Twelfth Int. Conf. on IFSs, Sofia, 17-18 May 2008
NIFS Vol. 14 (2008), 1, 48–56

Ranking Intuitionistic Fuzzy Alternatives

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We propose a method for ranking alternatives represented by Atanassov's intuitionistic fuzzy sets (A-IFSs) which takes into account not only the amount of information related to an alternative (expressed by a distance from the ideal positive alternative) but also the reliability of information (how sure the information is). We stress (like in our previous papers) that taking into account all three functions (membership, non-membership and hesitation) in the description of A-IFSs is the necessary condition to obtain results we intuitively expect.

1 Introduction

Atanassov's intuitionistic fuzzy sets (cf. Atanassov [1], [2], [3]), are a tool to better model imperfect information. An important "meta-problem" is how to rank alternatives (options). A set of alternatives is expressed as that each option fulfills a set of criteria to some extent μ, and it does not fulfill it to some extent ν. This can be represented by A-IFSs [cf. Section 2]. The ranking of intuitionistic fuzzy alternatives is non-trivial because there is no linear order among them as opposed to fuzzy sets (Zadeh [25]). For some approaches for ranking the intuitionistic fuzzy alternatives, cf. Chen and Tan [4], Hong and Choi [5], Li et al. [6], [7], and Liu and Wang [8].

Here we propose another method. First, we employ the representation of A-IFSs taking into account all three functions (the membership, non-membership, and hesitation margin). Second, we propose a ranking function which depends on two factors: the amount of information (expressed by the distance from the ideal positive alternative), and the reliability of information (expressed by the hesitation margin).

2 A Brief Introduction to Intuitionistic Fuzzy Sets

Atanassov's intuitionistic fuzzy set (Atanassov [1], [3]) A is:

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{1}$$

where: $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{2}$$

and $\mu_A(x), \nu_A(x) \in [0,1]$ denote the degree of membership and a degree of non-membership of $x \in A$, respectively, and the *hesitation margin* of $x \in A$ is:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{3}$$

The $\pi_A(x)$ expresses a lack of knowledge of whether x belongs to A or not (Atanassov [3]); obviously, $0 \leq \pi_A(x) \leq 1$, for each $x \in X$; $\pi_A(x)$ is important while considering distances, entropy, similarity etc. (Szmidt and Kacprzyk [12], [18], [14], [20], [21]) for the A-IFSs. In this paper $\pi_A(x)$ is indispensable, too – it indicates how reliable (sure) the information represented by an alternative is.

We use the normalized Euclidean distance between the A-IFSs A, B in X (Szmidt and Kacprzyk [12], [18], Szmidt and Baldwin [9]):

$$\begin{aligned} e_{IFS}(A,B) &= (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + \\ &+ (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2)^{\frac{1}{2}} \end{aligned} \tag{4}$$

For (4) we have:

$$0 \leq e_{IFS}(A,B) \leq 1 \tag{5}$$

The application of A-IFSs instead of fuzzy sets means the introduction of another degree of freedom (non-memberships) into a set description. Such a generalization of fuzzy sets gives us an additional possibility to represent imperfect knowledge i.e., describing many real problems in a more adequate way (cf. Szmidt and Kacprzyk [10], [13], [16], [15], [19], Szmidt and Kukier [23], [24]).

