neg B$; $\neg(A \veebar B) \Leftrightarrow \neg A \& \neg B$ and the law of double negation $\neg\neg A \Leftrightarrow A$. Moreover, we have $A \supset B \Leftrightarrow \neg A \veebar B$.

A conjunctive normal form (CNF) A is of the sort $D_i \& \cdots \& D_m$, where $D_i = 1_{i,1} \vee \cdots \vee 1_{i,k_i}$ – is a clause of literals (a literal is either a propositional variable or a negated variable $\neg p$). The literals p and $\neg p$ are called opposite. Two clauses C and D are called connected if they contain a common variable occuring in opposite literals (e.g. p in C and $\neg p$ in D).

Lemma 1. *A clause C is an IFT iff it is a classical two-valued tautology iff C contains a pair of opposite literals.*

Lemma 2. *A conjunction of two literals C and D which are IFTs is an IFT iff they are connected.*

Proof. If C and D are IFTs and are connected, then $C \& D$ is an IFT. Consider an arbitrary V. Let $C = p \vee A$, $D = \neg p \vee B$, where $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$ and let $V(p) = \langle \mu, \gamma\rangle$. Then

$$\max(\mu, a) \geqslant \mu \geqslant \min(\mu, d)$$

$$\max(\gamma, c) \geqslant \gamma \geqslant \min(\gamma, b)$$

$$\max(\mu, a) \geqslant \min(\gamma, b)$$

$$\max(\gamma, c) \geqslant \min(\mu, d)$$

and from

$$V(C \& D) = \langle \min(\max(\mu, a), \max(\gamma, c)), \max(\min(\gamma, b), \min(\mu, d))\rangle$$

and

$$\min(\max(\mu, a), \max(\gamma, c)) \geqslant \max(\min(\gamma, b), \min(\mu, d))$$

it follows that $C \& D$ is an IFT.

Let C and D be two IFT clauses. Let us define the following evaluation W. For variables p which occur in both positive and negative literals in C let $W(p) = \langle 0.2, 0.2\rangle$. For variables q that appear in both positive and negative in D: $W(q) = \langle 0.4, 0.4\rangle$. Note that the sets of such variables are disjoint. For variables which occur positively in C or D let W be $\langle 0.2, 0.4\rangle$ and for variables occuring negatively in C or D $\langle 0.4, 0.2\rangle$. It is a simple check that shows that $W(C \& D) = \langle 0.2, 0.4\rangle$. Thus, the conjunction of C and D is not an IFT.

A CNF A is called totally connected if every pair of clauses C, D in it are connected.

Theorem 6. *A CNF A is an IFT iff all clauses in it are IFTs and A is totally connected.*

Proof. Assume that all clauses of A are IFTs and that A is totally connected. If we assume that for some evaluation function $W : W(A) = \langle \mu, \gamma\rangle$ is such that $\mu < \gamma$, then it can be easily seen that there is a pair of clauses C and D of A such that $C \& D$ is already not IFT (due to W); but this is impossible by Lemma 2. In the opposite direction: if at least two clauses in A are not connected, then their conjunction will not be an IFT; hence A will not be an IFT, too.

Theorem 7 (for characterization of IFL). *The set of propositional IFTs is decidable.*

Proof. Follows easily from the above reduction of the notion of an IFT to classical validity and syntactic restrictions.

5. Variants of intuitionistic fuzzy modus ponens

Two variants of IFPC were given above, an sg-variant and a (max–min)-variant and there it was shown that the modus ponens (MP) is not valid in the case of (max–min) version of the operation "implication". Under "a valid rule" we understand here a rule that preserves the following property: if for a given valuation on all premises are with $\mu \geqslant \gamma$, then for the conclusion we have the same. Validity is essential if one is interested in passing from hypothesis to conclusions without loss in the degree of truth.

Following [6,13] we shall here introduce five different definitions of the implication (in the case of intuitionistic fuzziness) and point the ones for which MP is valid.

Definition 1. $V(A \supset B) = \langle \max(1-\mu(A), \gamma(B)), \min(\mu(A), 1-\gamma(B)) \rangle$.
Check for correctness of the definition:

$$\max(1-\mu(A),\gamma(B)) + \min(\mu(A),1-\gamma(B)) \leqslant \max(1-\mu(A),\gamma(B)) + 1 - \max(1-\mu(A),\gamma(B)) = 1.$$

Check for correctness of MP: Let $\mu(A) \geqslant \gamma(A)$ and let $\max(1-\mu(A),\gamma(B)) \geqslant \min(\mu(A),1-\gamma(B))$. If, for example, $\mu(A) = \gamma(A) = 0.5, \mu(B) = 0.0, \gamma(B) = 0.5$. Then both of the last inequalities are valid, but $\mu(B) < \gamma(B)$, i.e., MP is not valid.

