Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy G-modules

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Abstract: In this paper, the notion of intuitionistic fuzzy G-modules on a G-module *M* over a field *K* is introduced. The quotient intuitionistic fuzzy G-modules are defined and discussed. *A* homomorphism of G-module *M* onto *M** is established. Also we introduced (weak) intuitionistic fuzzy G-homomorphism (isomorphism). Intersection, Sum, Product and Cartesian product of two intuitionistic fuzzy G-modules are also discussed.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy G-module, (α, β)–cut set, Support, Quotient intuitionistic fuzzy G-module.
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1 Introduction

After the introduction of fuzzy sets by Zadeh [17], a number of generalizations of this fundamental concept have come up. The notion of intuitionistic fuzzy sets introduced by Atanassov [2] is one among them. Algebraic structures play a vital role in Mathematics and numerous applications of these structures are seen in many disciplines such as computer sciences, information sciences, theoretical physics, control engineering and so on. This inspires researchers to study and carry out research in various concepts of abstract algebra in fuzzy setting. Biswas [5] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. Fuzzy submodules of a module *M* over a ring *R* were first introduced by Naevoita and Ralescu [11]. Since then different types of fuzzy submodules were investigated in the last two decades. Shery Fernadez introduced the notion of fuzzy G-modules in [8]. Davvaz et.al. in 2006 [7] defined intuitionistic fuzzy submodules of a module *M* over a ring *R*. Many more results have been obtained by other researchers on intuitionistic fuzzy modules (see [9, 12, 13, 14]).

In this paper, we introduce the notion of intuitionistic fuzzy G-modules and established many results.

2 Preliminaries

In this section, we recall some definitions and results which will be used later.

Definition (2.1)[2] Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in X$, respectively, and for any $x \in X$, we have $\mu_A(x) + \nu_A(x) \leq 1$.

Definition (2.2)[4] For any IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, of X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \text{ for all } x \in X.$$

Then $\pi_A(x)$ is called the degree of indeterminacy of x in A.

Remark (2.3)(i) When $\pi_A(x) = 0$, i.e., when $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x)$, $\forall\, x \in X$. Then A is called a fuzzy set.
(ii) For convenience, we write the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition (2.4)[4] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$ be any two IFS's of X, then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$

(iii) $A^c = \{\langle x, (\mu_A{}^c)(x), (\nu_A{}^c)(x)\rangle : x \in X\}$, where

$$(\mu_A{}^c)(x) = \nu_A(x) \text{ and } (\nu_A{}^c)(x) = \mu_A(x) \text{ for all } x \in X$$

(iv) $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$ (called the Necessity operator on IFS A)

(v) $\lozenge A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$ (called the Possibility operator on IFS A)

(vi) $A \cap B = \{\langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cap \nu_B)(x)\rangle : x \in X\}$, where

$$(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$$

and

$$(\nu_A \cap \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\} = \nu_A(x) \vee \nu_B(x)$$

(vii) $A \cup B = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cup \nu_B)(x)\rangle : x \in X\}$, where

$$(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$$

and

$$(\nu_A \cup \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\} = \nu_A(x) \wedge \nu_B(x)$$

(viii) $A \times B = \{\langle (x, y), \mu_{A\times B}(x, y), \nu_{A\times B}(x, y)\rangle : x \in X \text{ and } y \in Y\}$, where

$$\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

and

$$\nu_{A\times B}(x, y) = \max\{\nu_A(x), \nu_B(y)\}.$$

Definition (2.5)[14] Let $(X, \,.\,)$ be a groupoid and A, B be two IFS's of X. Then the intuitionistic fuzzy sum and product of A and B are denoted by $A + B$ and AB and defined as: $(A + B)(x) = (\mu_{A+B}(x), \nu_{A+B}(x))$ and $AB(x) = (\mu_{AB}(x), \nu_{AB}(x))$, where

$$\mu_{A+B}(x)=\begin{cases}\left(\underset{x=a+b}{\vee}\{\mu_A(a)\wedge\mu_B(b)\},\ \underset{x=a+b}{\wedge}\{\nu_A(a)\vee\nu_B(b)\}\right) & ;\ if\ x=a+b\\ (0,1) & ;\ \text{otherwise}\end{cases}\quad ;\ \text{for all } x\in X$$

$$\mu_{AB}(x)=\begin{cases}\left(\underset{\substack{x=\sum a_i b_i\\ i<\infty}}{\vee}\left\{\underset{i}{\wedge}\left(\mu_A(a_i)\wedge\mu_B(b_i)\right)\right\},\ \underset{\substack{x=\sum a_i b_i\\ i<\infty}}{\wedge}\left\{\underset{i}{\vee}\left(\nu_A(a_i)\vee\nu_B(b_i)\right)\right\}\right) & ;\ if\ x=\sum_{i<\infty}a_i b_i\\ (0,1) & ;\ \text{otherwise}\end{cases}\quad ,\ \text{for all } x\in X$$

Definition (2.6)[13] Let A be an intuitionistic fuzzy set of a universe set X. Then (α, β)-cut of A is a crisp subset $C_{\alpha,\beta}(A)$ defined as $C_{\alpha,\beta}(A) = \{x : x\in X$ such that $\mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta\}$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$

Definition (2.7)[15] Let $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X\}$ be an IFS of a universe set X, then support of A in X is denoted by $Supp_X(A)$ and is defined as

$$Supp_X(A)=\{x\in X:\ \mu_A(x)>0 \text{ and } \nu_A(x)<1\}$$

Clearly, $\text{Supp}_X(A)=\bigcup\{C_{\alpha,\beta}(A):\ \text{for all } \alpha,\beta\in(0,1] \text{ such that } 0<\alpha+\beta\leq 1\}$.

Using definition (2.7), the following results are easy to verify

Proposition (2.8)[15] Let A and B be two IFS's of the universe set X, then the following results holds

(i) If $A\subseteq B \Rightarrow \text{Supp}_X(A)\subseteq\text{Supp}_X(B)$

(ii) $\text{Supp}_X(A\cap B)=\text{Supp}_X(A)\cap\text{Supp}_X(B)$

(iii) $\text{Supp}_X(A\cup B)=\text{Supp}_X(A)\cup\text{Supp}_X(B)$

(iv) $\text{Supp}_X(A\times B)=\text{Supp}_X(A)\times\text{Supp}_X(B)$

(v) $\text{Supp}_X(A+B)=\text{Supp}_X(A)+\text{Supp}_X(B)$, where $(X, +)$ is a groupoid

Theorem (2.9)[10] Let X be a non-empty set. For an intuitionistic fuzzy set A in X, we have

$$A=\bigcup_{\substack{\alpha,\beta\in[0,1]\\ \alpha+\beta\leq 1}}{}_{(\alpha,\beta)}A$$

where ${}_{(\alpha,\beta)}A$ denote the IFS defined by ${}_{(\alpha,\beta)}A(x)=\begin{cases}(\alpha,\beta) & ;if\ x\in C_{(\alpha,\beta)}(A)\\ (0,1) & otherwise\end{cases}\ ;\ \forall\, x\in X$

and $\bigcup$ *denote* the union given in the definition (2.4)(vii).

Definition (2.10)[12, 13] Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let A and B be IFS's of X and Y respectively. Then the image of A under the map f is denoted by f (A) and is defined as $f(A)(y)=\left(\mu_{f(A)}(y),\ \nu_{f(A)}(y)\right)$, where

$$\mu_{f(A)}(y)=\begin{cases}\vee\{\mu_A(x):x\in f^{-1}(y)\}\\ 0\ ;\ \text{otherwise}\end{cases}\quad \text{and}\quad \nu_{f(A)}(y)=\begin{cases}\wedge\{\nu_A(x):x\in f^{-1}(y)\}\\ 1\ ;\ \text{otherwise}\end{cases}$$

Also the pre-image of B under f is denoted by f^{-1}(B) and is defined as $f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)),\ where$
$\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))\ ;\ \forall\ x \in X$.

