Notes on Intuitionistic Fuzzy Sets
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Shrinking operators over interval-valued intuitionistic fuzzy sets

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In the memory of our friend,
Prof. Beloslav Riečan

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Abstract: Two new operators over interval-valued intuitionistic fuzzy sets are introduced and their basic properties are studied. The first of them is called shrinking operator and the second one, being an extension of the first one, is called (α, β)-shrinking operator.
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1 Introduction

Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs, see [2,4]) were introduced in 1989 as an extension of the Intuitionistic Fuzzy Sets (IFSs, see [1–3]) and Interval-Valued Fuzzy Sets (see [6]).

In a series of papers in recent years, a lot of new operators were introduced over IVIFSs. After the preliminary results, presented in Section 2, in Section 3 of the present paper, a new—shrinking—operator is defined and some of its basic properties are studied. In Section 4, it is further extended to the (α, β)-shrinking operator.

2 Preliminary remarks

Let us have a fixed universe E and its subset A. An IVIFS A over E is an object of the form: $A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\}$, where $M_A(x) \subseteq [0,1]$ and $N_A(x) \subseteq [0,1]$ are intervals and for all $x \in E$: $\sup M_A(x) + \sup N_A(x) \leq 1$. Obviously, each IFS can be represented as an IVIFS, as follows,

$$\{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\} = \{\langle x, [\mu_A(x), \mu_A(x)], [\nu_A(x), \nu_A(x)]\rangle \mid x \in E\}.$$

Also, each IVFS can be represented by an IVIFS as

$$\{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\} = \{\langle x, M_A(x), [1 - \sup M_A(x), 1 - \inf N_A(x)]\rangle \mid x \in E\}.$$

IVIFSs have geometrical interpretations similar to, but more complex than these of the IFSs. For example, the second geometrical interpretation of IFS (see [2]) now has the form from Fig. 1.

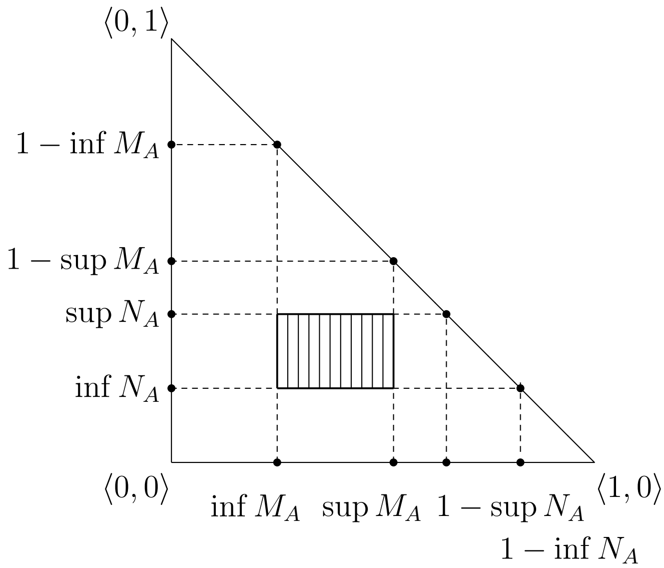
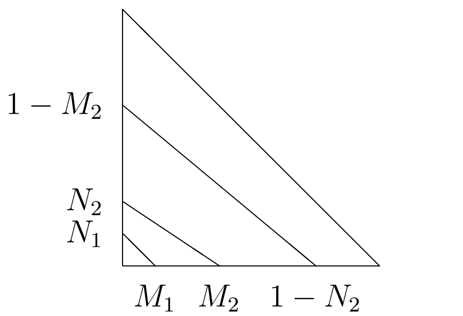


Figure 1. Graphical interpretation of an element of an IVIFS

Other geometrical interpretations of the IVIFSs are shown on Figures 2 and 3.



