Индексирани матрици
Index matrices

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1. Basic concepts

Let I **be a fixed set of indices and** $\mathcal{R}$ **be the set of the real numbers. By IM with index sets** K **and** L $(K, L \subset I)$**, we will denote the object:**

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \ldots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \ldots & a_{k_2,l_n} \\ \vdots & & & & \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, **for** $1 \leq i \leq m$, **and** $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R}$.

1. Basic concepts

For the IMs $A = [K, L, \{a_{k_i,l_j}\}], B = [P, Q, \{b_{p_r,q_s}\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other, specific ones.

(a) addition $A \oplus B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} + b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}$$

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(b) **termwise multiplication** $A \otimes B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}]$, **where**

$$c_{t_u,v_w} = a_{k_i,l_j}.b_{p_r,q_s}, \quad \text{for} \quad \begin{array}{l} t_u = k_i = p_r \in K \cap P \text{ and} \\ v_w = l_j = q_s \in L \cap Q; \end{array}$$

(c) **multiplication** $A \odot B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}]$, **where**

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \sum_{l_j = p_r \in L \cap P} a_{k_i,l_j}.b_{p_r,q_s}, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}$$

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(d) **structural subtraction** $A \ominus B = [K - P, L - Q, \{c_{t_u,v_w}\}]$, **where "–" is the set–theoretic difference operation and**

$$c_{t_u,v_w} = a_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

(e) **multiplication with a constant** $\alpha.A = [K, L, \{\alpha.a_{k_i,l_j}\}]$, **where** α **is a constant.**

(f) **termwise subtraction** $A - B = A \oplus (-1).B$.

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For example, if we have the IMs X and Y

$$X = \begin{array}{c|ccc} & c & d & e \\ \hline a & 1 & 2 & 3 \\ b & 4 & 5 & 6 \end{array}, \quad Y = \begin{array}{c|cc} & c & r \\ \hline a & 10 & 11 \\ p & 12 & 13 \\ q & 14 & 15 \end{array},$$

then

$$X \oplus Y = \begin{array}{c|cccc} & c & d & e & r \\ \hline a & 11 & 2 & 3 & 11 \\ b & 4 & 5 & 6 & 0 \\ p & 12 & 0 & 0 & 13 \\ q & 14 & 0 & 0 & 15 \end{array}.$$

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If IM Z has the form

$$Z = \begin{array}{c|c} & u \\ \hline c & 10 \\ d & 11 \\ s & 12 \\ t & 13 \end{array},$$

then

$$X \odot Z = \begin{array}{c|cc} & e & u \\ \hline a & 3 & 1 \times 10 + 2 \times 11 \\ b & 6 & 4 \times 10 + 5 \times 11 \\ s & 0 & 12 \\ t & 0 & 13 \end{array} = \begin{array}{c|cc} & e & u \\ \hline a & 3 & 32 \\ b & 6 & 95 \\ s & 0 & 12 \\ t & 0 & 13 \end{array}.$$

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Now, we can directly see that when

$$K = P = \{1, 2, ..., m\},$$

$$L = Q = \{1, 2, ..., n\}$$

we obtain the definitions for standard matrix operations. In the IM case we can use different symbols as indices of the rows and columns and they, as we saw above, give us additional information and possibilities for description.

Let $\mathcal{IM}_{\mathcal{R}}$ be the set of all IMs with their elements being real numbers, $\mathcal{IM}_{\{0,1\}}$ be the set of all $(0,1)$-IMs. i.e., IMs with elements only 0 or 1, and $\mathcal{IM}_{\mathcal{P}}$ be the class of all IMs with elements – predicates .

1. Basic concepts

The problem with the "zero"-IM is more complex than in the standard matrix case. We can introduce as "zero"-IM for $\mathcal{IM}_{\mathcal{R}}$ as the IM

$$I_0 = [K, L, \{0\}]$$

all matrix elements of which are equal to 0 and $K, L \subset I$ are arbitrary index sets, as well as the IM

$$I_{\emptyset} = [\emptyset, \emptyset, \{a_{k_i,l_j}\}].$$

In the second case there are no matrix cells where the elements a_{k_i,l_j} may be inserted. In both cases for each IM $A = [K, L, \{b_{k_i,l_j}\}]$ and for $I_0 = [K, L, \{0\}]$ with the same index sets, we will obtain:

$$A \oplus I_0 = A = I_0 \oplus A.$$

The situation with $\mathcal{IM}_{\{0,1\}}$ is similar, while in the case of $\mathcal{IM}_{\mathcal{P}}$ the "zero"-IM can be either IM $I_f = [K, L, \{\text{"}false\text{"}\}]$, or the IM $I_{\emptyset} = [\emptyset, \emptyset, \{a_{k_i,l_j}\}]$ where the elements a_{k_i,l_j} are arbitrary predicates.