Proposition (2.11)[15] Let $f: X \to Y$ be a mapping and A, B are IFS of X and Y respectively. Then the following result holds

(*i*) $f(\text{Supp}_X(A)) \subseteq \text{Supp}_Y(f(A))$ and equality hold when the map f is bijective

(*ii*) $f^{-1}(\text{Supp}_Y(B)) = \text{Supp}_X(f^{-1}(B))$

Definition (2.12)[6] Let G be a group and M be a vector space over a field K. Then M is called a G-module if for every $g \in$G and $m\in$M, $\exists$ a product (called the action of G on M), $gm\in$M satisfies the following axioms

i) $1_G \cdot m = m$, $\forall\ m\in$M (1_G being the identity of G)
ii) $(g{\cdot}h){\cdot}m = g{\cdot}(h{\cdot}m)$, $\forall\ m\in$M, $g, h\in$G
iii) $g{\cdot}(k_1m_1 + k_2m_2) = k_1(g{\cdot}m_1) + k_2(g{\cdot}m_2)$, $\forall\ k_1, k_2 \in$K; $m_1, m_2\in$M and $g\in$G.

Since G acts on M on the left hand side, M may be called a left G-module. In a similar way, we can define right G-module. We shall consider only left G-modules. *A* parallel study is possible using right G-modules also.

Definition (2.13)[6] Let G be a group and M be a G-module over the field K. Let N be a subspace of the vector space over K. Then N is called a G-submodule of M if $an_1 + bn_2 \in$N, for all $a, b\in$K and $n_1, n_2 \in$ N

Definition (2.14)[6] Let M and M^* be G-modules. *A* mapping $f: M \to M^*$ is a G-module homomorphism if

(i) $f(k_1m_1 + k_2m_2) = k_1 f(m_1) + k_2 f(m_2)$
(ii) $f(gm) = g\,f(m)$, $\forall\ k_1, k_2 \in$K ; $m, m_1, m_2\in$M and $g\in$G.

Definition (2.15)[6] Let M and M^* be G-modules and let $f : M \to M^*$ is a G-module homomorphism. Then Ker f ={$m\in$M : $f(m) = 0^*$} is a G-submodule of M and Im f ={ $f(m)$: $m\in$M} is a G-submodule of M*, and there is a G-isomorphism $M/\text{Ker} f \to \text{Im} f$ such that $m + \text{Ker} f \mapsto f(m)$.

Definition (2.16)[6] Let M be a G-module. *A* subspace N of M is a G-module if N is also a G-module under the same action of G.

Proposition (2.17)[6] If M is a G-module and N is a G-submodule of M, then M / N is a G-module.

3 Intuitionistic fuzzy G-modules

Definition (3.1) Let G be a group and M be a G-module over K, which is a subfield of **C.** Then a intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M
such that following conditions are satisfied

(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, $\forall\, a, b \in K$ and $x, y \in M$ and
(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall\, g \in G$; $m \in M$

Example (3.2) Let G = {1, -1}, $M = R^n$ over R. Then M is a G-module. Define the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ on M by

$$\mu_A(x) = \begin{cases} 1 & ;\ if\ x = 0 \\ 0.5 & ;\ if\ x \neq 0 \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & ;\ if\ x = 0 \\ 0.25 & ;\ if\ x \neq 0 \end{cases}$$

Where $x = (x_1, x_2, \ldots\ldots, x_n) \in R^n$. Then A is an intuitionistic fuzzy G-module on M.

Proof. Let $a, b \in R$ and $x = (x_1, x_2, \ldots\ldots, x_n)$, $y = (y_1, y_2, \ldots\ldots, y_n) \in R^n$. Then $ax + by = (ax_1 + by_1, ax_2 + by_2, \ldots\ldots, ax_n + by_n)$, we have

$$\mu_A(ax+by) = \begin{cases} 1 & ;if\ \ a = b = 0 \text{ or } x = y = 0 \\ 1 & ;if\ \ ax_i + by_i = 0, \text{ for all } i \\ 0.5 & ;if\ \ ax_i + by_i \neq 0, \text{ for some } i \end{cases} \quad \text{and}$$

$$\nu_A(ax+by) = \begin{cases} 0 & ;if\ \ a = b = 0 \text{ or } x = y = 0 \\ 0 & ;if\ \ ax_i + by_i = 0, \text{ for all } i \\ 0.25 & ;if\ \ ax_i + by_i \neq 0, \text{ for some } i \end{cases}$$

Case (i) when $x = y = 0$, then $\mu_A(ax + by) = 1 = 1 \wedge 1 \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(ax + by) = 0 = 0 \vee 0 \leq \nu_A(x) \vee \nu_A(y)$
Case (ii) when $x = 0$ and $y \neq 0$, i.e., $y_i \neq 0$, for some i, then
$\mu_A(ax + by) = \mu_A(by) = 0.5 = 1 \wedge 0.5 \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(ax + by) = \nu_A(by) = 0.25 = 0 \vee 0.25 \leq \nu_A(x) \vee \nu_A(y)$
Case (iii) when $x \neq 0$ and $y = 0$, i.e., $x_i \neq 0$, for some i, then
$\mu_A(ax + by) = \mu_A(ax) = 0.5 = 0.5 \wedge 1 \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(ax + by) = \nu_A(ax) = 0.25 = 0.25 \vee \ \leq \nu_A(x) \vee \nu_A(y)$
Case (iv) when $x \neq 0$ and $y \neq 0$

Subcase (i) when $ax_i + by_i = 0$, for all i, then
$\mu_A(ax + by) = 1 > 0.5 = 0.5 \wedge 0.5 \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(ax + by) = 0 < 0.25 = 0.25 \vee 0.25 \leq \nu_A(x) \vee \nu_A(y)$

Subcase (ii) when $ax_i + by_i \neq 0$, for some i, then
$\mu_A(ax + by) = 0.5 = 0.5 \wedge 0.5 \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(ax + by) = 0.25 = 0.25 \vee 0.25 \leq \nu_A(x) \vee \nu_A(y)$
Thus, in all the cases, we find that
$\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$
For, $g \in G$ and $x \in M$, we have $gx = (\pm x_1, \pm x_2, \ldots\ldots, \pm x_n)$, then

$$\mu_A(gx) = \begin{cases} 1 & ;\ if\ \ \pm x = 0 \\ 0.5 & ;if\ \ \pm x_i \neq 0 \text{ for some } i \end{cases} \quad \text{and} \quad \nu_A(gx) = \begin{cases} 0 & ;\ if\ \ \pm x = 0 \\ 0.25 & ;if\ \ \pm x_i \neq 0 \text{ for some } i \end{cases}$$

$\therefore \quad \mu_A(gx) \geq \mu_A(x)$ and $\nu_A(gx) \leq \mu_A(x)$.
Hence A is intuitionistic fuzzy G-module on M.

Example (3.3) Consider the G-module $M = R(i) = C$ over the field R and G ={1, -1}.

Define the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ by

$$\mu_A(z) = \begin{cases} 1 & ; \text{if } z = 0 \\ 0.5 & ; \text{if } z \in R\text{-}\{0\} \\ 0.25 & ; \text{if } z \in R(i)\text{-R} \end{cases} \quad \text{and} \quad \nu_A(z) = \begin{cases} 0 & ; \text{if } z = 0 \\ 0.25 & ; \text{if } z \in R\text{-}\{0\} \\ 0.5 & ; \text{if } z \in R(i)\text{-R} \end{cases}$$

Then A is a intuitionistic fuzzy G-module on M.

Proof. Let $a, b \in$R and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in$M, where $x_1, y_1, x_2, y_2 \in$R, then $a z_1 + b z_2 = (ax_1 + bx_2) + i(ay_1 + by_2)$, we have

$$\mu_A(az_1 + bz_2) = \begin{cases} 1 & ; \text{if } ax_1 + bx_2 = 0 \ \& \ ay_1 + by_2 = 0 \\ 0.5 & ; \text{if } ax_1 + bx_2 \neq 0 \ \& \ ay_1 + by_2 = 0 \\ 0.25 & ; \text{if } ay_1 + by_2 \neq 0 \end{cases} \quad \text{and}$$

$$\nu_A(az_1 + bz_2) = \begin{cases} 0 & ; \text{if } ax_1 + bx_2 = 0 \ \& \ ay_1 + by_2 = 0 \\ 0.25 & ; \text{if } ax_1 + bx_2 \neq 0 \ \& \ ay_1 + by_2 = 0 \\ 0.5 & ; \text{if } ay_1 + by_2 \neq 0 \end{cases}$$

Now, we have 16 cases of which 10 are essential

Case(i) when $x_1 = y_1 = x_2 = y_2 = 0$, then $z_1 = 0$ and $z_2 = 0$. Also, $ax_1 + bx_2 = 0$ and $ay_1 + by_2 = 0$