Definition 2. $V(A \supset B) = \langle \gamma(A) + \mu(A)\cdot\mu(B), \mu(A)\cdot\gamma(B) \rangle$.
Check for correctness of the definition:

$$0 \leqslant \gamma(A) + \mu(A)\cdot\mu(B) + \mu(A)\cdot\gamma(B) \leqslant \gamma(A) + \mu(A)\cdot(\mu(B)+\gamma(B)) \leqslant \mu(A) + \gamma(A) \leqslant 1.$$

Check for correctness of MP: Let $\mu(A) \geqslant \gamma(A)$ and let $\gamma(A) + \mu(A)\cdot\mu(B) \geqslant \mu(A)\cdot\gamma(B)$. If for example $\mu(A) = \gamma(A) = 0.5$, $\mu(B) = 0.0$, $\gamma(B) = 0.5$. Then both of the last inequalities are valid, but $\mu(B) < \gamma(B)$, i.e., MP is not valid.

Definition 3. $V(A \supset B) = \langle \min(1, \gamma(A) + \mu(B)), \max(0, 1-\gamma(A)-\mu(B)) \rangle$.

Check for correctness of the definition:

$$\min(1,\gamma(A)+\mu(B)) + \max(0,1-\gamma(A)-\mu(B)) \leqslant \min(1,\gamma(A)+\mu(B)) + 1 - \min(1,\gamma(A)+\mu(B)) = 1.$$

Check for correctness of MP: Let $\mu(A) \geqslant \gamma(A)$ and let $\min(1,\gamma(A)+\mu(B)) \geqslant \max(0,1-\gamma(A)-\mu(B))$. If, for example, $\mu(A) = \gamma(A) = 0.5$, $\mu(B) = 0.0$, $\gamma(B) = 0.5$. Then both of the last inequalities are valid, but $\mu(B) < \gamma(B)$, i.e., MP is not valid.

Definition 4.

$$V(A \supset B) = \begin{cases} \langle \gamma(A), \mu(A) \rangle & \text{if } \mu(B) \leqslant \gamma(B), \\ \langle \mu(B), \gamma(B) \rangle & \text{if } \mu(A) \geqslant \gamma(A), \\ \langle \max(\gamma(A), \mu(B)), \min(\mu(A), \gamma(B)) \rangle, & \text{otherwise.} \end{cases}$$

Check for correctness of the definition: Obviously,

$$0 \leqslant \max(\gamma(A),\mu(B)) + \min(\mu(A),\gamma(B)) \leqslant 1.$$

Check for correctness of MP: Let $\mu(A) \geqslant \gamma(A)$ and let $\mu(A \supset B) \geqslant \gamma(A \supset B)$. Then $\mu(B) = \mu(A \supset B) \geqslant \gamma(A \supset B) = \gamma(B)$, i.e., MP is valid.

Definition 5.

$$V(A \supset B) = \begin{cases} \langle 1,0 \rangle & \text{if } \mu(A) \leqslant \mu(B) \,\&\, \gamma(A) \geqslant \gamma(B), \\ \langle \mu(B), \gamma(A) \rangle & \text{if } \mu(A) > \mu(B) \,\&\, \gamma(A) \geqslant \gamma(B), \\ \langle \mu(A), \gamma(B) \rangle & \text{if } \mu(A) \leqslant \mu(B) \,\&\, \gamma(A) < \gamma(B), \\ \langle 0,1 \rangle & \text{if } \mu(A) > \mu(B) \,\&\, \gamma(A) < \gamma(B). \end{cases}$$

Check for correctness of the definition: obvious.

Check for correctness of MP: Let $\mu(A) \geqslant \gamma(A)$ and let $\mu(A \supset B) \geqslant \gamma(A \supset B)$.
If $\mu(A) \leqslant \mu(B)$ & $\gamma(A) \geqslant \gamma(B)$, then $\mu(B) \geqslant \mu(A) \geqslant \gamma(A) \geqslant (B)$;
if $\mu(A) > \mu(B)$ & $\gamma(A) \geqslant \gamma(B)$, then $\mu(B) \geqslant \gamma(A) \geqslant \gamma(B)$;
if $\mu(A) \leqslant \mu(B)$ & $\gamma(A) < \gamma(B)$, then $\mu(B) \geqslant \mu(A) \geqslant \gamma(B)$;
the case $\mu(A) > \mu(B)$ & $\gamma(A) < \gamma(B)$ is impossible by the second assumption, i.e., MP is valid.