$M_1 = \inf M_A(x)$, $M_2 = \sup M_A(x)$
$N_1 = \inf N_A(x)$, $N_2 = \sup N_A(x)$

Figure 2. Second graphical interpretation

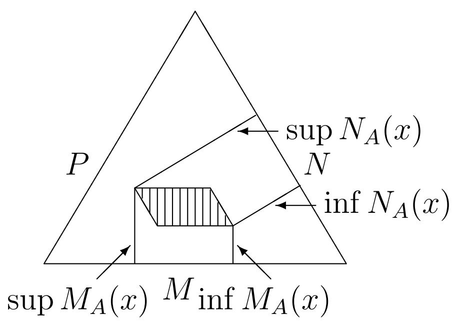


Figure 3. Third graphical interpretation

Following [2], we introduce some operations and relations over two IVIFSs A and B:

$$\begin{array}{lll}
A \subset_{\Box,\inf} B & \text{iff} & (\forall x \in E)(\inf M_A(x) \le \inf M_B(x)), \\
A \subset_{\Box,\sup} B & \text{iff} & (\forall x \in E)(\sup M_A(x) \le \sup M_B(x)), \\
A \subset_{\Diamond,\inf} B & \text{iff} & (\forall x \in E)(\inf N_A(x) \ge \inf N_B(x)), \\
A \subset_{\Diamond,\sup} B & \text{iff} & (\forall x \in E)(\sup N_A(x) \ge \sup N_B(x)), \\
A \subset_{\Box} B & \text{iff} & A \subset_{\Box,\inf} B \;\&\; A \subset_{\Box,\sup} B, \\
A \subset_{\Diamond} B & \text{iff} & A \subset_{\Diamond,\inf} B \;\&\; A \subset_{\Diamond,\sup} B, \\
A \subseteq B & \text{iff} & A \subset_{\Box} B \;\&\; B \subset_{\Diamond} A, \\
A = B & \text{iff} & A \subseteq B \;\&\; B \subseteq A,
\end{array}$$

$$\begin{aligned}
\neg A &= \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}, \\
A \cap B &= \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))], \\
&\qquad [\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}, \\
A \cup B &= \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x) \sup M_B(x))], \\
&\qquad [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\} \\
A@B &= \left\{\left\langle x, \left[\frac{\inf M_A(x) + \inf M_B(x)}{2}, \frac{\sup M_A(x) + \sup M_B(x)}{2}\right],\right.\right. \\
&\qquad \left.\left.\left[\frac{\inf N_A(x) + \inf N_B(x)}{2}, \frac{\sup N_A(x) + \sup N_B(x)}{2}\right]\right\rangle \mid x \in E\right\}.
\end{aligned}$$

Following [2], we mention that the first two (classical) intuitionistic fuzzy modal operators over the IVIFS A are:

$$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\},$$

$$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\},$$

and the standard intuitionistic fuzzy topological operators over the IVIFS A are:

$$C(A) = \{\langle x, [K'_{\inf}, K'_{\sup}], [L'_{\inf}, L'_{\sup}]\rangle \mid x \in E\},$$

$$I(A) = \{\langle x, [K''_{\inf}, K''_{\sup}], [L''_{\inf}, L''_{\sup}]\rangle \mid x \in E\},$$

where:

$$\begin{aligned}
K'_{\inf} &= \sup_{x \in E} \inf M_A(x), \\
K'_{\sup} &= \sup_{x \in E} \sup M_A(x), \\
L'_{\inf} &= \inf_{x \in E} \inf N_A(x), \\
L'_{\sup} &= \inf_{x \in E} \sup N_A(x), \\
K''_{\inf} &= \inf_{x \in E} \inf M_A(x), \\
K''_{\sup} &= \inf_{x \in E} \sup M_A(x),
\end{aligned}$$

$$L''_{\inf} = \sup_{x \in E} \inf N_A(x),$$
$$L''_{\sup} = \sup_{x \in E} \sup N_A(x),$$

and the four operators that transform an IVIFS A to an IFS are:

$$\begin{aligned} *_1 A = &\ \{\langle x, \inf M_A(x), \inf N_A(x)\rangle \mid x \in E\}, \\ *_2 A = &\ \{\langle x, \inf M_A(x), \sup N_A(x)\rangle \mid x \in E\}, \\ *_3 A = &\ \{\langle x, \sup M_A(x), \inf N_A(x)\rangle \mid x \in E\}, \\ *_4 A = &\ \{\langle x, \sup M_A(x), \sup N_A(x)\rangle \mid x \in E\}. \end{aligned}$$

3 The simplest shrinking operator over interval-valued intuitionistic fuzzy sets

Let A be an IVIFS. Then we define

$$\begin{aligned} S(A) &= \left\{\left\langle x, \frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2}\right\rangle |x \in E\right\} \\ &= \left\{\left\langle x, \left[\frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf M_A(x) + \sup M_A(x)}{2}\right],\right.\right. \\ &\qquad \left.\left.\left[\frac{\inf N_A(x) + \sup N_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2}\right]\right\rangle |x \in E\right\}. \end{aligned}$$

Obviously, for each IFS A:

$$S(A) = A.$$

The three geometrical interpretations of the new operator are given on Figures 4–6.