So, $\mu_A(az_1 + bz_2) = 1 = 1 \wedge 1 = \mu_A(z_1) \wedge \mu_A(z_2)$ and

$\nu_A(az_1 + bz_2) = 0 = 0 \vee 0 = \nu_A(z_1) \vee \nu_A(z_2)$

Case(ii) when $x_1 \neq 0$, $y_1 = x_2 = y_2 = 0$ (same as $x_2 \neq 0$, $y_1 = x_1 = y_2 = 0$), then $z_1 = x_1 \neq 0$ and $z_2 = 0$. *Also*, $ax_1 + bx_2 = ax_1$ and $ay_1 + by_2 = 0$

$$\mu_A(az_1 + bz_2) = \left.\begin{cases} 1 & ; a = 0 \\ 0.5 & ; a \neq 0 \end{cases}\right\} \geq 0.5 = 0.5 \wedge 1 = \mu_A(z_1) \wedge \mu_A(z_2) \text{ and}$$

$$\nu_A(az_1 + bz_2) = \left.\begin{cases} 0 & ; a = 0 \\ 0.25 & ; a \neq 0 \end{cases}\right\} \leq 0.25 = 0.25 \vee 0 = \nu_A(z_1) \vee \nu_A(z_2)$$

Case(iii) when $y_1 \neq 0$, $x_1 = x_2 = y_2 = 0$ (same as $y_2 \neq 0$, $x_1 = x_2 = y_1 = 0$), then $z_1 = iy_1 \neq 0$ and $z_2 = 0$. Also, $ax_1 + bx_2 = 0$ and $ay_1 + by_2 = ay_1$

$$\mu_A(az_1 + bz_2) = \left.\begin{cases} 1 & ; a = 0 \\ 0.25 & ; a \neq 0 \end{cases}\right\} \geq 0.25 = 0.25 \wedge 1 = \mu_A(z_1) \wedge \mu_A(z_2) \text{ and}$$

$$\nu_A(az_1 + bz_2) = \left.\begin{cases} 0 & ; a = 0 \\ 0.5 & ; a \neq 0 \end{cases}\right\} \leq 0.5 = 0.5 \vee 0 = \nu_A(z_1) \vee \nu_A(z_2)$$

Case(iv) when $x_1 \neq 0$, $y_1 \neq 0$, $x_2 = y_2 = 0$ (same as $x_2 \neq 0$, $y_2 \neq 0$, $x_1 = y_1 = 0$), then $z_1 = x_1 + iy_1 \neq 0$ and $z_2 = 0$, Also, $ax_1 + bx_2 = ax_1$ and $ay_1 + by_2 = ay_1$

$$\mu_A(az_1 + bz_2) = \left.\begin{cases} 1 & ; a = 0 \\ 0.25 & ; a \neq 0 \end{cases}\right\} \geq 0.25 = 0.25 \wedge 1 = \mu_A(z_1) \wedge \mu_A(z_2) \text{ and}$$

$$\nu_A(az_1 + bz_2) = \left.\begin{cases} 0 & ; a = 0 \\ 0.5 & ; a \neq 0 \end{cases}\right\} \leq 0.5 = 0.5 \vee 0 = \nu_A(z_1) \vee \nu_A(z_2)$$

Case(v) when $x_1 \neq 0$, $x_2 \neq 0$, $y_1 = y_2 = 0$, then $z_1 = x_1 \neq 0$ and $z_2 = x_2 \neq 0$,

Also, $ax_1 + bx_2 = ax_1 + bx_2$ and $ay_1 + by_2 = 0$

$$\mu_A(az_1+bz_2)=\left\{\begin{array}{ll} 1 & ;\left\{\begin{array}{l} a=0\,\&\,b=0 \\ or\ ax_1+bx_2=0 \end{array}\right\} \\ 0.5 & ;\left\{\begin{array}{l} (a\neq 0, b=0)\ or\ (b\neq 0, a=0) \\ or\quad ax_1+bx_2\neq 0 \end{array}\right\} \end{array}\right\}\geq 0.5=0.5\wedge 0.5=\mu_A(z_1)\wedge\mu_A(z_2) \text{ and}$$

$$\nu_A(az_1+bz_2)=\left\{\begin{array}{ll} 0 & ;\left\{\begin{array}{l} a=0\,\&\,b=0 \\ or\ ax_1+bx_2=0 \end{array}\right\} \\ 0.25 & ;\left\{\begin{array}{l} (a\neq 0, b=0)\ or\ (b\neq 0, a=0) \\ or\quad ax_1+bx_2\neq 0 \end{array}\right\} \end{array}\right\}< 0.25=0.25\vee 0.25=\nu_A(z_1)\vee\nu_A(z_2)$$

Case(vi) when $y_1 \neq 0$, $y_2 \neq 0$, $x_1 = x_2 = 0$, then $z_1 = iy_1 \neq 0$ and $z_2 = iy_2 \neq 0$, Also, $ax_1 + bx_2 = 0$ and $ay_1 + by_2 = ay_1 + by_2$

$$\mu_A(az_1+bz_2)=\left\{\begin{array}{ll} 1 & ;\left\{\begin{array}{l} a=0\,\&\,b=0 \\ or\ ay_1+by_2=0 \end{array}\right\} \\ 0.25 & ;\left\{\begin{array}{l} (a\neq 0\ \&\ b=0) \\ or(b\neq 0\ \&\ a=0) \\ or\ ay_1+by_2\neq 0 \end{array}\right\} \end{array}\right\}\geq 0.25=0.5\wedge 0.25=\mu_A(z_1)\wedge\mu_A(z_2) \text{ and}$$

$$\nu_A(az_1+bz_2)=\left\{\begin{array}{ll} 0 & ;\left\{\begin{array}{l} a=0\,\&\,b=0 \\ or\ ay_1+by_2=0 \end{array}\right\} \\ 0.5 & ;\left\{\begin{array}{l} (a\neq 0\ \&\ b=0) \\ or(b\neq 0\,\&\,a=0) \\ or\ ay_1+by_2\neq 0 \end{array}\right\} \end{array}\right\}\leq 0.5=0.25\vee 0.5=\nu_A(z_1)\vee\nu_A(z_2)$$

Case(vii) when $x_1 \neq 0$, $y_1 = 0$, $x_2 = 0$, $y_2 \neq 0$ (same as $x_2 \neq 0$, $y_2 = 0$, $x_1 = 0$, $y_1 \neq 0$), then $z_1 = x_1 \neq 0$ and $z_2 = iy_2 \neq 0$, Also, $ax_1 + bx_2 = ax_1$ and $ay_1 + by_2 = by_2$

$$\mu_A(az_1+bz_2)=\left\{\begin{array}{ll} 1 & ;\ a=0\ \&\ b=0 \\ 0.5 & ;\ a\neq 0\ \&\ b=0 \\ 0.25 & ;\ b\neq 0 \end{array}\right\}\geq 0.25=0.5\wedge 0.25=\mu_A(z_1)\wedge\mu_A(z_2) \text{ and}$$

$$\nu_A(az_1+bz_2)=\left\{\begin{array}{ll} 0 & ;\ a=0\,\&\,b=0 \\ 0.25 & ;\ a\neq 0\ \&\ b=0 \\ 0.5 & ;\ b\neq 0 \end{array}\right\}\leq 0.5=0.25\vee 0.5=\nu_A(z_1)\vee\nu_A(z_2)$$

Case(viii) when $x_1 \neq 0$, $y_1 \neq 0$, $x_2 \neq 0$, $y_2 \neq 0$, then $z_1 = x_1 + iy_1 \neq 0$ and $z_2 = x_2 + iy_2 \neq 0$, Also, $ax_1 + by_1 = ax_1 + by_1$ and $ay_1 + by_2 = ay_1 + by_2$

$$\mu_A(az_1+bz_2)=\begin{Bmatrix} 1 & ;\begin{Bmatrix}(a=0\,\&\,b=0)\\ or\ (ax_1+bx_2=0\,\&\,ay_1+by_2=0)\end{Bmatrix} \\ 0.5 & ;\ ax_1+bx_2\neq 0\,\&\,ay_1+by_2=0 \\ 0.25 & ;\ ay_1+by_2\neq 0 \end{Bmatrix}\geq 0.25=0.25\wedge 0.25=\mu_A(z_1)\wedge\mu_A(z_2) \text{ and}$$

$$\nu_A(az_1+bz_2)=\begin{Bmatrix} 0 & ;\begin{Bmatrix}(a=0\,\&\,b=0)\\ or\ (ax_1+bx_2=0\,\&\,ay_1+by_2=0)\end{Bmatrix} \\ 0.25 & ;\ ax_1+bx_2\neq 0\,\&\,ay_1+by_2=0 \\ 0.5 & ;\ ay_1+by_2\neq 0 \end{Bmatrix}\leq 0.5=0.5\vee 0.5=\nu_A(z_1)\vee\nu_A(z_2)$$