1. Basic concepts

Let $I_1 = [K, L, \{1\}]$ denote the IM, which elements are equal to 1, and where $K, L \subset I$ are arbitrary index sets.

The operations defined above are oriented to IMs, whose elements are real or complex numbers. Let us note these operations, respectively, by $\oplus_{+}$, $\otimes_{\times}$, $\odot_{+,\cdot}$, $\ominus_{-}$. In Section 3.1 other versions of the same operations will be given and in the Conclusion some areas of their applications will be discussed.

Theorem 1. (a) $\langle \mathcal{IM}_{\mathcal{R}}, \oplus_{+} \rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\mathcal{R}}, \otimes_{\times} \rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\mathcal{R}}, \odot_{+,\times} \rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\mathcal{R}}, \oplus_{+}, I_{\emptyset} \rangle$ is a commutative monoid.

2. Modifications of the IM-operations

It is well-known that the $(0,1)$-matrices have applications in the areas of discrete mathamatics and combinatorial analysis. When we choose to work with this kind of matrices, the above operations have the following forms.

(a′) $A \oplus_{\max} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \max(a_{k_i,l_j}, b_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}$$

2. Modifications of the IM-operations

(**b′**) $A \otimes_{\min} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}]$, **where**

$$c_{t_u,v_w} = \min(a_{k_i,l_j}, b_{p_r,q_s}), \ \textbf{for} \quad \begin{array}{l} t_u = k_i = p_r \in K \cap P \ \textbf{and} \\ v_w = l_j = q_s \in L \cap Q; \end{array}$$

(**c′**) $A \odot_{\max,\min} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}]$, **where**

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \textbf{if } t_u = k_i \in K \ \textbf{and} \ v_w = l_j \in L - P \\ b_{p_r,q_s}, & \textbf{if } t_u = p_r \in P - L \ \textbf{and} \ v_w = q_s \in Q \\ \max\limits_{l_j = p_r \in L \cap P} \min(a_{k_i,l_j}, b_{p_r,q_s}), & \textbf{if } t_u = k_i = p_r \in K \ \textbf{and} \ v_w = q_s \in Q \\ 0, & \textbf{otherwise} \end{cases}$$

2. Modifications of the IM-operations

Operation (d) from Section 2 preserves its form, operation (e) is possible only in the case when $\alpha \in \{0,1\}$, while operation (f) is impossible.

The three operations are applicable also to the case, when the elements of the IMs are real numbers.

Theorem 2. (a) $\langle \mathcal{IM}_{\{0,1\}}, \oplus_{\max} \rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\{0,1\}}, \otimes_{\min} \rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\{0,1\}}, \odot_{\max,\min} \rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\{0,1\}}, \oplus_{\max}, I_{\emptyset} \rangle$ is a commutative monoid.

2. Modifications of the IM-operations

When we choose to work with matrices, which elements are sentences or predicates, the forms of the above operations change. Now, they become

(a″) $A \oplus_{\vee} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} \vee b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ false, & \text{otherwise} \end{cases}$$

2. Modifications of the IM-operations

$$(\mathbf{b}'')\quad A \otimes_{\wedge} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}], \text{ where}$$

$$c_{t_u,v_w} = a_{k_i,l_j} \wedge b_{p_r,q_s}, \quad \textbf{for} \quad t_u = k_i = p_r \in K \cap P \textbf{ and } v_w = l_j = q_s \in L \cap Q;$$

$$(\mathbf{c}'')\quad A \odot_{\vee,\wedge} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}], \text{ where}$$

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \textbf{if } t_u = k_i \in K \textbf{ and } v_w = l_j \in L - P \\ b_{p_r,q_s}, & \textbf{if } t_u = p_r \in P - L \textbf{ and } v_w = q_s \in Q \\ \bigvee_{l_j = p_r \in L \cap P} (a_{k_i,l_j} \wedge b_{p_r,q_s}), & \textbf{if } t_u = k_i = p_r \in K \textbf{ and } v_w = q_s \in Q \\ false, & \textbf{otherwise} \end{cases}$$

2. Modifications of the IM-operations

Operation (d) preserves its form, while operations (e) and (f) are impossible.

