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Compass-and-straightedge constructions in the intuitionistic fuzzy interpretational triangle: two new intuitionistic fuzzy modal operators

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Abstract: The idea about the two new intuitionistic fuzzy modal operators, proposed here, was inspired by a review of the modal operators defined over intuitionistic fuzzy sets and the observation that graphically all of them are constructed by orthogonal projections. Here for the first time, we propose a new method of constructing two different modal operators, using a compass-and-straightedge construction, producing for each point from the intuitionistic fuzzy interpretational triangle, the two points onto the triangle's hypothenuse that are respectively equidistant from the Truth and the Falsity as the point itself. The properties of these so-constructed new intuitionistic fuzzy operators are studied and formulated in two theorems.

Keywords: Intuitionistic fuzzy modal operator, Intuitionistic fuzzy interpretational triangle, Ruler-and-compass construction, InterCriteria Analysis.
AMS Classification: 03E72.

1 Introduction

The idea about the two new intuitionistic fuzzy modal operators, proposed here for the first time, was inspired by a review of the modal operators defined over intuitionistic fuzzy sets and the observation that graphically all of them are constructed by orthogonal projections. Traditionally, taking an element x of an intuitionistic fuzzy set, we can graphically represent it as a point plotted onto the intuitionistic fuzzy interpretational triangle, and we can draw its orthogonal projections, parallel to the abscissa (representing the membership), and the ordinate (representing the non-membership). The points where these two projections intersect the hypotenuse of the interpretational triangle, are the points representing the two *modal operators* Necessity and Possibility, where the hypotenuse itself is the projection of the universe of all intuitionistic fuzzy sets onto the universe of all fuzzy sets.

In addition to the modal operators Necessity and Possibility, there are multiple *extended modal operators* $F_{\alpha,\beta}$, $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, $J^*_{\alpha,\beta}$, and many more generalizing the concept of extended modal operator over intuitionistic fuzzy sets, and all of them are constructed using orthogonal projections. Extensive research by Atanassov, Dencheva, Çuvalcıoğlu (see [3–10] and [1], Chapter 3) has been dedicated on stepwise generalization of the extended modal operators, and constructing elaborate hierarchies demonstrating the existing relations between them.

Here for the first time, we propose a new method of constructing two different modal operators, using a compass-and-straightedge construction (CSC), producing for each point from the intuitionistic fuzzy interpretational triangle two points onto the triangle's hypotenuse that are respectively equidistant from the Truth and the Falsity as the point itself.

The paper is organized as follows: Section 2 gives the formal definitions and notations. Section 3 contains the definitions of the so-constructed new intuitionistic fuzzy operators and some of their researched properties. Section 4 provides discussion and conclusions.

2 Preliminaries

In the intuitionistic fuzzy logic (see, e.g., [2]), each proposition, variable or formula is evaluated with two degrees – "truth degree" or "degree of validity", "degree of membership", and 'falsity degree" or "degree of non-validity" or "degree of non-membership". Thus, to each one of these objects, e.g., x, two real numbers, traditionally denoted by $\mu(x)$ and $\nu(x)$, are assigned with the following constraints: $\mu(x), \nu(x) \in [0, 1]$ and $\mu(x) + \nu(x) \leq 1$.

Let an evaluation function V be defined over a set of propositions $\mathcal{S}$ in such a way that for $p \in \mathcal{S}$: $V(x) = \langle \mu(x),\ \nu(x) \rangle$. Hence the function $V : \mathcal{S} \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$ – the set of logical objects that we use (in general case – formulas).

In [3], the object $\langle a, b \rangle$ was called *intuitionistic fuzzy pair (IFP)*, if $a, b, a + b \in [0, 1]$. For the sake of simplicity and brevity, below, we will stick to this notation, instead of the (equivalent but longer) notation $\langle \mu(x),\ \nu(x) \rangle$.

Concerning the boundary cases, we assume that the evaluation function V assigns to the logical truth T the value $V(T) = \langle 1, 0 \rangle$, to the logical falsity F, the value $V(F) = \langle 0, 1 \rangle$, and to the logical uncertainty U, the value $V(U) = \langle 0, 0 \rangle$.

As it was discussed in [3], if the IFP $\langle a, b\rangle$, then the classical (originally defined) *intuitionistic fuzzy negation* is $V(\neg x) = \langle b, a\rangle$. Numerous other negations have been defined as well, exhibiting non-classical behaviour.

The geometrical interpretation (for IFS interpretation triangle, see [1]) of the IFP $\langle a, b\rangle$ is shown on Figure 1.