Case(ix) when $x_1 = 0$, $y_1 \neq 0$, $x_2 \neq 0$, $y_2 \neq 0$ (same as $x_2 = 0$, $y_1 \neq 0$, $x_1 \neq 0$, $y_2 \neq 0$), then $z_1 = iy_1 \neq 0$ and $z_2 = x_2 + iy_2 \neq 0$, Also, $ax_1 + bx_2 = ax_1$ and $ay_1 + by_2 = ay_1 + by_2$

$$\mu_A(az_1+bz_2)=\begin{Bmatrix} 1 & ;\ a=0\,\&\,b=0 \\ 0.5 & ;\ a\neq 0\ \&\ ay_1+by_2=0 \\ 0.25 & ;\ ay_1+by_2\neq 0 \end{Bmatrix}\geq 0.25=0.25\wedge 0.25=\mu_A(z_1)\wedge\mu_A(z_2) \text{ and}$$

$$\nu_A(az_1+bz_2)=\begin{Bmatrix} 0 & ;\ a=0\,\&\,b=0 \\ 0.25 & ;\ a\neq 0\ \&\ ay_1+by_2=0 \\ 0.5 & ;\ ay_1+by_2\neq 0 \end{Bmatrix}\leq 0.5=0.5\vee 0.5=\nu_A(z_1)\vee\nu_A(z_2)$$

Case(x) when $x_1 \neq 0$, $y_1 = 0$, $x_2 \neq 0$, $y_2 \neq 0$ (same as $x_1 \neq 0$, $y_2 = 0$, $x_2 \neq 0$, $y_1 \neq 0$), then $z_1 = x_1 \neq 0$ and $z_2 = x_2 + iy_2 \neq 0$, Also, $ax_1 + by_1 = ax_1$ and $ax_2 + by_2 = ax_2 + by_2$

$$\mu_A(az_1+bz_2)=\begin{Bmatrix} 1 & ;\ a=0\,\&\,b=0 \\ 0.5 & ;\ a\neq 0\ \&\ ax_2+by_2=0 \\ 0.25 & ;\ ax_2+by_2\neq 0 \end{Bmatrix}\geq 0.25=0.5\wedge 0.25=\mu_A(z_1)\wedge\mu_A(z_2) \text{ and}$$

$$\nu_A(az_1+bz_2)=\begin{Bmatrix} 0 & ;\ a=0\,\&\,b=0 \\ 0.5 & ;\ a\neq 0\ \&\ ax_2+by_2=0 \\ 0.25 & ;\ ax_2+by_2\neq 0 \end{Bmatrix}\leq 0.5=0.25\vee 0.5=\nu_A(z_1)\vee\nu_A(z_2)$$

Thus, in all the cases, we find that

$\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$

For, $g \in G$ and $z \in M$, we have

$\mu_A(gz) = \mu_A(\pm 1.z) \geq \mu_A(z)$ and $\nu_A(gz) = \nu_A(\pm 1.z) \leq \nu_A(z)$;

Hence A is intuitionistic fuzzy G-module on M.

Remark (3.4) In example (3.3), if we take the group as $G = \{1, -1, i, -i\}$, then A is not an intuitionistic fuzzy G-module on M, for the conditions

$\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall\, g \in G$; $m \in M$, are not satisfied

e.g., take $m = 3$ and $g = i$, then

$\mu_A(3i) = 0.25 \not\geq 0.5 = \mu_A(3)$ also, $\nu_A(3i) = 0.5 \not\leq 0.25 = \nu_A(3)$.

Lemma (3.5) Let *M* be a G-module over K and *A* be intuitionistic fuzzy G-module on M. Then either $C_{(\alpha,\beta)}(A) = \varnothing$ or a G-submodule of M, where $\alpha,\ \beta \in [0,1]$ such that $\alpha + \beta \le 1$.

Proof. If $C_{(\alpha,\beta)}(A) = \varnothing$, then nothing to prove. Therefore, suppose that $C_{(\alpha,\beta)}(A) \ne \varnothing$.
Consider $x, y \in C_{(\alpha,\beta)}(A)$ and $a, b \in$ K be any elements, then we have
$\mu_A(ax+by) \ge \mu_A(x) \wedge \mu_A(y) \ge \alpha$ and $\nu_A(ax+by) \le \nu_A(x) \vee \nu_A(y) \le \beta$
$\Rightarrow\ ax+by \in C_{(\alpha,\beta)}(A).$
Also, for any $g \in G$ and $x \in C_{(\alpha,\beta)}(A)$, we have
$\mu_A(gx) \ge \mu_A(x) \ge \alpha$ and $\nu_A(gx) \le \nu_A(x) \le \beta$
$\Rightarrow\ gx \in C_{(\alpha,\beta)}(A)$. Hence $C_{(\alpha,\beta)}(A)$ is a G-submodule of M.

Proposition (3.6) Let *M* be a G-module over K and *A* be IFS of M, then *A* is intuitionistic fuzzy G-module on *M* if and only if either $C_{(\alpha,\beta)}(A) = \varnothing$ or $C_{(\alpha,\beta)}(A)$, for all α, β ∈[0, 1] such that α + β ≤ 1, is a G-submodule of M.
Proof. Firstly, let *A* be intuitionistic fuzzy G-module on M. Then either $C_{(\alpha,\beta)}(A) = \varnothing$ or $C_{(\alpha,\beta)}(A)$ is a G-submodule of *M* follows from Lemma (3.5).
Conversely, let $C_{(\alpha,\beta)}(A)$ be a G-submodule of *M* for all α, β ∈[0, 1] such that α + β ≤ 1.
To show that *A* is intuitionistic fuzzy G-module on M. Let $x, y \in M$ and $a, b \in K$ be any elements. Wlog, let $x \in C_{(\alpha_1,\beta_1)}(A)$ and $y \in C_{(\alpha_2,\beta_2)}(A)$ for some $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_i + \beta_i \le 1$, *where* $i = 1, 2$.
Case(i) When $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$, then $x, y \in C_{(\alpha_1,\beta_1)}(A)$ and so $ax + by \in C_{(\alpha_1,\beta_1)}(A)$
$\Rightarrow\ \mu_A(ax + by) \ge \alpha_1$ and $\nu_A(ax + by) \le \beta_1$.
But $\alpha_1 = \alpha_1 \wedge \alpha_2 = \mu_A(x) \wedge \mu_A(y)$ and $\beta_1 = \beta_1 \vee \beta_2 = \nu_A(x) \vee \nu_A(y)$.
Therefore, $\mu_A(ax + by) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \le \nu_A(x) \vee \nu_A(y)$

Case(ii) When $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, then $x, y \in C_{(\alpha_1,\beta_2)}(A)$ and so $ax + by \in C_{(\alpha_1,\beta_2)}(A)$, for
$\alpha_1 < \alpha_2\ \Rightarrow\ \alpha_1 + \beta_2 < \alpha_2 + \beta_2 \le 1$ i.e., $\alpha_1 + \beta_2 \le 1$.
$\Rightarrow\ \mu_A(ax + by) \ge \alpha_1$ and $\nu_A(ax + by) \le \beta_2$.
But $\alpha_1 = \alpha_1 \wedge \alpha_2 = \mu_A(x) \wedge \mu_A(y)$ and $\beta_2 = \beta_1 \vee \beta_2 = \nu_A(x) \vee \nu_A(y)$.
Therefore, $\mu_A(ax + by) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \le \nu_A(x) \vee \nu_A(y)$
Case(iii) When $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$, then $x, y \in C_{(\alpha_2,\beta_2)}(A)$ and so $ax + by \in C_{(\alpha_2,\beta_2)}(A)$
$\Rightarrow\ \mu_A(ax + by) \ge \alpha_2$ and $\nu_A(ax + by) \le \beta_2$.
But $\alpha_2 = \alpha_1 \wedge \alpha_2 = \mu_A(x) \wedge \mu_A(y)$ and $\beta_2 = \beta_1 \vee \beta_2 = \nu_A(x) \vee \nu_A(y)$
Therefore, $\mu_A(ax + by) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \le \nu_A(x) \vee \nu_A(y)$
Case(iv) When $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, then $x, y \in C_{(\alpha_2,\beta_1)}(A)$ and so $ax + by \in C_{(\alpha_2,\beta_1)}(A)$, for
$\alpha_2 < \alpha_1\ \Rightarrow\ \alpha_2 + \beta_2 < \alpha_1 + \beta_2 \le 1$ i.e., $\alpha_2 + \beta_1 \le 1$
$\Rightarrow\ \mu_A(ax + by) \ge \alpha_2$ and $\nu_A(ax + by) \le \beta_1$.
But $\alpha_2 = \alpha_1 \wedge \alpha_2 = \mu_A(x) \wedge \mu_A(y)$ and $\beta_1 = \beta_1 \vee \beta_2 = \nu_A(x) \vee \nu_A(y)$