In [7] the operation "symmetric sum" is used. A sixth variant of implication is based on this operation.

Definition 6. (Let $V(A) = \langle a,b \rangle$ and $V(B) = \langle c,d \rangle$)

$$V(A \supset B) = \left\langle \frac{b \cdot c}{b \cdot c + (1-b)\cdot(1-c)}, \frac{a \cdot d}{a \cdot d + (1-a)\cdot(1-d)} \right\rangle.$$

Check for correctness of the definition: From

$$\begin{aligned} &(b\cdot c + (1-b)\cdot(1-c))\cdot(a\cdot d + (1-a)\cdot(1-d)) - a\cdot d\cdot(b\cdot c + (1-b)\cdot(1-c)) \\ &\quad - b\cdot c\cdot(a\cdot d + (1-a)\cdot(1-d)) \\ &= (1-a)\cdot(1-b)\cdot(1-c)\cdot(1-d) - a\cdot b\cdot c\cdot d \quad (\text{from } a+b \leqslant 1 \text{ and } c+d \leqslant 1) \\ &\geqslant b\cdot a\cdot d\cdot c - a\cdot b\cdot c\cdot d = 0. \end{aligned}$$

Check for correctness of MP: Let $\mu(A) \geqslant \gamma(A)$ and let $\mu(A \supset B) \geqslant \gamma(A \supset B)$, i.e.,

$$a \geqslant b \quad \text{and} \quad \frac{b\cdot c}{b\cdot c + (1-b)\cdot(1-c)} \geqslant \frac{a\cdot d}{a\cdot d + (1-a)\cdot(1-d)}$$

and let us assume that $c < d$. Then

$$\begin{aligned} 0 &\leqslant b\cdot c\cdot(a\cdot d + (1-a)\cdot(1-d)) - a\cdot d\cdot(b\cdot c + (1-b)\cdot(1-c)) \\ &= b\cdot c\cdot(1-a)\cdot(1-d) - a\cdot d\cdot(1-b)\cdot(1-c) \quad (\text{from } a \geqslant b \text{ and } d > c) \\ &< b\cdot c\cdot(1-a)\cdot(1-d) - b\cdot c\cdot(1-b)\cdot(1-c) \\ &< b\cdot c\cdot(1-a)\cdot(1-d) - b\cdot c\cdot(1-a)\cdot(1-d) = 0, \end{aligned}$$

which is a contradiction, i.e. $c \geqslant d$. Therefore, MP is valid.

A modification of Definition 6 is

Definition 7.

$$V(A \supset B) = \left\langle \frac{b\cdot c}{(1-a)\cdot(1-d) + (1-b)\cdot(1-c)}, \frac{a\cdot d}{(1-b)\cdot(1-c) + (1-a)\cdot(1-d)} \right\rangle.$$

Check for correctness of the definition:

$$\frac{b\cdot c}{(1-a)\cdot(1-d)+(1-b)\cdot(1-c)}+\frac{a\cdot d}{(1-b)\cdot(1-c)+(1-a)\cdot(1-d)}$$
$$=\frac{b\cdot c+a\cdot d}{(1-a)\cdot(1-d)+(1-b)\cdot(1-c)}\leqslant\frac{(1-a)\cdot(1-d)+(1-b)\cdot(1-c)}{(1-a)\cdot(1-d)+(1-b)\cdot(1-c)}=1.$$

Check for correctness of MP: Let $\mu(A)\geqslant\gamma(A)$ and let $\mu(A\supset B)\geqslant\gamma(A\supset B)$, i.e.,

$$a\geqslant b \quad\text{and}\quad \frac{b\cdot c}{(1-a)\cdot(1-d)+(1-b)\cdot(1-c)}\geqslant\frac{a\cdot d}{(1-b)\cdot(1-c)+(1-a)\cdot(1-d)}.$$

Therefore, $b\cdot c\geqslant a\cdot d\geqslant b\cdot d$, i.e. $c\geqslant d$. Hence, MP is valid.