2.1 Geometrical representation

A possible geometrical representations of an A-IFS is as in Fig. 1 (cf. Atanassov [3]). It is worth noticing that although we use a 2D figure, we still adopt our approach (e.g., Szmidt and Kacprzyk [12], [18], [14], [20], [21]) with the membership, non-membership and hesitation margin. Any element in an A-IFS may be represented inside MNO. Each point belonging to MNO is described by: (μ, ν, π). Points M and N represent crisp elements. Point $M(1,0,0)$ represents elements fully belonging to an A-IFS as $\mu = 1$, and may be seen as the ideal positive element. Point $N(0,1,0)$ represents elements fully not belonging to an A-IFS as $\nu = 1$. Point $O(0,0,1)$ represents elements unsure as to if they belong or not to an A-IFS as $\pi = 1$. Segment MN, with $\pi = 0$, represents elements belonging to the classic fuzzy sets ($\mu + \nu = 1$). For example, point $A(0.2, 0.8, 0)$ (Fig. 1), like any element from segment MN represents an element of a fuzzy set. A line parallel to MN describes elements with the same hesitation margin. In Fig. 1 we can see point $F(0.5, 0.1, 0.4)$ representing an element with the hesitation margin equal 0.4, like $B(0.2, 0, 0.8)$, with the hesitation margin equal 0.8. The closer a parallel line to MN is to O (Fig. 1), the higher the hesitation margin.

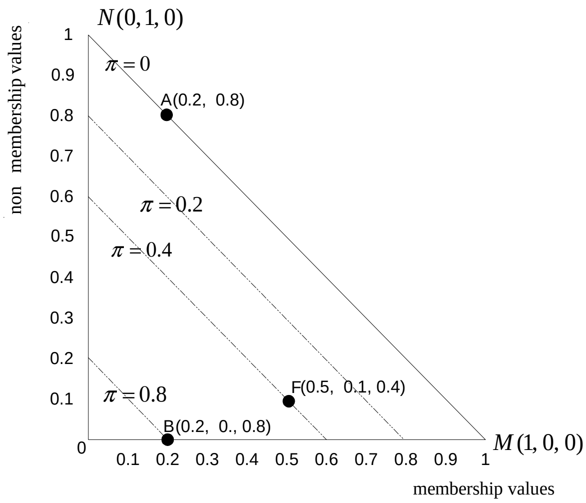

Figure 1: Geometrical representation

3 A New Method for the Ranking of Intuitionistic Fuzzy Alternatives

Let an x belonging to an A-IFS characterized via (μ, ν, π) expresses a voting situation: μ is the proportion (from $[0, 1]$) of voters who vote for x, ν the proportion of those who vote against, and π of those who abstain. The simplest ranking of the alternatives may be to use a distance measure from the ideal voting situation (ideal positive alternative M). Let $A = (x, 0.2, 0.8, 0)$ – 20% vote for, 80% against, and 0% abstain, $B = (x,\ 0.2,\ 0,\ 0.8)$ – 20% vote for, 0% vote against and 80% abstain, The normalized Euclidean distance (4) gives:

$$e_{IFS}(M, A) = (0.5((1 - 0.2)^2 + (0 - 0.8)^2 + (0 - 0)^2))^{0.5} = 0.8 \tag{6}$$

$$e_{IFS}(M, B) = (0.5((1 - 0.2)^2 + (0 - 0)^2 + (0 - 0.8)^2))^{0.5} = 0.8 \tag{7}$$

The results seems to be counterintuitive as (4) suggests [cf. (6)–(7)] that the alternatives (represented by) A, B seem to be "the same". A general explanation of the above counterintuitive result follows from Fig. 2 as the results of (4) are not univocally given for a given membership value μ; for clarity, the distances (4) for any x from M (Fig. 2a) are presented for μ and ν for $[0, 1]$ instead of for $\mu + \nu \leq 1$ only. For the same reason (to better see the effect), in Fig. 2b the contour plot of the distances (4) is given only for the range of μ and ν for which $\mu + \nu \leq 1$). It is obvious that the results of (4) are not univocally given for a given membership value μ. So, the distances (cf. also Szmidt and Kacprzyk [22]) from the ideal positive alternative alone do not make it possible to rank the alternatives in the intended way.