Therefore, $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$.
Thus, in all cases, we see that $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$.
Further, let $g \in G$ and $\mu_A(x) = \alpha$ and $\nu_A(x) = \beta$, where $\alpha + \beta \leq 1$, then $x \in C_{(\alpha,\beta)}(A)$.
But $C_{(\alpha,\beta)}(A)$ is a G-submodule of M and so, $g\,x \in C_{(\alpha,\beta)}(A)$.
Thus $\mu_A(gx) \geq \alpha = \mu_A(x)$ and $\nu_A(gx) \leq \beta = \nu_A(x)$.
Hence A is an intuitionistic fuzzy is G-module on M.

Theorem (3.7) Let M be a G-module over K and A be an intuitionistic fuzzy G-module on M. Then $\mathrm{Supp}_M(A)$ is a G-submodule of M.

Proof. Let $x, y \in \mathrm{Supp}_M(A)$ and $a, b \in$ K, then $\mu_A(x) > 0$, $\nu_A(x) < 1$ and $\mu_A(y) > 0$, $\nu_A(y) < 1$
$\Rightarrow \mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y) > 0$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y) < 1$.
Therefore, $ax + by \in \mathrm{Supp}_M(A)$.
Also, $\mu_A(gx) \geq \mu_A(x) > 0$ and $\nu_A(gx) \leq \nu_A(x) < 1$, for every $g \in G$ and $x \in M$.
Therefore, $gx \in \mathrm{Supp}_M(A)$. Hence $\mathrm{Supp}_M(A)$ is a G-submodule of M.

Remark (3.8) The converse of Theorem (3.7) need not be true, i.e., $\mathrm{Supp}_M(A)$ may be G-submodule of M, but A may not be intuitionistic fuzzy G-module of an G-module M over K.

Example (3.9) Take M = C and G = {1, -1, i, - i}, then M is a G-module over R. Take the intuitionistic fuzzy set A as in example (3.3). Clearly, $\mathrm{Supp}_M(A) =$ M, is a G-submodule of M, but A is not intuitionistic fuzzy G-module on M (see Remark (3.4)).

Theorem (3.10) Let M be a G-module over K and A, B be two intuitionistic fuzzy G-modules on M. Then $A \cap$ B is also an intuitionistic fuzzy G-module on M.

Proof. Let $a, b \in$ K and $x, y \in$ M, then

$$\begin{aligned}\mu_{A\cap B}(ax+by) &= \mu_A(ax+by) \wedge \mu_B(ax+by)\\ &\geq \{\mu_A(x) \wedge \mu_A(y)\} \wedge \{\mu_B(x) \wedge \mu_B(y)\}\\ &= \{\mu_A(x) \wedge \mu_B(x)\} \wedge \{\mu_A(y) \wedge \mu_B(y)\}\\ &= \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)\end{aligned}$$

Thus, $\mu_{A\cap B}(ax+by) \geq \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)$
Similarly, we can show that $\nu_{A\cap B}(ax+by) \leq \nu_{A\cap B}(x) \vee \nu_{A\cap B}(y)$
For, $g \in G$ *and* $z \in M$, *we have*
$\mu_{A\cap B}(gz) = \mu_A(gz) \wedge \mu_B(gz) \geq \mu_A(z) \wedge \mu_B(z) = \mu_{A\cap B}(z)$, i.e., $\mu_{A\cap B}(gz) \geq \mu_{A\cap B}(z)$
Similarly, we can show that $\nu_{A\cap B}(gz) \leq \nu_{A\cap B}(z)$
Hence $A \cap B$ is intuitionistic fuzzy G-module on M.

Theorem (3.11) Let M be a G - module over K and $\{A_i : i = 1, 2, \ldots, n\}$, be a family of intuitionistic fuzzy G-modules on M. Then $\bigcap_{i=1}^{n} A_i$ is also intuitionistic fuzzy G-module on M.

Theorem (3.12) Let M_1, M_2 be G-modules over K and A, B be intuitionistic fuzzy G-modules on M_1 and M_2 respectively. Then $A \times$ B is also intuitionistic fuzzy G-module on $M_1 \times M_2$.

Proof. Let $a, b \in$ K and $x = (x_1, y_1)$, $y = (x_2, y_2) \in M_1 \times M_2$, then

$$\mu_{A\times B}(ax+by)=\mu_{A\times B}\{a(x_1,y_1)+b(x_2,y_2)\}$$
$$=\mu_{A\times B}\{(ax_1+bx_2),\ (ay_1+by_2)\}$$
$$=\mu_A(ax_1+bx_2)\wedge\mu_B(ay_1+by_2)$$
$$\geq\{\mu_A(x_1)\wedge\mu_A(x_2)\}\wedge\{\mu_B(y_1)\wedge\mu_B(y_2)\}$$
$$=\{\mu_A(x_1)\wedge\mu_B(y_1)\}\wedge\{\mu_A(x_2)\wedge\mu_B(y_2)\}$$
$$=\mu_{A\times B}(x_1,y_1)\wedge\mu_{A\times B}(x_2,y_2)$$

Thus, $\mu_{A\times B}(ax+by)\geq\mu_{A\times B}(x)\wedge\mu_{A\times B}(y)$

Similarly, we can show that $\nu_{A\times B}(ax+by)\leq\nu_{A\times B}(x)\vee\nu_{A\times B}(y)$

For, $g\in G$ *and* $z=(x,y)\in M_1\times M_2$, *we have*

$$\mu_{A\times B}(gz)=\mu_{A\times B}\{g(x,y)\}=\mu_{A\times B}(gx,gy)=\mu_A(gx)\wedge\mu_B(gy)\geq\mu_A(x)\wedge\mu_B(y)=\mu_{A\times B}(z),$$

i.e., $\mu_{A\times B}(gz)\geq\mu_{A\times B}(z)$. *Similarly*, we can show that $\nu_{A\times B}(gz)\leq\nu_{A\times B}(z)$

Hence $A\times B$ is intuitionistic fuzzy G-module on $M_1\times M_2$.

Definition (3.13) Let M be a G-module over K and A = (μ_A, ν_A) and B = (μ_B, ν_B) be intuitionistic fuzzy G-modules on M, then their sum $A + B$ = (μ_{A+B}, ν_{A+B}) is defined as $\mu_{A+B}(x)=\vee_{x=a+b}\{\mu_A(a)\wedge\mu_B(b)\}$ and $\nu_{A+B}(x)=\wedge_{x=a+b}\{\nu_A(a)\vee\nu_B(b)\}$, for all $x\in M$.

Theorem (3.14) Let M be a G-module over K and A, B be two intuitionistic fuzzy G-modules on M. Then A + B is also intuitionistic fuzzy G-module on M.

Proof. Let $x, y\in M$ be any two elements and let min{ $\mu_{A+B}(x)$, $\mu_{A+B}(y)$} = α (say)

Let $\varepsilon > 0$ be given, then

$$\alpha-\varepsilon<\mu_{A+B}(x)=\vee_{x=a+b}\{\mu_A(a)\wedge\mu_B(b)\} \text{ and } \alpha-\varepsilon<\mu_{A+B}(y)=\vee_{y=c+d}\{\mu_A(c)\wedge\mu_B(d)\}$$

so there exists a representation $x=a+b,\ y=c+d$, where $a, b, c, d\in$ M such that

$\alpha-\varepsilon<\mu_A(a)\wedge\mu_B(b)$ and $\alpha-\varepsilon<\mu_A(c)\wedge\mu_B(d)$

$\Rightarrow$ $\alpha-\varepsilon<\mu_A(a)$, $\alpha-\varepsilon<\mu_B(b)$ and $\alpha-\varepsilon<\mu_A(c)$, $\alpha-\varepsilon<\mu_B(d)$

$\Rightarrow$ $\alpha-\varepsilon<\mu_A(a)\wedge\mu_A(c)\leq\mu_A(a+c)$ and $\alpha-\varepsilon<\mu_B(b)\wedge\mu_B(d)\leq\mu_B(b+d)$

Thus, we get $x+y=(a+b)+(c+d)=(a+c)+(b+d)$ such that

$\Rightarrow$ $\alpha-\varepsilon<\mu_A(a+c)\wedge\mu_B(b+d)$

$\Rightarrow$ $\alpha-\varepsilon<\vee_{x+y=(a+c)+(b+d)}\{\mu_A(a+c)\wedge\mu_B(b+d)\}=\mu_{A+B}(x+y)$

Since ε is arbitrary, it follows that $\mu_{A+B}(x+y)\geq\alpha=\mu_{A+B}(x)\wedge\mu_{A+B}(y)$.