Theorem 3. (a) $\langle \mathcal{IM}_{\mathcal{P}}, \oplus_{\vee} \rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\mathcal{P}}, \otimes_{\wedge} \rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\mathcal{P}}, \odot_{\vee,\wedge} \rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\mathcal{P}}, \oplus_{\vee}, I_{\emptyset} \rangle$ is a commutative monoid.

3. Relations over IMs

Let the two IMs $A = [K, L, \{a_{k,l}\}]$ **and** $B = [P, Q, \{b_{p,q}\}]$ **be given. We shall introduce the following (new) definitions where** $\subset$ **and** $\subseteq$ **denote the relations** *"strong inclusion"* and *"weak inclusion"*.

Definition 1: The strict relation "inclusion about dimension" is

$$A \subset_d B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$

$$\&(\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

Definition 2: The non-strict relation "inclusion about dimension" is

$$A \subseteq_d B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

3. Relations over IMs

Definition 3: The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

Definition 4: The non-strict relation "inclusion about value" is

$$A \subseteq_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}).$$

Definition 5: The strict relation "inclusion" is

$$A \subset B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$

$$\&(\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

Definition 6: The non-strict relation "inclusion" is

$$A \subseteq B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}).$$

3. Relations over IMs

It can be directly seen that for every two IMs A and B,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

4. Operations “reduction” over an IM

First, we shall introduce operations $(k,*)$- and $(*,l)$-reduction of a given IM $A = [K, L, \{a_{k_i,l_j}\}]$:

$$A_{(k,*)} = [K - \{k\}, L, \{c_{t_u,v_w}\}]$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \quad \text{for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L$$

and

$$A_{(*,l)} = [K, L - \{l\}, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \quad \text{for } t_u = k_i \in K \text{ and } v_w = l_j \in L - \{l\}.$$

4. Operations “reduction” over an IM

Second, we define

$$A_{(k,l)} = (A_{(k,*)})_{(*,l)} = (A_{(*,l)})_{(k,*)},$$

i.e.,

$$A_{(k,l)} = [K - \{k\}, L - \{l\}, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \quad \text{for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L - \{l\}.$$

Theorem 4. For every IM A and for every $k_1, k_2 \in K$, $l_1, l_2 \in L$,

$$(A_{(k_1,l_1)})_{(k_2,l_2)} = (A_{(k_2,l_2)})_{(k_1,l_1)}.$$

4. Operations “reduction” over an IM

Third, let $P = \{k_1, k_2, ..., k_s\} \subseteq K$ and $Q = \{q_1, q_2, ..., q_t\} \subseteq L$. Finally, we define the following three operations:

$$A_{(P,l)} = (...((A_{(k_1,l)})_{(k_2,l)})...)_{(k_s,l)},$$

$$A_{(k,Q)} = (...((A_{(k,l_1)})_{(k,l_2)})...)_{(k,l_t)},$$

$$A_{(P,Q)} = (...((A_{(p_1,Q)})_{(p_2,Q)})...)_{(p_s,Q)} = (...((A_{(P,q_1)})_{(P,q_2)})...)_{(P,q_t)}.$$

Obviously,

$$A_{(K,L)} = I_\emptyset,$$

$$A_{(\emptyset,\emptyset)} = A.$$

4. Operations "reduction" over an IM

Theorem 5. For every two IMs $A = [K, L, \{a_{k_i,l_j}\}], B = [P, Q, \{b_{p_r,q_s}\}]$:

$$A \subseteq_d B \text{ iff } A = B_{(P-K,Q-L)}.$$

For the opposite direction we obtain, that if $A = B_{(P-K,Q-L)}$, then

$$A = B_{(P-K,Q-L)} \subseteq_d B_{\emptyset,\emptyset} = B.$$

5. Operation "projection" over an IM

Let $M \subseteq K$ and $N \subseteq L$. Then,

$$pr_{M,N}A = [M, N, \{b_{k_i,l_j}\}],$$

where

$$(\forall k_i \in M)(\forall l_j \in N)(b_{k_i,l_j} = a_{k_i,l_j}).$$

Theorem 6. For every IM A and sets $M_1 \subseteq M_2 \subseteq K$ and $N_1 \subseteq N_2 \subseteq L$ equality

$$pr_{M_1,N_1} pr_{M_2,N_2} A = pr_{M_1,N_1} A$$

holds.