Definition 8.

$$V(A\supset B)=\left\langle\frac{b+c}{2-(a+d)},\frac{a+d}{2-(b+c)}\right\rangle.$$

Check for correctness of the definition:

$$\frac{b+c}{2-(a+d)}+\frac{a+d}{2-(b+c)}=\frac{2\cdot(a+b+c+d)-(b+c)^2-(a+d)^2}{(2-(a+d))\cdot(2-(b+c))}\leqslant 1,$$

because

$$(2-(a+d))\cdot(2-(b+c))-2\cdot(a+b+c+d)+(b+c)^2+(a+d)^2$$
$$=4-(a+d)\cdot(b+c)+2\cdot((b+c)-(a+d))^2\geqslant 0.$$

Check for correctness of MP: Let $\mu(A)\geqslant\gamma(A)$ and let $\mu(A\supset B)\geqslant\gamma(A\supset B)$, i.e.,

$$a\geqslant b \quad\text{and}\quad \frac{b+c}{2-(a+d)}\geqslant\frac{a+d}{2-(b+c)}.$$

Let us assume that $c<d$. Then

$$0\leqslant(b+c)\cdot(2-(b+c))-(a+d)\cdot(2-(a+d))$$
$$<(a+d)\cdot(2-(b+c))-(a+d)\cdot(2-(a+d))=(a+d)\cdot((b+c)-(a+d))<0,$$

which is a contradiction, i.e. $c\geqslant d$. Hence, MP is valid.

It should be noted that the last five definitions of the operation "implication" have the following drawbacks: they are what is usually called "external" operations (unlike conjunction, negation, etc.) and their evaluation requires exact comparison of real numbers, i.e. they are not continuous; on the other hand they do not seem to be very functional, in the sense that in logical calculations they are very cumbersome (introduce a lot of cases).

Compared with the standard implication ($\neg A\vee B$) implications 4 and 5 are stronger: if $A\supset B$ (in the sense of Definitions 4 or 5) is an IFT, then $\neg A\vee B$ is also an IFT.

Really, let $\mu(A\supset B)\geqslant\gamma(A\supset B)$ and let us assume that $\max(\mu(B),\gamma(A))<\min(\mu(A),\gamma(B))$, i.e. (max–min)-variant of the implication is not valid. If $\mu(B)\leqslant\gamma(B)$, then from Definition 4 $\gamma(A)\geqslant\mu(A)$ and $\gamma(A)\leqslant\max(\mu(B),\gamma(A))<\min(\mu(A),\gamma(B))\leqslant\gamma(A)$, which is a contradiction; if $\mu(A)\geqslant\gamma(A)$, then from Definition 4 $\mu(B)\geqslant\gamma(B)$ and $\mu(B)\leqslant\max(\mu(B),\gamma(A))<\min(\mu(A),\gamma(B))\leqslant\gamma(B)$, which is a contradiction; if $\mu(B)>\gamma(B)$ or $\mu(A)<\gamma(A)$, then from Definition 4: $\max(\mu(B),\gamma(A))\geqslant\mu(A)$, which is a contradiction.

Therefore, the (max–min)-variant of the implication is valid. For the implication from Definition 5 the assertion is proved analogically.

Unfortunately it is the opposite direction that would be useful.

An open problem is the characterization of all IFTs in the propositional language extended with the implications introduced above and possibly a proof of a completeness theorem: If A is an IFT, then it is provable from a set of axioms by means of the rule MP. Although such a theorem is very probable, the open problem is to find suitable axioms (especially for the implication). The above implications do not have some of the expected properties, e.g. a well known axiom for the classical implication $A \supset (B \supset A)$ is not an IFT, while for example $A \supset A$ is an IFT.

6. Intuitionistic fuzzy predicate logic

We can extend our consideration to the full language of first-order predicate logic (cf. [5, 10, 11]). Let us assume that the language is without functional symbols (for simplicity of presentation), i.e. atomic formulae are of the kind $P(x, y, \ldots, z)$, where P is an n-ary predicate symbol, $x, y, \ldots, z$ are n individual variables. Predicate logic formulae are built up from atomic formulae by means of the propositional operations "&", "$\veebar$", "$\supset$", "$\neg$" and by application of quantifiers, i.e. if A is a formula, x – a variable, then $\forall x A$ and $\exists x A$ are formulae.

Truth values of predicate formulae are obtained, if a domain of interpretation E is fixed, called usually the universe of the interpretation. Atomic formulae get their meaning through interpretation functions i which assign to each variable x an element $i(x) \in E$. The truth value of a given atomic formula $P(x, y, \ldots, z)$ under the interpretation function i is determined by an evaluation function V which assigns to each n-ary predicate symbol P a function $V(P): E^n \to [0,1] \times [0,1]$. The pair (E, V) is called a model. In this situation we have (for a given i): $V(P(x, y, \ldots, z)) = V(P)(i(x), i(y), \ldots, i(z))$. The evaluation V can be extended for arbitrary formulae by the inductive clauses for "&", "$\veebar$", "$\supset$", "$\neg$".