Similarly, we can show that $\nu_{A+B}(x+y)\leq\nu_{A+B}(x)\vee\nu_{A+B}(y)$

Further, let $\beta=\mu_{A+B}(x)\vee\mu_{A+B}(y)=\mu_{A+B}(x)$, and let ε >0, then

β-ε < $\mu_{A+B}(x)=\vee_{x=a+b}\{\mu_A(a)\wedge\mu_B(b)\}$, *so* there exists a representation

$x=a+b$ such that β-ε < $\mu_A(a)\wedge\mu_B(b)\Rightarrow\beta$-$\varepsilon$ <$\mu_A(a)$, β-ε <$\mu_B(b)$

$\Rightarrow$ β - ε < $\mu_A(a)\leq\mu_A(ka)$, β - ε < $\mu_B(b)\leq\mu_B(kb)$, for any $k\in$K

$\Rightarrow$ β - ε < $\mu_A(ka)\wedge\mu_B(kb)$, for any $k\in$K

Now, $kx = k(a + b) = ka + kb$ so that β - ε < $\mu_A(ka)\wedge\mu_B(kb)$

$\Rightarrow \quad \beta\text{-}\varepsilon < \bigvee_{kx=k(a+b)} \{\mu_A(ka) \wedge \mu_B(kb)\} = \mu_{A+B}(kx)$

Since ε is arbitrary, it follows that

$\mu_{A+B}(kx) \geq \beta = \mu_{A+B}(x)$. Similarly, we can show that $\nu_{A+B}(kx) \leq \nu_{A+B}(x)$

Further, let $g \in$G and $x \in$M be any element, then $\mu_{A+B}(x) = \bigvee_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\}$

Now, $\mu_A(a) \leq \mu_A(ga)$, $\mu_B(b) \leq \mu_B(gb) \Rightarrow \mu_A(a) \wedge \mu_B(b) \leq \mu_A(ga) \wedge \mu_B(gb)$

Also, $gx = g(a+b) = ga + gb$

$\mu_{A+B}(x) = \bigvee_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} \leq \bigvee_{gx=ga+gb} \{\mu_A(ga) \wedge \mu_B(gb)\} = \mu_{A+B}(gx)$

i.e., $\mu_{A+B}(gx) \geq \mu_{A+B}(x)$. Similarly, we can show that $\nu_{A+B}(gx) \leq \nu_{A+B}(x)$.

Hence A + B is intuitionistic fuzzy G-module on M.

Definition (3.15) Let *M* be a G-module over K and $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy G-modules on M, then their product $AB = (\mu_{AB}, \nu_{AB})$ is defined as

$\mu_{AB}(x) = \bigvee_{x=\sum_{i<\infty} a_i b_i} \left\{ \bigwedge_i \left(\mu_A(a_i) \wedge \mu_B(b_i) \right) \right\}$ and $\nu_{AB}(x) = \bigwedge_{x=\sum_{i<\infty} a_i b_i} \left\{ \bigvee_i \left(\nu_A(a_i) \vee \nu_B(b_i) \right) \right\}$, for all $x \in M$.

Theorem (3.16) Let *M* be a G - module over K and A, B be intuitionistic fuzzy G-modules on M. Then AB is also intuitionistic fuzzy G-module on M.

Proof. Let $x, y \in M$ be any two elements and let $\mu_{AB}(x) \wedge \mu_{AB}(y) = \alpha$ (say)

Let $\varepsilon > 0$ be given, then

$\alpha - \varepsilon < \mu_{AB}(x) = \bigvee_{x=\sum_{i<\infty} a_i b_i} \left\{ \bigwedge_i \left(\mu_A(a_i) \wedge \mu_B(b_i) \right) \right\}$ and $\alpha - \varepsilon < \mu_{AB}(y) = \bigvee_{y=\sum_{i<\infty} p_i q_i} \left\{ \bigwedge_i \left(\mu_A(p_i) \wedge \mu_B(q_i) \right) \right\}$

$\Rightarrow \alpha - \varepsilon < \bigwedge_i \{\mu_A(a_i) \wedge \mu_B(b_i)\}$ and $\alpha - \varepsilon < \bigwedge_i \{\mu_A(p_i) \wedge \mu_B(q_i)\}$, for all i .

$\Rightarrow \alpha - \varepsilon < \mu_A(a_i) \wedge \mu_B(b_i)$ and $\alpha - \varepsilon < \mu_A(p_i) \wedge \mu_B(q_i)$, for all i .

$\Rightarrow \alpha - \varepsilon < \mu_A(a_i)$, $\alpha - \varepsilon < \mu_B(b_i)$ and $\alpha - \varepsilon < \mu_A(p_i)$, $\alpha - \varepsilon < \mu_B(q_i)$, for all i .

$\Rightarrow \alpha - \varepsilon < \mu_A(a_i) \wedge \mu_A(p_i) \leq \mu_A(a_i + p_i)$ and $\alpha - \varepsilon < \mu_B(b_i) \wedge \mu_B(q_i) \leq \mu_B(b_i + q_i)$, for all *i*.

Thus, we get $x + y = \sum (a_i b_i + p_i q_i)$, where $a_i, b_i, p_i, q_i \in$ M such that

$\alpha - \varepsilon < \mu_A(a_i + p_i) \wedge \mu_B(b_i + q_i)$, for all $i \Rightarrow \alpha - \varepsilon < \bigwedge_i \{\mu_A(a_i + p_i) \wedge \mu_B(b_i + q_i)\}$

$\Rightarrow \alpha - \varepsilon < \bigvee_{x+y=\sum_{i<\infty}(a_i b_i + p_i q_i)} \bigwedge_i \{\mu_A(a_i + p_i) \wedge \mu_B(b_i + q_i)\} = \mu_{AB}(x+y)$

Since $\varepsilon > 0$ is arbitrary, so we have $\mu_{AB}(x+y) \geq \alpha = \mu_{AB}(x) \wedge \mu_{AB}(y)$

Similarly, we can show that $\nu_{AB}(x+y) \leq \nu_{AB}(x) \vee \nu_{AB}(y)$

Further, let $\beta = \mu_{AB}(x) \vee \mu_{AB}(y) = \mu_{AB}(x)$, and let $\varepsilon > 0$, then

$\beta\text{-}\varepsilon < \mu_{AB}(x) = \bigvee_{x=\sum_{i<\infty} a_i b_i} \bigwedge_i \{\mu_A(a_i) \wedge \mu_B(b_i)\}$, *so* there exists a representation

$x = \sum_{i<\infty} a_i b_i$ such that $\beta\text{-}\varepsilon < \bigwedge_i \{\mu_A(a_i) \wedge \mu_B(b_i)\}$

$\Rightarrow \quad \beta\text{-}\varepsilon < \mu_A(a_i) \wedge \mu_B(b_i)$, for all *i*.