6. Hierarchical operations over IMs

Let A be an ordinary IM and let its element a_{k_f,e_g} be an IM by itself:

$$a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}],$$

where

$$K \cap P = L \cap Q = \emptyset.$$

Here, we will introduce the following hierarchical operation:

$$A|(a_{k_f,l_g}) = [(K - \{k_f\}) \cup P, (L - \{l_g\}) \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K - \{k_f\} \text{ and } v_w = l_j \in L - \{l_g\} \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}$$

6. Hierarchical operations over IMs

Let us assume that in the case when a_{k_f,l_g} is not an element of IM A, then

$$A|(a_{k_f,l_g}) = A.$$

Let for $i = 1, 2, ..., m$:

$$a^i_{k_{i,f},l_{i,g}} = [P_i, Q_i, \{b^i_{p_{i,r},q_{i,s}}\}],$$

where for every i, j $(1 \le i, j \le m)$:

$$P_i \cap P_j = Q_i \cap Q_j = \emptyset,$$

$$P_i \cap K = Q_i \cap L = \emptyset.$$

Then, for $k_{1,f}, k_{2,f}, ..., k_{m,f} \in K$ and $l_{1,g}, l_{2,g}, ..., l_{m,g} \in L$:

$$A|(a^1_{k_{1,f},l_{1,g}}, a^2_{k_{2,f},l_{2,g}}, ..., a^m_{k_{m,f},l_{m,g}}) = (...((A|(a^1_{k_{1,f},l_{1,g}}))|(a^2_{k_{2,f},l_{2,g}}))...)|(a^m_{k_{m,f},l_{m,g}}).$$

6. Hierarchical operations over IMs

Theorem 7. Let the IM A be given and let for $i = 1, 2$:

$$a^i_{k_{i,f},l_{i,g}} = [P_i, Q_i, \{b^i_{p_{i,r},q_{i,s}}\}],$$

where

$$P_1 \cap P_2 = Q_1 \cap Q_2 = \emptyset,$$

$$P_i \cap K = Q_i \cap L = \emptyset.$$

Then,

$$A|(a^1_{k_{1,f},l_{1,g}}, a^2_{k_{2,f},l_{2,g}}) = A|(a^2_{k_{2,f},l_{2,g}}, a^1_{k_{1,f},l_{1,g}}).$$

6. Hierarchical operations over IMs

Let A and a_{k_f,l_g} be as above, let b_{m_d,n_e} be the element of the IM a_{k_f,l_g}, and let

$$b_{m_d,n_e} = [R, S, \{c_{t_u,v_w}\}],$$

where

$$K \cap R = L \cap S = P \cap R = Q \cap S = K \cap P = L \cap Q = \emptyset.$$

6. Hierarchical operations over IMs

Then,

$$(A|(a_{k_f,l_g}))|(b_{m_d,n_e}) = [(K-\{k_f\})\cup(P-\{m_d\})\cup R, (L-\{l_g\})\cup(Q-\{n_e\}\cup S\,\{\alpha_{\beta_\gamma,\delta_\varepsilon}\}],$$

where

$$\alpha_{\beta_\gamma,\delta_\varepsilon} = \begin{cases} a_{k_i,l_j}, & \text{if } \beta_\gamma = k_i \in K - \{k_f\} \text{ and } \delta_\varepsilon = l_j \in L - \{l_g\} \\ b_{p_r,q_s}, & \text{if } \beta_\gamma = p_r \in P - \{m_d\} \text{ and } \delta_\varepsilon = q_s \in Q - \{n_e\} \\ c_{t_u,v_w}, & \text{if } \beta_\gamma = t_u \in R \text{ and } \delta_\varepsilon = v_w \in S \\ 0, & \text{otherwise} \end{cases}$$

Theorem 8. For the above IMs A, a_{k_f,l_g} and b_{m_d,n_e}

$$(A|(a_{k_f,l_g}))|(b_{m_d,n_e}) = A|((a_{k_f,l_g})|(b_{m_d,n_e})).$$

7. Operation “substitution” over an IM

Let IM $A = [K, L, \{a_{k,l}\}]$ be given.