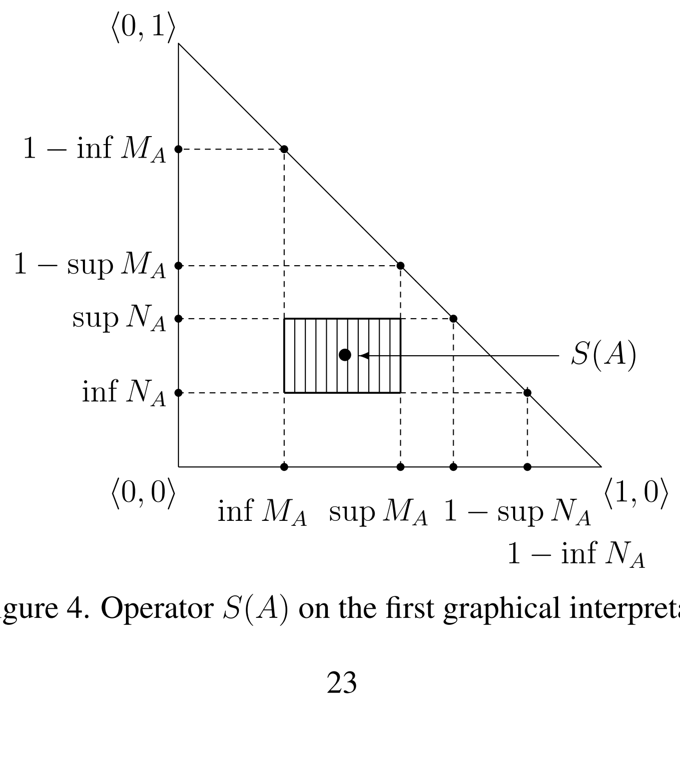
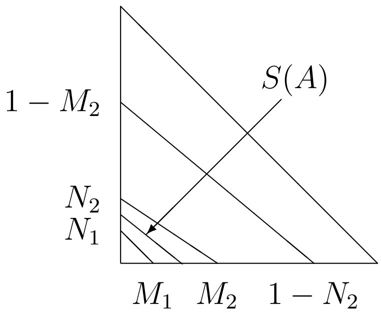


Figure 4. Operator $S(A)$ on the first graphical interpretation



$M_1 = \inf M_A(x)$, $M_2 = \sup M_A(x)$
$N_1 = \inf N_A(x)$, $N_2 = \sup N_A(x)$

Figure 5. Operator $S(A)$ on the second interpretation

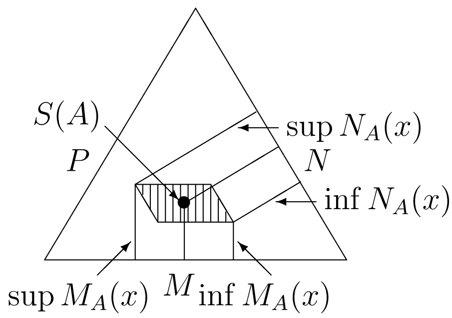


Figure 6.

Theorem 1. *For each IVIFS A:*

(a) $S(S(A)) = S(A)$,

(b) $\neg S(\neg A) = S(A)$.

Proof. Let A be an IVIFS. Then for (a) we obtain

$$S(S(A)) = S\left(\left\{\left\langle x, \left[\frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf M_A(x) + \sup M_A(x)}{2}\right],\right.\right.\right.$$
$$\left.\left.\left.\left[\frac{\inf N_A(x) + \sup N_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2}\right]\right\rangle |x \in E\right\}\right)$$
$$= \left\{\left\langle x, \left[\frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf M_A(x) + \sup M_A(x)}{2}\right],\right.\right.$$
$$\left.\left.\left[\frac{\inf N_A(x) + \sup N_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2}\right]\right\rangle |x \in E\right\} = S(A).$$

