Eighth Int. Conf. on IFSs, Varna, 20-21 June 2004
NIFS Vol. 10 (2004), 3, 15-28

Entropy for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory

Eulalia Szmidt * and Jim F. Baldwin **

* Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01–447 Warsaw, Poland
E-mail: szmidt@ibspan.waw.pl

** Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, England
E-mail: Jim.Baldwin@bristol.ac.uk

Abstract

In this article we remind parallels for intuitionistic fuzzy sets and mass assignment theory, and propose a non-probabilistic-type entropy measure for both of them. The proposed measure is a result of a common geometric interpretation valid for these theories, and uses a ratio of distances between considering elements/support pairs and crisp elements. It is also shown that the proposed measure can be defined in terms of the ratio of cardinalities: of $X \cap X^c$ and $X \cup X^c$.
Keywords: intuitionistic fuzzy sets, mass assignment theory, entropy.

1 Introduction

Fuzziness, a feature of imperfect information, results from the lack of crisp distinction between the elements belonging and not belonging to a set (i.e. the boundaries of the set under consideration are not sharply defined). A measure of fuzziness often used and cited in the literature is an entropy first mentioned by Zadeh [26]. The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon entropy (Jaynes [16]). However, the two functions measure fundamentally different types of uncertainty. Basically, the Shannon entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment.

De Luca and Termini [15] introduced some requirements which capture our intuitive comprehension of the degree of fuzziness. Kaufmann [17] proposed to measure the degree of fuzziness of any fuzzy set A by a metric distance between its membership function and the membership function (charecteristic function) of its nearest crisp set. Another way given by Yager [25] was to view the degree of fuzziness in terms of a lack of distinction between the fuzzy set and its complement. Indeed, it is the lack of distinction between sets and their complements that distinguishes fuzzy sets from crisp sets. The less the set differs from its complement, the fuzzier it is. Kosko [19] investigated the fuzzy entropy in relation to a measure of subsethood.

In this paper we propose a measure of fuzziness for intuitionistic fuzzy sets (Atanassov [1], [2]) and mass assignment theory (Baldwin [4], [7]), [8]). First, we remind parallels and a common geometrical representation for both theories (cf. Szmidt and Baldwin [21]). The geometrical representation makes it possible to discuss the essence of the proposed measure of entropy and illustrates the first way how to calculate it. It is also shown that the proposed measure can be stated as the ratio of the cardinalities: that of $X \cap X^c$ and that of $X \cup X^c$, where X^c is the complement of X. For the different approaches we refer the interested reader to Burillo and Bustince [13], Ban [12], Cornelis and Kerre [14].

2 Brief introduction to intuitionistic fuzzy sets

As opposed to a fuzzy set in X(Zadeh [26]) , given by

$$A' = \{< x, \mu_{A'}(x) > | x \in X\} \tag{1}$$

"where $\mu_{A'}(x) \in [0,1]$ is the membership function of the fuzzy set A', an intuitionistic fuzzy set (Atanassov [1], [2]) A is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where: $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0,1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

Obviously, each fuzzy set may be represented by the following intuitionistic fuzzy set

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \tag{4}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

an *intuitionistic fuzzy index* (or a *hesitation margin*) of $x \in A$ and, it expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [2]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

3 Brief introduction to mass assignment theory

The theory of mass assignment has been developed by Baldwin [4], [7], [8] to provide a formal framework for manipulating both probabilistic and fuzzy uncertainty.

A fuzzy set can be converted into a mass assignment (Baldwin [3]). This mass assignment represents a family of probability distributions.

Definition 1 *Let A' be a normalized fuzzy set in $X = \{x\}$ such that*

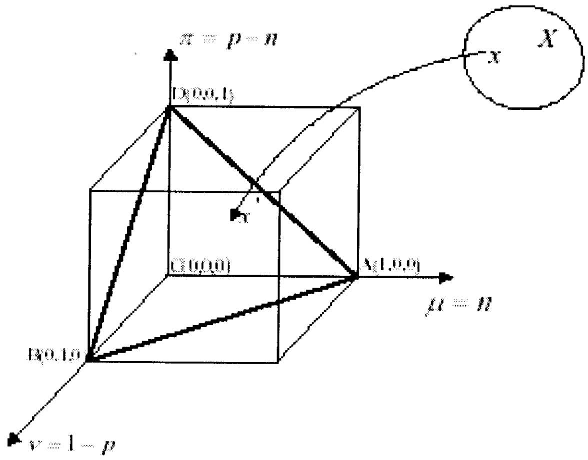

Figure 1: A common geometrical representation

$$A^{'} = \sum_{x_i \in X} x_i/\mu(x_i)$$

$$\mu(x_1) = 1, \ \mu(x_i) \leq \mu(x_j) \quad for \quad i > j$$

where $\mu(x)$ is the membership function.
The mass assignment associated with $A^{'}$ is (Baldwin [5])