$\leq \mu_A(ka_i) \wedge \mu_B(b_i)\}$, for all i and for all $k \in K$

Hence $\beta\text{-}\varepsilon < \wedge_i \{\mu_A(ka_i) \wedge \mu_B(b_i)\} < \vee_{kx=\sum_{i<\infty}(ka_i)b_i} \wedge_i \{\mu_A(ka_i) \wedge \mu_B(b_i)\} = \mu_{AB}(kx)$

As ε >0 is arbitrary , so we have $\mu_{AB}(kx) \geq \beta = \mu_{AB}(x)$

Similarly, we can show that $\nu_{AB}(kx) \leq \nu_{AB}(x)$

Further, let $g \in$G and $x\in$M be any element, then $\mu_{AB}(x) = \vee_{x=\sum_{i<\infty} a_ib_i} \left\{\wedge_i\left(\mu_A(a_i) \wedge \mu_B(b_i)\right)\right\}$

Now, $\mu_A(a_i) \leq \mu_A(ga_i) \Rightarrow \mu_A(a_i) \wedge \mu_A(b_i) \leq \mu_A(ga_i) \wedge \mu_A(b_i)$, for all i

$\Rightarrow \quad \wedge_i\left(\mu_A(a_i) \wedge \mu_B(b_i)\right) \leq \wedge_i\left(\mu_A(ga_i) \wedge \mu_B(b_i)\right)$, for all i

$\Rightarrow \quad \vee_{x=\sum_{i<\infty} a_ib_i} \left\{\wedge_i\left(\mu_A(a_i) \wedge \mu_B(b_i)\right)\right\} \leq \vee_{gx=\sum_{i<\infty}(ga_i)b_i} \left\{\wedge_i\left(\mu_A(ga_i) \wedge \mu_B(b_i)\right)\right\}$

i.e., $\mu_{AB}(x) = \vee_{x=\sum_{i<\infty} a_ib_i} \left\{\wedge_i\left(\mu_A(a_i) \wedge \mu_B(b_i)\right)\right\} \leq \vee_{gx=\sum_{i<\infty}(ga_i)b_i} \left\{\wedge_i\left(\mu_A(ga_i) \wedge \mu_B(b_i)\right)\right\} = \mu_{AB}(gx)$

Similarly, *we* can show that $\nu_{AB}(gx) \leq \nu_{AB}(x)$.

Hence AB is intuitionistic fuzzy G-module on M.

Definition (3.17) Let *M* be a G-module over K and *A* be an intuitionistic fuzzy G-modules on M. Let N be a G-submodule of M. Then the restriction of *A* on N is denoted by $A|_N$ is an IFS on N defined as $(A|_N)(x) = \left(\mu_{A|_N}(x),\ \nu_{A|_N}(x)\right)$, where

$\mu_{A|_N}(x) = \mu_A(x)$ and $\nu_{A|_N}(x) = \nu_A(x)$, $\forall\ x \in N$.

Proposition (3.18) If *A* be an intuitionistic fuzzy G-module of a G-module *M* over K and let N be a G-submodule of M. Then $A|_N$ is an intuitionistic fuzzy G-module of N.

Proof. Let *a*, *b* $\in$ K and *x*, *y* $\in$ N, then

$\mu_{A|N}(ax+by) = \mu_A(ax+by) \qquad [\because\ ax+by \in \text{N}]$

$\geq \mu_A(x) \wedge \mu_A(y)$

$= \mu_{A|N}(x) \wedge \mu_{A|N}(x)$

Thus, $\mu_{A|N}(ax+by) \geq \mu_{A|N}(x) \wedge \mu_{A|N}(y)$.

Similarly, we can show that $\nu_{A|N}(ax+by) \leq \nu_{A|N}(x) \vee \nu_{A|N}(y)$.

For, $g \in G$ *and* $z \in N$, *we have*

$\mu_{A|N}(gz) = \mu_A(gz) \geq \mu_A(z) \quad [\because\ gz \in N]$

Similarly, we can show that $\nu_{A|N}(gz) \leq \nu_A(z)$.

Hence $A|N$ is intuitionistic fuzzy G-module on N.

Proposition (3.19) Let *M* be a G-module over K and N be a G-submodule of M. Then the IFS A_N on *M* / N, defined by

$\mu_{A_N}(x + \text{N}) = \vee\{\mu_A(x + n): n\in\text{N}\}$ and $\nu_{A_N}(x + \text{N}) = \wedge\{\nu_A(x + n): n\in \text{N}\}$, $\forall\ x\in$M, is an intuitionistic fuzzy G-module on *M* / N.

Proof. For *a*, *b* $\in$ K and *x*, *y* $\in$ M, we have

$\mu_{A_N}\{a(x + \text{N}) + b(y + \text{N})\} = \mu_{A_N}\{(ax + by) + \text{N}\}$

$= \vee\{\ \mu_A(\{ax + by\}+ n) :\ n \in\text{N}\}$

$= \vee\{ \mu_A(\{ax + by\}+ an_1 + bn_2) : n_1, n_2 \in N \}$, where $n = an_1 + bn_2$, for some $n_1, n_2 \in N$

$= \vee\{ \mu_A(\{a(x + n_1) + b(y + n_2)\}) : n_1, n_2 \in N \}$

$\geq \vee \{ \mu_A\{a(x + n_1)\} \wedge \mu_A\{b(y + n_2)\} : n_1, n_2 \in N \}$

$\geq \vee \{ \mu_A(x + n_1) \wedge \mu_A(y + n_2) : n_1, n_2 \in N \}$

$\geq [\vee\{\mu_A(x + n_1) : n_1 \in N \}] \wedge [\vee\{ \mu_A(y + n_2) : n_2 \in N \}]$

$= \mu_A(x + N) \wedge \mu_A(y + N)$

Thus, $\mu_{A_N} [a(x + N)+ b(y + N)] \geq \mu_{A_N} (x + N) \wedge \mu_{A_N} (y + N)$.

Similarly, we can show that $\nu_{A_N} [a(x + N) + b(y + N)] \leq \nu_{A_N} (x + N) \vee \nu_{A_N} (y + N)$.

Also, $\mu_{A_N} [g(x + N)] = \mu_{A_N} (gx + N) = \vee\{\mu_A(gx + n) : n \in N\}= \vee\{ \mu_A(gx + gn_3) : n_3 \in N\}$

$= \vee\{ \mu_A(g(x + n_3)) : n_3 \in N\}$

$\geq \vee\{ \mu_A(x + n_3) : n_3 \in N\}$

$= \mu_{A_N} (x + N)$

Thus $\mu_{A_N} [g(x + N)] \geq \mu_{A_N} (x + N)$

Similarly, we can show that $\nu_{A_N} [g(x + N)] \leq \nu_{A_N} (x + N)$.

Therefore, $A_N = (\mu_{A_N}, \nu_{A_N})$ is intuitionistic fuzzy G-module on M / N.

Remark (3.20) The intuitionistic fuzzy G-module A_N defined on M / N, as defined above, is called the *quotient intuitionistic fuzzy G-module* or *factor intuitionistic fuzzy G-module* of *A* on *M* relative to G-submodule *N*.

4 Homomorphism and intuitionistic fuzzy G-homomorphism of intuitionistic fuzzy G-modules

In this section, we study the behaviour of intuitionistic fuzzy G-modules under the G-module homomorphism. We also defined and discuss the notion of (weak) intuitionistic fuzzy G-homomorphism (isomorphism).

Theorem (4.1) Let *M* and *M** be G-modules and let *f* be a G-module homomorphism from *M* into M*. If B is a intuitionistic fuzzy G-module on M*, then f^{-1}(B) is a intuitionistic fuzzy G-module of M.

Proof. We know that f^{-1}(B) is an intuitionistic fuzzy subset of *M* when B is a intuitionistic fuzzy subset of *M** and *f* is a G-module homomorphism of *M* into M*.

Now, for $a, b \in K$, $x, y \in M$ and $g \in G$, we have

$f^{-1}(B)(ax + by) = \left(\mu_{f^{-1}(B)}(ax + by),\ \nu_{f^{-1}(B)}(ax + by)\right)$, where

$\mu_{f^{-1}(B)}(ax + by) = \mu_B(f(ax + by))$

$= \mu_B(af(x) + bf(y))$ [$\because f$ is a homomorphism]

$\geq \mu_B(f(x)) \wedge \mu_B(f(y))$ [$\because$ B is a intuitionistic fuzzy G-module]

$\geq \mu_{f^{-1}(B)}(x) \wedge \mu_{f^{-1}(B)}(y)$

Thus, $\mu_{f^{-1}(B)}(ax + by) \geq \mu_{f^{-1}(B)}(x) \wedge \mu_{f^{-1}(B)}(y)$

Similarly, *we* can show that $\nu_{f^{-1}(B)}(ax + by) \leq \nu_{f^{-1}(B)}(x) \vee \nu_{f^{-1}(B)}(y)$

Also, $f^{-1}(B)(gm) = \left(\mu_{f^{-1}(B)}(gm),\ \nu_{f^{-1}(B)}(gm)\right)$, where

$\mu_{f^{-1}(B)}(gm) = \mu_B(f(gm)) = \mu_B(g\,f(m)) \geq \mu_B(f(m)) = \mu_{f^{-1}(B)}(m)$

Thus, $\mu_{f^{-1}(B)}(gm) \geq \mu_{f^{-1}(B)}(m)$.