First, local substitution over the IM is defined for the couples of indices (p, k) and/or (q, l), respectively, by

$$[\frac{p}{k}]A = [(K - \{k\}) \cup \{p\}, L, \{a_{k,l}\}],$$

$$[\frac{q}{l}]A = [K, (L - \{l\}) \cup \{q\}, \{a_{k,l}\}],$$

Second,

$$[\frac{p}{k}\frac{q}{l}]A = [\frac{p}{k}][\frac{q}{l}]A,$$

i.e.

$$[\frac{p}{k}\frac{q}{l}]A = [(K - \{k\}) \cup \{p\}, (L - \{l\}) \cup \{q\}, \{a_{k,l}\}].$$

Obviously, for the above indices k, l, p, q:

$$[\frac{k}{p}]([\frac{p}{k}]A) = [\frac{l}{q}]([\frac{q}{l}]A) = [\frac{k}{p}\frac{l}{q}]([\frac{p}{k}\frac{q}{l}]A) = A,$$

7. Operation "substitution" over an IM

Let the sets of indices $P = \{p_1, p_2, ..., p_m\}$, $Q = \{q_1, q_2, ..., q_n\}$ be given.

Third, for them we define sequentially:

$$[\frac{P}{K}]A = [\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_n}{k_n}]A,$$

$$[\frac{Q}{L}]A = ([\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}]A),$$

$$[\frac{K}{P}\frac{Q}{L}]A) = [\frac{P}{K}][\frac{Q}{L}]A,$$

i.e.,

$$[\frac{P}{K}\frac{Q}{L}]A = [\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_m}{k_m}\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}]A = [P, Q, \{a_{k,l}\}]$$

Obviously, for the sets K, L, P, Q:

$$[\frac{K}{P}]([\frac{P}{K}]A) = [\frac{L}{Q}]([\frac{Q}{L}]A) = [\frac{K}{P}\frac{L}{Q}]([\frac{P}{K}\frac{Q}{L}]A) = A.$$

7. Operation "substitution" over an IM

Theorem 9. For every four sets of indices P_1, P_2, Q_1, Q_2

$$[\frac{P_2}{P_1}\frac{Q_2}{Q_1}][\frac{P_1}{K}\frac{Q_1}{L}]A = [\frac{P_2}{K}\frac{Q_2}{L}]A.$$

8. Properties of intuitionistic fuzzy IMs

The IFIM has the form

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \langle \mu_{k_1,l_2}, \nu_{k_1,l_2} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ k_2 & \langle \mu_{k_2,l_1}, \nu_{k_2,l_1} \rangle & \langle \mu_{k_2,l_2}, \nu_{k_2,l_2} \rangle & \dots & \langle \mu_{k_2,l_n}, \nu_{k_2,l_n} \rangle \\ \vdots & & & & \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \langle \mu_{k_m,l_2}, \nu_{k_m,l_2} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$.

For the IMs $A = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], B = [P, Q, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other specific ones.

8. Properties of intuitionistic fuzzy IMs

(a) **addition** $A \oplus B = [K \cup P, L \cup Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$, where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

8. Properties of intuitionistic fuzzy IMs

(b) **termwise multiplication** $A \otimes B = [K \cap P, L \cap Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$,
where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$= \begin{cases} \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \end{cases}$$

8. Properties of intuitionistic fuzzy IMs

(c) multiplication $A \odot B = [K \cup (P - L), Q \cup (L - P), \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle \}]$,
where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \langle \max\limits_{l_j = p_r \in L \cap P} (\min(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \min\limits_{l_j = p_r \in L \cap P} (\max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \rangle, & \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

8. Properties of intuitionistic fuzzy IMs

(d) **structural subtraction**

$$A \ominus B = [K - P, L - Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where "–" is the set–theoretic difference operation and

$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$, for $t_u = k_i \in K - P$ and $v_w = l_j \in L - Q$.

(e) **negation of an IFIM** $\neg A = [K, L, \{\neg\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$, where $\neg$ is one of the above (or another) negations.

(f) **termwise subtraction** $A - B = A \oplus \neg B$.