The definition for the quantifiers is as follows:

$$V(\forall x A) = \left\langle \min_{a \in E} \mu(A(i(x) = a), \max_{a \in E} \gamma(A(i(x) = a) \right\rangle,$$

$$V(\exists x A) = \left\langle \max_{a \in E} \mu(A(i(x) = a), \min_{a \in E} \gamma(A(i(x) = a) \right\rangle,$$

which can be denoted simply (where "x ranges over E") as

$$V(\forall x A) = \left\langle \min_x \mu(A), \max_x \gamma(A) \right\rangle \quad \text{and} \quad V(\exists x A) = \left\langle \max_x \mu(A), \min_x \gamma(A) \right\rangle.$$

Predicate IFTs can be defined just as their propositional counterparts: these are the formulae which get the valuation with $\mu \geqslant \gamma$ for every model and interpretation.

Theorem 8. *The logical axioms of the theory K (see [14]):*

(a) $A \supset (B \supset A)$,

(b) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,

(c) $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$

(d) $\forall x A(x) \supset A(t)$ *for the fixed variable* t,

(e) $\forall x(A \supset B) \supset (A \supset \forall x B)$.

are IFTs.

Proof. (e) $V(\forall x(A \supset B) \supset (A \supset \forall xB)) = V(\forall x(A \supset B)) \to V(A \supset \forall xB) = \langle \min_x \max(\mu(B), \gamma(A)), \max_x \min(\mu(A), \gamma(B)) \rangle \to \langle \max(\gamma(A), \min_x \mu(B)), \min(\mu(A), \max_x \gamma(B)) \rangle = \langle \max(\gamma(A), \min_x \mu(B), \max_x \min(\mu(A), \gamma(B))), \min(\mu(A), \max_x \gamma(B), \min_x \max(\mu(B), \gamma(A))) \rangle$ and $\max(\gamma(A), \min_x \mu(B), \max_x \min(\mu(A), \gamma(B))) \geqslant \max(\gamma(A), \min_x \mu(B)) = \min_x \max(\mu(B), \gamma(A)) \geqslant \min(\mu(A), \max_x \gamma(B), \min_x \max(\mu(B), \gamma(A)))$. □

Below we list some assertions, which are theorems of classical first-order logic (see [14]).

Theorem 9. *The following formulae are IFTs*:
(a) $(\forall x A(x) \supset B) \equiv \exists x(A(x) \supset B)$,
(b) $\exists x A(x) \supset B \equiv \forall x(A(x) \supset B)$,
(c) $B \supset \forall x A(x) \equiv \forall x(B \supset A(x))$,
(d) $B \supset \exists x A(x) \equiv \exists x(B \supset A(x))$,
(e) $(\forall x A \;\&\; \forall x B) \equiv \forall x(A \,\&\, B)$,
(f) $(\forall x A \vee \forall x B) \supset \forall x(A \vee B)$,
(g) $\neg \forall x A \equiv \exists x \neg A$,
(h) $\neg \exists x A \equiv \forall x \neg A$,
(i) $\forall x \forall y A \equiv \forall y \forall x A$,
(j) $\exists x \exists y A \equiv \exists y \exists x A$,
(k) $\exists x \forall y A \supset \forall y \exists x A$,
(l) $\forall x(A \supset B) \supset (\forall x A \supset \forall x B)$.

These results are extensions of the results from Section 2 and in fuzzy set theory. The link between the interpretations of quantifiers and the topological operators C (closure) and I (interior) defined over IFS [1] is obvious. The question how to relate the above results and the results from [4], where the modal operators $\square$ and $\diamond$ are interpreted in the terms of IFS, is of some interest and clearly, the basic problem which remains unsolved is the characterization of predicate IFTs by means of a calculus.

A partial solution of the problem of giving a calculus which generates all predicate IFTs is presented in the next theorem.

Theorem 10. *A prenex normal form A is an IFT iff it is a classical predicate tautology and its quantifier free matrix is a propositional IFT.*

Here a prenex form means (see [14]) a predicate formula in which all quantifiers are moved to the left. The proof is based on the fact that all predicate transformations leading to prenex form in classical logic are valid for the IF case also.

7. Conclusions

In this part of the series, written in 1990, we included all published and unpublished results in the area of the IFPC. IFPC contains some principally new elements compared to the standard fuzzy logics. Its greater descriptive power will be demonstrated better, in the subsequent parts of our series because there essentially new elements will be introduced for the fuzzy as well as for the standard logics.

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