Similarly, we can show that $\nu_{f^{-1}(B)}(gm) \leq \nu_{f^{-1}(B)}(m)$.

Hence $f^{-1}(B)$ is intuitionistic fuzzy G-module on M.

Theorem (4.2) Let *M* and *M** be G-modules and let *f* be a G-module homomorphism from *M* onto M*. If *A* is an intuitionistic fuzzy G-module on M, then *f* (A) is an intuitionistic fuzzy G-module of M*.

Proof. It may be recalled that if *A* is a intuitionistic fuzzy subset of M, then *f* (A) is also intuitionistic fuzzy subset of M*. Now we show that *f* (A) is a intuitionistic G-module on M*. For *a*, *b* ∈K and *x*, *y* ∈M* and *g*∈G, we have

$f(A)(ax + by) = \left(\mu_{f(A)}(ax + by),\ \nu_{f(A)}(ax + by)\right)$, where

$\mu_{f(A)}(ax + by) = \vee\left\{\mu_A(z) : z \in f^{-1}(ax + by)\right\}$

$= \vee\left\{\mu_A(z) : f(z) = ax + by,\ x, y \in M^*, z \in M\right\}$

$= \vee\left\{\mu_A(a'z' + b'z'') : f(z') = x, f(z'') = y\right\}$ [$\because f$ is a homomorphism onto]

$= \vee\left\{\mu_A(a'z' + b'z'') : z' \in f^{-1}(x), z'' \in f^{-1}(y)\right\}$

$\geq \vee\left[\left\{\mu_A(z') \wedge \mu_A(z'')\right\} : z' \in f^{-1}(x), z'' \in f^{-1}(y)\right]$

$\geq \left[\vee\left\{\mu_A(z') : z' \in f^{-1}(x)\right\}\right] \wedge \left[\vee\left\{\mu_A(z'') : z'' \in f^{-1}(y)\right\}\right]$

$\geq \mu_{f(A)}(x) \wedge \mu_{f(A)}(y)$

Thus, $\mu_{f(A)}(ax + by) \geq \mu_{f(A)}(x) \wedge \mu_{f(A)}(y)$

Similarly, *we* can show that $\nu_{f(A)}(ax + by) \leq \nu_{f(A)}(x) \vee \nu_{f(A)}(y)$

Also, $f(A)(gm^*) = \left(\mu_{f(A)}(gm^*),\ \nu_{f(A)}(gm^*)\right)$, where

$$\mu_{f(A)}(gm^*) = \vee\left\{\mu_A(x):\ x \in f^{-1}(gm^*)\right\}$$
$$= \vee\left\{\mu_A(x):\ f(x) = gm^*\ ,\ g \in \mathrm{G}\ ,\ m^* \in \mathrm{M}^*\right\}$$
$$= \vee\left\{\mu_A(gm):\ gm\ \in \mathrm{M}\ ,\ f(gm) = gm^*,\ g \in \mathrm{G}\ ,\ m^* \in \mathrm{M}^*,\ m \in \mathrm{M}\right\}$$
$$\geq \vee\left\{\mu_A(m):\ m\ \in \mathrm{M}\ ,\ f(m) = m^* \in \mathrm{M}^*\right\}$$
$$\geq \vee\left\{\mu_A(m):\ m \in f^{-1}(m^*)\right\}$$
$$= \mu_{f(A)}(m^*)$$

Thus, $\mu_{f(A)}(gm^*) \geq \mu_{f(A)}(m^*)$.

Similarly, we can show that $\nu_{f(A)}(gm^*) \leq \nu_{f(A)}(m^*)$.

Hence $f(A)$ is intuitionistic fuzzy G-module on M*.

Definition (4.3) Let *M* and *M** be G-modules and let A, B be two intuitionistic fuzzy G-submodules on *M* and *M** respectively. Let $f: M \to M^*$ be a G-module homomorphism. Then *f* is called a ***weak intuitionistic fuzzy* G-*homomorphism*** of *A* into B if *f* (A) $\subseteq$ B. The homomorphism *f* is an intuitionistic fuzzy G-homomorphism of *A* onto B if *f* (A) = B. We say that *A* is intuitionistic fuzzy G-homomorphic to B and we write $A \approx$ B.
Let $f: M \to M^*$ be an G-module isomorphism. Then *f* is called a ***weak intuitionistic fuzzy*** G-***isomorphism*** *f* (A) $\subseteq$ B and *f* is a ***intuitionistic fuzzy*** G-***isomorphism*** if *f* (A) = B.

Example (4.4) Let G ={1, -1} and *M* = C, *M** = R be G-modules over R. Define the IFSs *A* and B on *M* and *M** respectively as:

$$\mu_A(x+iy) = \begin{cases} 1 & ; if\ x=y=0 \\ 0.5 & ; if\ x\neq 0\ \&\ y=0 \\ 0.25 & ; if\ y\neq 0 \end{cases} \quad \text{and} \quad \nu_A(z) = \begin{cases} 0 & ; if\ x=y=0 \\ 0.25 & ; if\ x\neq 0\ \&\ y=0 \\ 0.5 & ; if\ y\neq 0 \end{cases} \quad \text{and}$$

$$\mu_B(x) = \begin{cases} 1 & ; if\ x=0 \\ 0.5 & ; if\ x\neq 0 \end{cases} \quad \text{and} \quad \nu_B(x) = \begin{cases} 0 & ; if\ x=0 \\ 0.25 & ; if\ x\neq 0 \end{cases} ,\forall\, x\in R.$$

Then *A* and B are intuitionistic fuzzy G-modules on *M* and *M** respectively.
Define the mapping $f: M \to M^*$ by $f(x + iy) = x + y$, where *x*, *y* $\in$R.
For *a*, *b*$\in$R, and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, where $x_1, y_1, x_2, y_2 \in$R, we have

$$f(az_1 + bz_2) = f\{(a\,x_1 + bx_2) + i(a\,y_1 + by_2)\} = a\,x_1 + bx_2 + a\,y_1 + by_2$$
$$= a\,(x_1 + y_1) + b(x_2 + y_2)$$
$$= a\,f(z_1) + b\,f(z_2).$$

For *g*$\in$G and *z* = *x* + *iy* $\in$M, we have
$f(gz) = f\{g(x + iy)\} = f(gx + igy) = gx + gy = g(x + y) = g\,f(z)$.
$\therefore$ *f* is a G-module homomorphism.
We know that the image of *A* under *f* is given by

$$\mu_{f(A)}(r)=\begin{cases}\vee\{\ \mu_A(x+iy) : x+iy\ \in f^{-1}(r)\} \\ \quad 0\quad ;\quad \text{otherwise}\end{cases} \text{ and } \nu_{f(A)}(r)=\begin{cases}\wedge\{\ \nu_A(x+iy) : x+iy\ \in f^{-1}(r)\} \\ \quad 1\quad ;\quad \text{otherwise}\end{cases}$$

where $r \in M^*$ and $x + iy \in$ M.

$$\mu_{f(A)}(0) = \underset{x+iy\in M}{\vee}\{\ \mu_A(x+iy) : x+iy\ \in f^{-1}(0)\} = \underset{x+iy\in M}{\vee}\{\ \mu_A(x+iy) : f(x+iy)=0\}$$

$$= \mu_A(0+i0) \underset{x\in R-\{0\}}{\vee}\{\mu_A(x+i(-x)\} = 1 \underset{x\in R-\{0\}}{\vee}\{0.5\} = 1$$

Similarly, we have $\nu_{f(A)}(0) = 0.$

Also, $\mu_{f(A)}(r) = \underset{x+iy\in M}{\vee}\{\ \mu_A(x+iy) : x+iy\ \in f^{-1}(r)\} = \underset{x+iy\in M}{\vee}\{\ \mu_A(x+iy) : f(x+iy)=r\}$

$$= \mu_A(r+i0) \vee \mu_A(0+ir) \underset{\substack{p,q\in R \\ st.\ p+q=r}}{\vee}\{\mu_A(p+iq)\}$$

$$= (0.5) \vee (0.25) \underset{\substack{p,q\in R, \\ p+q=r}}{\vee}\{0.25\} = 0.5$$

Similarly, we have $\nu_{f(A)}(r) = 0.25.$

$\therefore$ *f* (A) = B. Hence *f* is an intuitionistic fuzzy G - homomorphism of *A* onto B.

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