$$\{x_1\} : 1 - \mu(x_2), \quad \{x_1, ..., x_i\} : \mu(x_i) - \mu(x_{i+1}) \quad for \quad i = 2, ...; \tag{6}$$

with $\mu(x_k) = 0 \quad for \quad x_k \notin X$

Support Pairs (the basic representation of uncertainty in the language FRIL (Baldwin at al. [7], [11]) are associated with mass assignments and represent an interval containing an unknown probability. Support Pairs are used to characterize uncertainty in facts and conditional probabilities in rules. A Support Pair (n, p) comprises a necessary and possible support and can be interpreted as an interval in which the unknown probability lies. A voting interpretation is also useful (Baldwin and Pilsworth [6]): the lower (necessary) support n represents the proportions of a sample population voting in favour of a proposition, whereas $(1 - p)$ represents the proportion voting against; $(p - n)$ represents the proportion abstaining.

For intuitionistic fuzzy sets (cf. Section 2) we have

- the proportion of a sample population voting in favour of a proposition is equal to μ (membership function),
- the proportion voting against is equal to ν (non-membership function),
- π represents the proportion abstaining.

Table 1:

	Baldwin's voting model	IFS voting model
voting in favour	n	μ
voting against	$1-p$	ν
abstaining	$p-n$	π

In Table 1 equality of parameters from Baldwin's voting model and from intuitionistic fuzzy set (IFS) voting model is presented.

So we can represent a Support Pair (n, p) using notation of intuitionistic fuzzy sets in the following way

$$(n,p) = (n, n+p-n) = (\mu, \mu+\pi) \tag{7}$$

i.e.: a Support Pair in Baldwin's voting model can be expressed by using notation of intuitionistic fuzzy sets. But it is necessary to stress that it is not just simple equivalence. In our considerations a simplyfing assumption was done - we put a sign of equalty to probabilities (mass assignment theory) and memberships/non-memberships. This assumption is valid under the condition that each value of membership/non-membership occurs with the same probability for each element x_i. In this paper, for the sake of simplicity we follow this assumption. Hovewer, in general, probabilities for intuitionistic fuzzy sets are calculated as was discussed in (Szmidt [20], Szmidt and Kacprzyk [22], Szmidt and Baldwin [21]).

Let us look at three Support Pairs (7) of special interests (Baldwin and Pilsworth [6])

- (1, 1) which represents total support for the associated statement,
- (0, 0) which represents total support against, and
- (0, 1) which characterizes complete uncertainty in the support.

  Of course the above Support Pairs have exactly the same meaning in intuitionistic fuzzy set models (under the assumption that we consider probabilities for intuitionistic fuzzy memberships/non-memberships as it was explained in (Szmidt [20], Szmidt and Kacprzyk [22], Szmidt and Baldwin [21]):

- (1, 1) means that $\mu = 1$ and $\pi = 0$, i.e. total support,
- (0, 0) means $\mu = 0$ and $\pi = 0$ what involves $\nu = 1$, i.e. total support against,
- (0, 1) means $\mu = 0$ and $\pi = 1$ i.e.: complete uncertainty in the support.

In other words both Support Pairs and intuitionistic fuzzy set models give the same intervals containing the probability of the fact being true, and the difference between the upper and lower values of intervals is a measure of the uncertainty associated with the fact.

The mass assignment structure is best used to represent knowledge that is statistically based such that the values can be measured, even if the measurements themselves are approximate or uncertain (Baldwin [9]).

4 Common geometrical interpretation

Having in mind that the parameters characteristic for intuitionistic fuzzy sets add up to one, i.e.

$$\mu + \nu + \pi = 1 \tag{8}$$

and the same for their counterparts for mass assignment theory (see (7) and Table 1), i.e.