Now, let us analyze the essense of a voting alternative (an intuitionistic fuzzy element) using the operators of (cf. Atanassov [3]): *necessity* ($\Box$), *possibility* ($\Diamond$), D_α, and $F_{\alpha,\beta}$

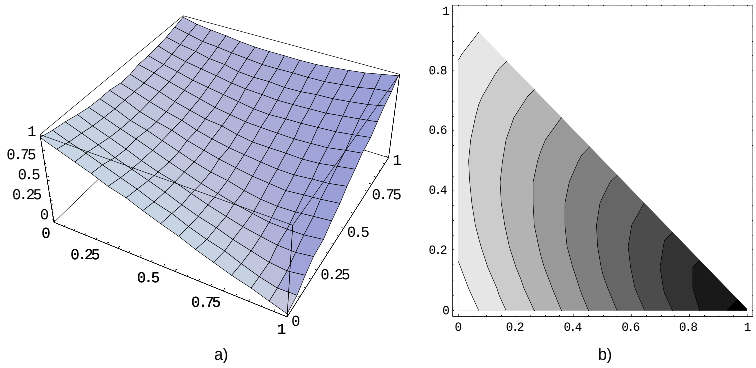

Figure 2: a) Distances (4) of any IFS element from ideal alternative M; b) contour plot

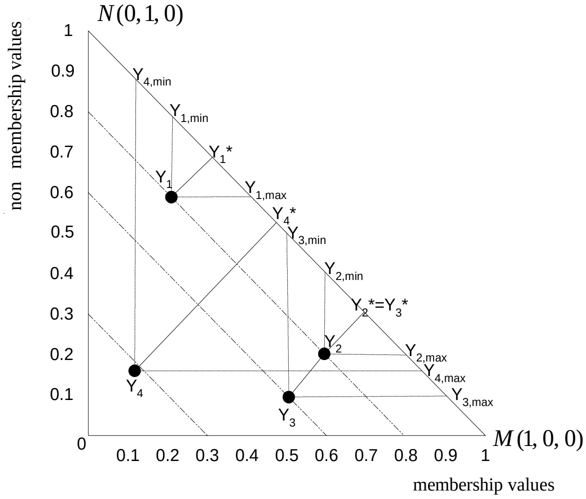

Figure 3: Ranking alternatives Y_i

(where $\alpha, \beta \in [0,1]$; $\alpha + \beta \leq 1$) given as:

$$\square A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\} \tag{8}$$

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in X\} \tag{9}$$

$$D_\alpha(A) \;=\; \{\langle x,\ \mu_A(x) + \alpha\pi_A(x),\ \nu_A(x) + (1-\alpha)\pi_A(x)\rangle\, | x \in X\} \tag{10}$$

$$F_{\alpha,\beta}(A) \;=\; \{\langle x,\ \mu_A(x) + \alpha\pi_A(x),\ \nu_A(x) + \beta\pi_A(x)\rangle\, | x \in X\} \tag{11}$$

For example, for alternative Y_1 we obtain $\square Y_1 = Y_{1,min}$, and $\Diamond Y_1 = Y_{1,max}$ (cf. Fig. 3). Operator $F_{\alpha,\beta}$ makes it possible for alternative Y_1 to become any alternative within the

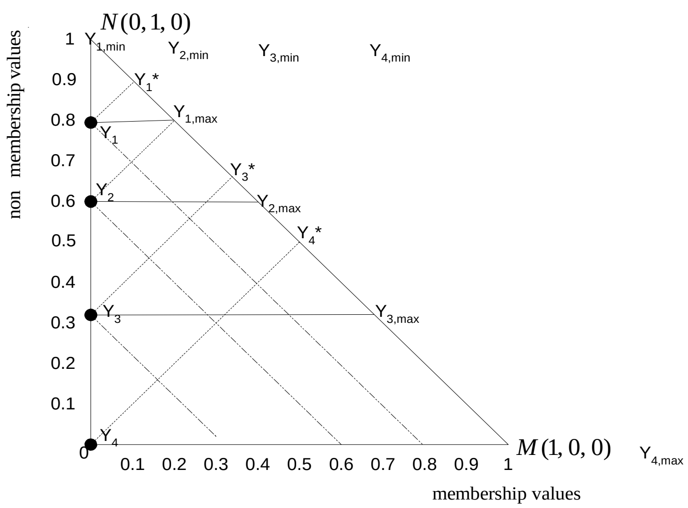

Figure 4: a) $R_E(Y_i^*)$ as a function of a distance Y_i^* from M and a hesitation margin; b) contour plot

triangle $Y_1Y_{1,max}Y_{1,min}$. By a similar reasoning, alternative $O(0,0,1)$ (because $\pi = 1$) may become any alternative (i.e. from within the whole area of MNO).