For (b) we obtain

$$\neg S(\neg A) = \neg S\left(\neg\{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}\right)$$
$$= \neg\left\{\left\langle x, \frac{\inf N_A(x) + \sup N_A(x)}{2}, \frac{\inf M_A(x) + \sup M_A(x)}{2}\right\rangle |x \in E\right\}$$
$$= \left\{\left\langle x, \frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2}\right\rangle |x \in E\right\} = S(A).$$

The proof of the next assertions is made in a similar way. □

Theorem 2. *For every two IVIFSs A and B:*

(a) $S(A \cup B) \supseteq S(A) \cup S(B)$,

(b) $S(A \cap B) \subseteq S(A) \cap S(B)$,

(c) $S(A@B) = S(A)@S(B)$.

Theorem 3. *For each IVIFS A:*

(a) $S(\Box A) = \Box S(A)$,

(b) $S(\Diamond A) = \Diamond S(A)$.

In [2], operator D_α is defined for each $\alpha \in [0,1]$ by:

$$\begin{aligned} D_\alpha(A) = \; & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\ & [\inf N_A(x), \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}. \end{aligned}$$

We can see that $S(D_\alpha(A)) \neq D_\alpha(S(A))$. Really,

$$\begin{aligned} S(D_\alpha(A)) = \; & S(\{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\ & [\inf N_A(x), \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}) \\ = \; & \Bigg\{\Bigg\langle x, \Bigg[\frac{\inf M_A(x) + \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))}{2}, \\ & \frac{\inf M_A(x) + \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))}{2}\Bigg], \\ & \Bigg[\frac{\inf N_A(x) + \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))}{2}, \\ & \frac{\inf N_A(x) + \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))}{2}\Bigg]\Bigg\rangle \mid x \in E\Bigg\} \\ \neq \; & \Bigg\{\Bigg\langle x, \Bigg[\frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf M_A(x) + \sup M_A(x)}{2} \\ & + \alpha.\left(1 - \frac{\inf M_A(x) + \sup M_A(x)}{2} - \frac{\inf N_A(x) + \sup N_A(x)}{2}\right)\Bigg], \\ & \Bigg[\frac{\inf N_A(x) + \sup N_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2} \\ & + (1-\alpha).\left(1 - \frac{\inf M_A(x) + \sup M_A(x)}{2} - \frac{\inf N_A(x) + \sup N_A(x)}{2}\right)\Bigg]\Bigg\rangle \mid x \in E\Bigg\} \\ = \; & D_\alpha\Bigg(\Bigg\{\Bigg\langle x, \Bigg[\frac{\inf M_A(x) + \sup M_A(x)}{2}, \frac{\inf M_A(x) + \sup M_A(x)}{2}\Bigg], \\ & \Bigg[\frac{\inf N_A(x) + \sup N_A(x)}{2}, \frac{\inf N_A(x) + \sup N_A(x)}{2}\Bigg]\Bigg\rangle \mid x \in E\Bigg\}\Bigg) \\ = \; & D_\alpha(S(A)). \end{aligned}$$

On the other hand, for operator $G_{\alpha,\beta}$, defined (see, [2]) for every $\alpha, \beta \in [0,1]$ by:

$$G_{\alpha,\beta}(A) = \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},$$

is valid the equality

$$S(G_{\alpha.\beta}(A)) = G_{\alpha.\beta}(S(A)).$$

Theorem 4. *For each IVIFS A:*

(a) $S(C(A)) \subseteq C(S(A))$,

(b) $S(I(A)) \supseteq I(S(A))$.