$$n + (1 - p) + (p - n) = 1 \tag{9}$$

and each of the parameters is from interval [0, 1], we can imagine a unit cube (Figure 1) inside which there is ABD triangle where the above equations are fulfilled. In other words, ABD triangle represents a surface where coordinates of any element belonging to an intuitionistic fuzzy set or representing any Support Pair can be represented. Each point belonging to ABD triangle is described via three coordinates: $(\mu, \nu, \pi) = (n, 1-p, p-n)$ – respectively for intuitionistic fuzzy set theory and mass assignment theory. Points A and B represent crisp elements. Point $A(1,0,0)$ - represents elements fully belonging to an intuitionistic fuzzy set as $\mu = 1$, or equivalently, 100% population voting for (as $n = 1$). Point $B(0,1,0)$ represents elements fully not belonging to an intuitionistic fuzzy set as $\nu = 1$ or equivalently, 100% population voting against (as $1 - p = 1$). Point $D(0,0,1)$ represents elements about which we are not able to say if they belong or not belong to an intuitionistic fuzzy set (intuitionistic fuzzy index $\pi = 1$) or equivalently, the proportion abstaining $p - n = 1$. Segment AB (where $\pi = 0$) represents elements belonging to classical fuzzy sets ($\mu + \nu = 1$), or the situation when $p - n = 0$ what means in terms of mass assignments that there is not uncertainty in the voting model.

The geometrical representation made it possible to introduce proper formulas for calculating distances between intuitionistic fuzzy sets, and between support pairs. We remind here only the formulas needed in out further considerations. For more details we refer readers to (Szmidt and Baldwin [21], Szmidt and Kacprzyk [23]).

- the normalized Hamming distance between any two intuitionistic fuzzy sets A and B containing k elements

$$l_{IFS}(A,B) = \frac{1}{2n}\sum_{i=1}^{k}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \tag{10}$$

  and its counterpart, i.e.

- the normalized Hamming distance for two sets of facts A and B reprezented via support pairs

$$\begin{aligned} l_{MASS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{k}(|n_A(x_i) - n_B(x_i)| + |(1 - p_A(x_i)) - (1 - p_B(x_i))| + \\ &+ |(p_A(x_i) - n_A(x_i)) - (p_B(x_i) - n_B(x_i))|) \end{aligned} \tag{11}$$

For (10) and (11), there holds, respectively: $0 \leq l_{IFS}(A,B) \leq 1$ and $0 \leq l_{MASS}(A,B) \leq 1$.

In our further considerations on entropy, besides the distances, the concept of cardinality will be also useful. In (Szmidt [20], Szmidt and Kacprzyk [24]) a definition of cardinalities for intuitionistic fuzzy sets is given. Having in mind that definition and the above considerations concerning parallels of intuitionistic fuzzy set theory and mass assignment theory, we give one definition expressed in terms of both theories.

Definition 2 *Let A be an intuitionistic fuzzy set with k elements (intuitionistic fuzzy set theory) or equivalently, a mass assignment with k suport pairs. First, we define the following two cardinalities:*

- *the least ("sure") cardinality of A is equal to the so-called sigma-count (cf. Zadeh [26]), and is called here the* $\min \sum Count$*:*

$$\min Card(A) = \min \sum Count(A) = \sum_{i=1}^{k} \mu_A(x_i) = \sum_{i=1}^{k} n(x_i) \tag{12}$$

- *the biggest cardinality of A, which is possible due to π_A, is called the* $\max \sum Count$*, and is equal to*

$$\begin{aligned} \max Card(A) &= \max \sum Count(A) = \\ &= \sum_{i=1}^{k} (\mu_A(x_i) + \pi_A(x_i)) = \sum_{i=1}^{k} p_A(x_i) \end{aligned} \tag{13}$$

and, clearly, for A^c (where A^c is a complement of A) we have

$$\min Card(A^c) = \min \sum Count(A^c) = \sum_{i=1}^{k} \nu_A(x_i) = \sum_{i=1}^{k} (1 - p_A(x_i)) \tag{14}$$

$$\begin{aligned} \max Card(A^c) &= \max \sum Count(A^c) = \\ &= \sum_{i=1}^{k} (\nu_A(x_i) + \pi_A(x_i)) = \sum_{i=1}^{k} (1 - n_A(x_i)) \end{aligned} \tag{15}$$

Then the cardinality of an intuitionistic fuzzy set or a respective mass assignment is defined as a number from the interval:

$$CardA \in [\min \sum Count(A), \max \sum Count(A)] \tag{16}$$

5 Entropy

Entropy we examine here is a non-probabilistic-type entropy measure. It is entropy in the sense of De Luca and Termini [15] axioms which are intuitive and have been widely employed in the fuzzy literature.

De Luca and Termini (1972) first axiomatized non-probabilistic entropy. The axioms were formulated in the following way. Let E be a set-to-point mapping $E : F(2^x) \longrightarrow [0, 1]$.