Therefore, we could say that the smaller the area of the triangle $Y_iY_{i,min}Y_{i,max}$ (Fig. 3), the better the alternative Y_i from Y. Alternatives on MN are the best in the sense that: 1) $\pi = 0$ here which means that the alternatives are fully reliable in the sense of the information represented, and 2) the alternatives are ordered – the closer an alternative to the ideal positive alternative $M(1,0,0)$, the better it is. This suggests that for the ranking of any intuitionistic fuzzy alternative Y_i, for a fixed π_i, we may convert them into fuzzy alternatives (naturally ordered) but still preserve knowledge of how sure this information is. For a fixed and specified π_i, each $Y_iY_{i,min}Y_{i,max}$ is univocally given by: $Y_i^* = 0.5(Y_{i,min} + Y_{i,max})$ (Fig. 3). These Y_i^*'s are the orthogonal projections of Y_i on MN. (cf. Szmidt and Kacprzyk [11]). These orthogonal projections may be obtained via D_α (10) with α equal 0.5. In this context the most natural way of ranking the alternatives seems to be making use of Y_i^*'s and their distances from the ideal alternative M, preserving also the information about the hesitation margin π_{Y_i}, i.e.:

$$R_E(Y_i^*) = 0.5(1 + \pi_{Y_i})e_{IFS}(M, Y_i^*) \tag{12}$$

Unfortunately, the results of (12) do not meet our expectations in the sense of their connections with the areas of the triangles $Y_iY_{i,min}Y_{i,max}$. Let us consider the alternatives Y_i, $i = 1, \ldots, 4$. – Fig. 4. We might expect that the alternatives are ordered by (12) from Y_1 to Y_4 as just such an order renders the areas of the respective triangles. But the results from (12) for the different alternatives seem to be "the same". For example, for Y_1=(0, 0.8, 0.2), $R_E(Y_1^*)$=0.54, for Y_2=(0, 0.6, 0.4), $R_E(Y_2^*)$=0.56, for Y_3=(0, 0.3, 0.7), $R_E(Y_3^*)$=0.55, for Y_4=(0, 0, 1), $R_E(Y_4^*)$=0.5.

This phenomenon is presented in general in Fig. 5 – for the alternatives for which the

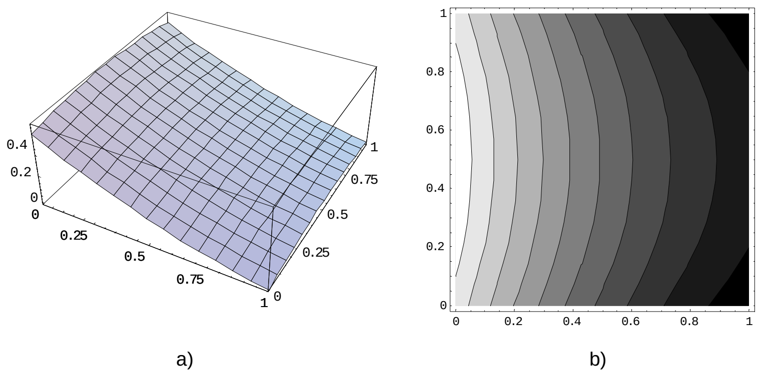

Figure 5: a) $R_E(Y_i^*)$ as a function of a distance Y_i^* from M and a hesitation margin; b) contour plot