Theorem 5. *For each IVIFS A:*

(a) $*_1 A \subset_{\Box,\inf} S(A) \subset_{\Diamond,\inf} *_1 A$,

(b) $*_1 A \subset_{\Box,\sup} S(A) \subset_{\Diamond,\sup} *_1 A$,

(c) $*_1 A \subset_{\Box} S(A) \subset_{\Diamond} *_1 A$,

(d) $*_2 A \subset_{\Box,\inf} S(A)$,

(e) $*_2 A \subset_{\Box,\sup} S(A)$,

(f) $*_2 A \subset_{\Box} S(A)$,

(g) $*_2 A \subset_{\Diamond,\inf} S(A)$,

(h) $*_2 A \subset_{\Diamond,\sup} S(A)$,

(i) $*_2 A \subset_{\Diamond} S(A)$,

(j) $S(A) \subset_{\Box,\inf} *_4 A \subset_{\Diamond,\inf} S(A)$,

(k) $S(A) \subset_{\Box,\sup} *_4 A \subset_{\Diamond,\sup} S(A)$,

(l) $S(A) \subset_{\Box} *_4 A \subset_{\Diamond} S(A)$,

(m) $S(A) \subset_{\Box,\inf} *_3 A$,

(n) $S(A) \subset_{\Box,\sup} *_3 A$,

(o) $S(A) \subset_{\Box} *_3 A$,

(p) $S(A) \subset_{\Diamond,\inf} *_3 A$,

(q) $S(A) \subset_{\Diamond,\sup} *_3 A$,

(r) $S(A) \subset_{\Diamond} *_3 A$.

4 (α, β)-Shrinking operator over interval-valued intuitionistic fuzzy sets

Let A be an IVIFS and $\alpha, \beta \in [0, 0.5]$. Then we define

$$\begin{aligned} S_{\alpha,\beta}(A) = \;& \{\langle x, \alpha(\inf M_A(x) + \sup M_A(x)), \beta(\inf N_A(x) + \sup N_A(x))\rangle | x \in E\} \\ = \;& \{\langle x, [\alpha(\inf M_A(x) + \sup M_A(x)), \alpha(\inf M_A(x) + \sup M_A(x))], \\ & [\beta(\inf N_A(x) + \sup N_A(x)), \beta(\inf N_A(x) + \sup N_A(x))]\rangle | x \in E\}. \end{aligned}$$

Obviously,

$$S(A) = S_{0.5,0.5}(A).$$

First, we must check that the definition is correct. Really,

$$\alpha(\inf M_A(x) + \sup M_A(x)) + \beta(\inf N_A(x) + \sup N_A(x))$$
$$\leq 2\alpha \sup M_A(x) + 2\beta \sup N_A(x) \leq \sup M_A(x) + \sup N_A(x) \leq 1.$$

Theorem 6. *For each IVIFS A and for every $\alpha, \beta \in [0, 0.5]$:*

(a) $S_{\alpha,\beta}(S_{\alpha,\beta}(A)) = S_{2\alpha^2,2\beta^2}(A)$,

(b) $\neg S_{\alpha,\beta}(\neg A) = S_{\beta,\alpha}(A)$.

Theorem 7. *For every two IVIFSs A and B, and for every $\alpha, \beta \in [0, 0.5]$:*

(a) $S_{\alpha,\beta}(A \cup B) \supseteq S_{\alpha,\beta}(A) \cup S_{\alpha,\beta}(B)$,

(b) $S_{\alpha,\beta}(A \cap B) \subseteq S_{\alpha,\beta}(A) \cap S_{\alpha,\beta}(B)$,

(c) $S_{\alpha,\beta}(A@B) = S_{\alpha,\beta}(A)@S_{\alpha,\beta}(B)$.

Theorem 8. *For each IVIFS A and for every two real numbers $\alpha, \beta \in [0, 1]$:*

(a) if $\alpha \leq \beta$, then $S_{\alpha,\beta}(\Box A) = \Box S_{\alpha,\beta}(A)$,

(b) if $\alpha \geq \beta$, then $S_{\alpha,\beta}(\Diamond A) = \Diamond S_{\alpha,\beta}(A)$.

Theorem 9. *For each IVIFS A and for every $\alpha, \beta \in [0, 0.5]$:*

(a) $S_{\alpha,\beta}(C(A)) \subseteq C(S_{\alpha,\beta}(A))$,

(b) $S_{\alpha,\beta}(I(A)) \supseteq I(S_{\alpha,\beta}(A))$.

Theorem 10. *For each IVIFS A, for every $\alpha, \beta \in [0, 0.5]$, and for every $\gamma, \delta \in [0, 1]$:*

$$S_{\alpha,\beta}(G_{\gamma,\delta}(A)) = S_{\alpha\gamma,\beta\delta}(A) = G_{\gamma,\delta}(S_{\alpha,\beta}(A)).$$

Similar equality is not valid for the other modal types of operators.

5 Conclusion

In a next research, other properties of the two new operators will be studied. Consequent extension of operator $S_{\alpha,\beta}$ will be introduced and its properties will be studied, too. Each of these operators will be redefined for interval-valued intuitionistic fuzzy pairs (see [5]).

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