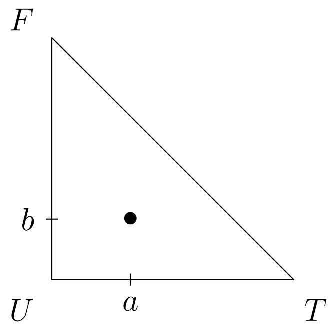

Figure 1. Geometrical interpretation of an element of an intuitionistic fuzzy set

In what follows, we will need comparisons between pairs of IFPs. For two IFPs $\langle a, b\rangle$ and $\langle c, d\rangle$, it is defined (cf. [2, p. 32 Eq. (1.5.13)]):

$$\langle a, b\rangle \leq \langle c, d\rangle \text{ iff } a \leq c \text{ and } b \geq d,$$

$$\langle a, b\rangle \geq \langle c, d\rangle \text{ iff } a \geq c \text{ and } b \leq d.$$

For other types of orderings between IFPs, the reader is referred to [15, 16, 17].

Finally, we need to formally introduce the *modal operators Necessity* (denoted by $\Box$) and *Possibility* (denoted by $\Diamond$). While originally defined in [8], for the IFP $\langle a, b\rangle$ are defined intuitionistic fuzzy modal operators, [2, 3]:

$$\Box\langle a, b\rangle = \langle a, 1-a\rangle,$$

$$\Diamond\langle a, b\rangle = \langle 1-b, b\rangle.$$

3 Main results

Let us have the variable x for which $V(x) = \langle a, b\rangle$. Therefore, in some applications like InterCriteria Analysis [9, 10] we may find it useful to work with distances $d(F, x) = \sqrt{a^2 + (1-b)^2}$ and $d(T, x) = \sqrt{(1-a)^2 + b^2}$ that are respectively the distances of the element of the IFS x to the points $\langle 0, 1\rangle$ ("Falsity") and $\langle 1, 0\rangle$ ("Truth"). Hence, on the IF interpretational triangle's hypotenuse we can construct two points $\mathrm{CSC_F}(x)$ and $\mathrm{CSC_T}(x)$, which are equidistant from F and T, respectively (see Figure 2), for which we have:

$$V(\mathrm{CSC_F}(x)) = \left\langle 1 - \frac{\sqrt{2((1-a)^2 + b^2)}}{2}, \frac{\sqrt{2((1-a)^2 + b^2)}}{2} \right\rangle$$

and

$$V(\text{CSC}_{\text{T}}(x)) = \left\langle \frac{\sqrt{2(a^2+(1-b)^2)}}{2}, 1 - \frac{\sqrt{2(a^2+(1-b)^2)}}{2} \right\rangle.$$

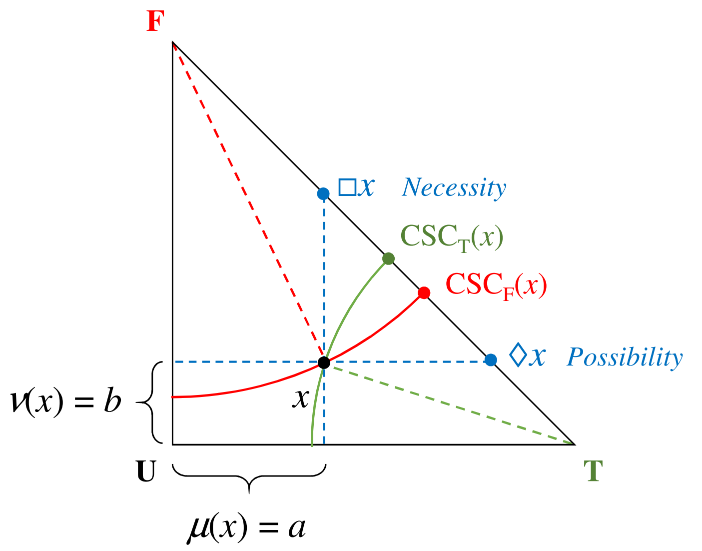

Figure 2. The new intuitionistic fuzzy modal operators $\text{CSC}_{\text{T}}(x)$ and $\text{CSC}_{\text{F}}(x)$.

First, we see that

$$\frac{\sqrt{2(a^2+(1-b)^2)}}{2} \le \frac{\sqrt{4(1-b)^2}}{2} = 1 - b \le 1$$

and, obviously,

$$\frac{\sqrt{2(a^2+(1-b)^2)}}{2} + 1 - \frac{\sqrt{2(a^2+(1-b)^2)}}{2} = 1.$$

Now, we can formulate and prove the following three statements.