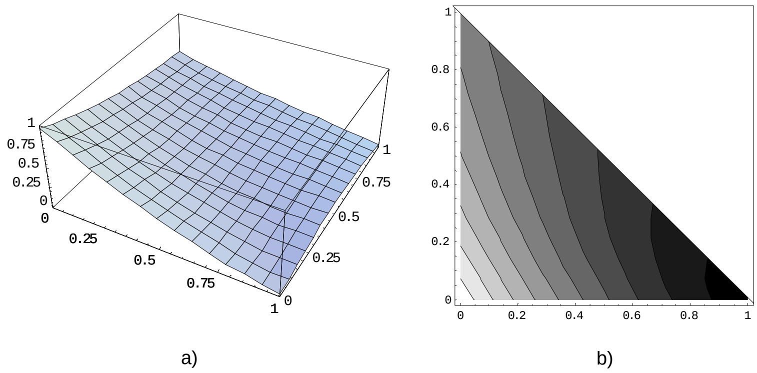

Figure 6: a) $R_E(Y_i)$ as a function of a distance (4) from M and a hesitation margin; b) contour plot

membership values are equal to zero, ranking (12) gives "the same" result (white area in Figure 5 b). It is the reason that instead of (12) we use the following measure R_E for ranking the alternatives Y_i

$$R_E(Y_i) = 0.5(1 + \pi_{Y_i})e_{IFS}(M, Y_i) \tag{13}$$

where $e_{IFS}(M, Y_i)$ is the distance (4) from the ideal positive alternative $M(1, 0, 0)$. The constant 0.5 was introduced in (13) to ensure that $0 < R_E(Y_i) \leq 1$. The values of R_E for any intuitionistic fuzzy element are as in Fig. 6a, and the counterpart contour plot – in Fig. 6b.

Equation (13) reflects the "quality" of an alternative – the lower $R_E(Y_i)$, (13), the better the alternative in the sense of the amount and reliability of information.

Let us rank at the beginning the same alternatives using (13) as we did for (12), i.e. Y_1=(0, 0.8, 0.2), Y_2=(0, 0.6, 0.4), Y_3=(0, 0.3, 0.7), and Y_4=(0, 0, 1). We obtain $R_E(Y_1)$=0.55, $R_E(Y_2)$=0.61, $R_E(Y_3)$=0.85, $R_E(Y_4)$=1. The results seem to render our intuition now.

The best one is alternative $M(1,0,0)$ ($R_E(M) = 0$). For alternative $N(0,1,0)$ we obtain $R_E(N) = 0.5$ (N is fully reliable as the hesitation margin is equal 0 but the distance $e_{IFS}(M,N) = 1$). In general, on MN, the values of R_E decrease from 0.5 (for alternative N) to 0 (for the best alternative M). The maximal value of R_E, i.e. 1, is for $O(0,0,1)$ for which $e_{IFS}(M,O), \pi_O = 1$ (alternative O "indicates" the whole triangle MNO). All other alternatives Y_i "indicate" smaller triangles $Y_i Y_{i,min} Y_{i,max}$ (Fig. 3), so that their R_E's are smaller (better as to the amount of the reliable information).

It is worth noticing that the results obtained via (13) which render our expectations for ranking the alternatives are obtained using all three functions describing intuitionistic fuzzy alternatives, i.e., membership function, non-membership function, and the hesitation margin. Also the distances in (13) are calculated taking into account all three functions. In other words, we use 3D representation of A-IFSs.

4 Conclusions

We discussed a method of ranking intuitionistic fuzzy alternatives. The method takes into account the amount and reliability of information connected with an alternative. We discussed two possibilities of the measure – first we tried to simplify the problem by boiling down the intuitionistic fuzzy alternatives into fuzzy alternatives, and take into account how reliable the alternative is (which was expressed via hesitation margins). Unfortunately, the method turned out not enough.

Only the full representation of an alternative (via membership, non-membership, and hesitation margin), and calculating its distance from an ideal alternative, whereas additionally the information about the reliability is taken into account, guarantee intuitively acceptable results.

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