8. Properties of intuitionistic fuzzy IMs

For example, if we have the IFIMs X and Y

$$X = \begin{array}{c|cc} & c & d \\ \hline a & \langle 0.5, 0.3\rangle & \langle 0.4, 0.2\rangle \\ b & \langle 0.1, 0.8\rangle & \langle 0.7, 0.1\rangle \end{array}, \qquad Y = \begin{array}{c|cc} & c & g \\ \hline a & \langle 0.3, 0.1\rangle & \langle 0.6, 0.2\rangle \\ e & \langle 0.3, 0.6\rangle & \langle 0.3, 0.6\rangle \\ f & \langle 0.5, 0.2\rangle & \langle 0.6, 0.1\rangle \end{array},$$

then

$$X \oplus Y = \begin{array}{c|ccc} & c & d & g \\ \hline a & \langle 0.5, 0.1\rangle & \langle 0.4, 0.2\rangle & \langle 0.6, 0.2\rangle \\ b & \langle 0.1, 0.8\rangle & \langle 0.7, 0.1\rangle & \langle 0.0, 1.0\rangle \\ e & \langle 0.3, 0.6\rangle & \langle 0.0, 1.0\rangle & \langle 0.3, 0.6\rangle \\ f & \langle 0.5, 0.2\rangle & \langle 0.0, 1.0\rangle & \langle 0.6, 0.1\rangle \end{array}.$$

8. Properties of intuitionistic fuzzy IMs

Now, we can directly see that when

$$K = P = \{1, 2, ..., m\},\ L = Q = \{1, 2, ..., n\}$$

we obtain the definitions for standard matrix operations with intuitionistic fuzzy pairs. In the IFIM case we can use different symbols as indices of the rows and columns and they, as we saw above, give us additional information and possibilities for description.

Let $\mathcal{IM}_{\mathcal{IF}}$ be the set of all IFIMs with their elements being intuitionistic fuzzy pairs. The problem with the "zero"-IFIM is more complex than in the standard matrix case. We can introduce as "zero"-IFIM for $\mathcal{IM}_{\mathcal{IF}}$ as the IFIM

$$I_0 = [K, L, \{\langle 0.0, 1.0\rangle\}]$$

all matrix elements of which are equal to $\langle 0.0, 1.0\rangle$ and $K, L \subset I$ are arbitrary index sets, as well as the IFIM

$$I_\emptyset = [\emptyset, \emptyset, \{a_{k_i, l_j}\}].$$

8. Properties of intuitionistic fuzzy IMs

In the second case there are no matrix cells where the elements a_{k_i,l_j} may be inserted. In both cases for each IFIM $A = [K, L, \{b_{k_i,l_j}\}]$ and for I_0 with the same index sets, we will obtain:

$$A \oplus I_0 = A = I_0 \oplus A.$$

Let $I_1 = [K, L, \{\langle 1.0, 0.0\rangle\}]$ denote the IFIM, which elements are equal to $\langle 1.0,\ 0.0\rangle$, and where $K, L \subset I$ are arbitrary index sets.

Theorem 10. (a) $\langle \mathcal{IM}_{\mathcal{IF}}, \oplus\rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\mathcal{IF}}, \otimes\rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\mathcal{IF}}, \odot\rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\mathcal{IF}}, \oplus, I_\emptyset\rangle$ is a commutative monoid.

9. Relations over IFIMs

Let the two IFIMs $A = [K, L, \{\langle a_{k,l}, b_{k,l}\rangle\}]$ and $B = [P, Q, \{\langle c_{p,q}, d_{p,q}\rangle\}]$ be given. We shall introduce the following (new) definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion"*.

Definition 1: The strict relation "inclusion about dimension" is

$$A \subset_d B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$
$$\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 2: The non-strict relation "inclusion about dimension" is

$$A \subseteq_d B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

9. Relations over IFIMs

Definition 3: The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 4: The non-strict relation "inclusion about value" is

$$A \subseteq_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 5: The strict relation "inclusion" is

$$A \subset B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$
$$\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 6: The non-strict relation "inclusion" is

$$A \subseteq B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

9. Relations over IFIMs

It can be directly seen that for every two IFIMs A and B,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

Operations "reduction", "projection" and "substitution" coincide with the respective operations defined over IMs, while hierarhical operations over IMs are not applied here.

Благодаря ви за вниманието!

Thank you for your attention!

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