Theorem 1. For every variable x:

$$V(\Box x) \le V(\text{CSC}_{\text{T}}(x)) \le V(\text{CSC}_{\text{F}}(x)) \le V(\Diamond x).$$

Proof. Let the IFP $x = \langle a, b\rangle$ be given.

First, we must prove that

$$a \le 1 - \frac{\sqrt{2((1-a)^2+b^2)}}{2} \iff 2 - 2a \ge \sqrt{2((1-a)^2+b^2)}$$

$$\iff 2(1-a)^2 \ge (1-a)^2 + b^2 \iff (1-a)^2 \ge b^2,$$

that is valid.

Second, we check sequentially:

$$\frac{\sqrt{2(a^2+(1-b)^2)}}{2} \geq 1 - \frac{\sqrt{2((1-a)^2+b^2)}}{2}$$

$$\iff 2(a^2+(1-b)^2) \geq 4 - 4\sqrt{2((1-a)^2+b^2)} + 2((1-a)^2+b^2)$$

$$\iff 2a^2+2-4b+2b^2 \geq 4 - 4\sqrt{2((1-a)^2+b^2)} + 2 - 4a + 2a^2 + 2b^2$$

$$\iff \sqrt{2((1-a)^2+b^2)} \geq 1 - a + b$$

$$\iff 2 - 4a + 2a^2 + 2b^2 \geq 1 + a^2 + b^2 - 2a + 2b - 2ab$$

$$\iff 1 - 2a - 2b + a^2 + b^2 + 2ab \geq 0$$

$$\iff 1 - 2(a+b) + (a+b)^2 \geq 0 \iff (1-(a+b))^2 \geq 0,$$

that is valid. The remaining checks are similar. □

Theorem 2. For every IFP x:

$$V(\neg \text{CSC}_{\text{T}}(\neg x)) = V(\text{CSC}_{\text{F}}(x)),$$

$$V(\neg \text{CSC}_{\text{F}}(\neg x)) = V(\text{CSC}_{\text{T}}(x)).$$

Proof. Let the IFP $x = \langle a, b \rangle$ be given.

Then, the classical negation gives $\neg x = \langle b, a \rangle$ and

$$V(\neg \text{CSC}_{\text{T}}(\neg x)) = \neg \left\langle 1 - \frac{\sqrt{2((1-b)^2+a^2)}}{2}, \frac{\sqrt{2((1-b)^2+a^2)}}{2} \right\rangle$$

$$= \left\langle \frac{\sqrt{2(a^2+(1-b)^2)}}{2}, 1 - \frac{\sqrt{2(a^2+(1-b)^2)}}{2} \right\rangle = V(\text{CSC}_{\text{F}}(x)).$$

The second equality is proved analogously. □

Theorem 3. For every variable $x = \langle a, 1-a \rangle$

$$V(\text{CSC}_{\text{F}}(x)) = V(\text{CSC}_{\text{T}}(x)) = \langle a, 1-a \rangle.$$

Proof. Let the IFP $x = \langle a, 1-a \rangle$ be given. Then

$$V(\text{CSC}_{\text{F}}(x)) = \left\langle \frac{\sqrt{2(a^2+a^2)}}{2}, 1 - \frac{\sqrt{2(a^2+a^2)}}{2} \right\rangle = \langle a, 1-a \rangle$$

and

$$V(\text{CSC}_{\text{T}}(x)) = \left\langle 1 - \frac{\sqrt{2((1-a)^2+(1-a)^2)}}{2}, \frac{\sqrt{2((1-a)^2+(1-a)^2)}}{2} \right\rangle$$

$$= \langle 1-(1-a), 1-a \rangle = \langle a, 1-a \rangle.$$

This completes the proof. □

As a corollary, we obtain that the application of any of the operators onto itself exhibits the following properties:

$$\text{CSC}_\text{F}(\text{CSC}_\text{F}(x)) = \text{CSC}_\text{F}(\text{CSC}_\text{T}(x)).$$

$$\text{CSC}_\text{T}(\text{CSC}_\text{T}(x)) = \text{CSC}_\text{T}(\text{CSC}_\text{F}(x)).$$

Therefore, the basic properties of the standard modal logic operators are valid here, too.

4 Conclusion

In a next leg of research, we will study the possibility to introduce extended modal operators for the new operators by analogy with the existing extended modal operators for operators $\Box$ and $\Diamond$. Currently, this seems as a promising direction of research, given that with the compass-and-straightedge construction, segments cut regions from the triangle that are closely located respectively to the vertices T and F, standing for the "truth" and "falsity" constants. Showing how the geometrical approach to IFS can evoke ideas in the logical domain, we also note that it has further proven useful in applications relying on the geometric interpretation of the intuitionistic fuzzy sets, like in InterCriteria Analysis.

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