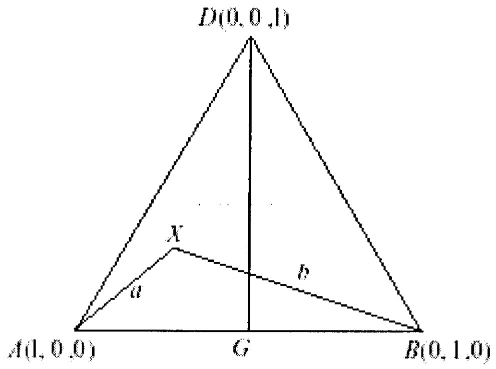

Figure 2: The triangle ABD (cf. Fig. 1) explaining a ratio-based measure of fuzziness

Hence E is a fuzzy set defined on fuzzy sets. E is an entropy measure if it satisfies the four De Luca and Termini axioms:

$$E(A) = 0 \text{ iff } A \in 2^x \text{ (}A\text{ non-fuzzy)} \tag{17}$$

$$E(A) = 1 \text{ iff } \mu_A(x_i) = 0.5 \text{ for all } i \tag{18}$$

$$E(A) \leq E(B) \text{ if } A \text{ is less fuzzy than } B \tag{19}$$

i.e., if

$$\mu_A(x) \leq \mu_B(x) \text{ when } \mu_B(x) \leq 0.5$$

and

$$\mu_A(x) \geq \mu_B(x) \text{ when } \mu_B(x) \geq 0.5$$

$$E(A) = E(A^c) \tag{20}$$

Since the De Luca and Termini axioms (17)–(20) were formulated for fuzzy sets (given only by their membership functions, and describing the situation depicted by the segment AB in Figure 2), they were reformulated for the intuitionistic fuzzy sets as follows (Szmidt [20], Szmidt and Kacprzyk [24]):

$$E(A) = 0 \text{ iff } A \in 2^x \text{ (}A\text{ non-fuzzy)} \tag{21}$$

$$E(A) = 1 \text{ iff } \mu_A(x_i) = \nu_A(x_i) \text{ for all } i \tag{22}$$

$$E(A) \leq E(B) \text{ if } A \text{ is less fuzzy than } B \tag{23}$$

i.e., if

$$\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \text{ for } \mu_B(x) \leq \nu_B(x)$$

or

$$\mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \text{ for } \mu_B(x) \geq \nu_B(x)$$

$$E(A) = E(A^c) \tag{24}$$

For mass assignment theory axioms (21) and (24) are identical, the counterparts of axioms (22) and (23) are as follows:

$$E(A) = 1 \text{ iff } n_A(x_i) = 1 - p_A(x_i) \text{ for all } i \tag{25}$$

$$E(A) \leq E(B) \text{ if } A \text{ is less fuzzy than } B \tag{26}$$

i.e., if

$$n_A(x) \leq n_B(x) \text{ and } 1 - p_A(x) \geq 1 - p_B(x) \text{ for } n_B(x) \leq 1 - p_B(x)$$

or

$$n_A(x) \geq n_B(x) \text{ and } 1 - p_A(x) \leq 1 - p_B(x) \text{ for } n_B(x) \geq 1 - p_B(x)$$

Differences between (18)-(19), and both (22)-(23) and (25)-(26) occur as we demand that the counterparts of the axioms (18)-(19) for fuzzy sets are fulfilled (for intuitionistic fuzzy sets and mass assignments) not only for point G (Figure 2), but for the whole segment DG.

The fuzziness of a fuzzy set, or its entropy, answers the question: how fuzzy is a fuzzy set. The same question may be posed in a case of an intuitionistic fuzzy set or an mass assignment. We will discuss the term fuzziness having in mind our geometric interpretation of intuitionistic fuzzy sets and mass assignments (Figure 1), concentrating mainly on the triangle ABD - Figure 2.

As was discussed earlier, non-fuzzy set (a crisp set) corresponds to the point A [point A represents the elements fully belonging to a set as $(\mu_A, \nu_A, \pi_A) = (1,0,0)$ or $(n_A, 1 - p_A, p_A - n_A) = (1,0,0)$] and the point B [point B represents the elements which fully does not belong to a set as $(\mu_B, \nu_B, \pi_B) = (0,1,0)$ or $(n_A, 1 - p_A, p_A - n_A) = (0,1,0)$]. Points A and B representing a crisp set have the degree of fuzziness equal to 0.

A fuzzy set corresponds to the segment AB. When we move from point A towards point B (along the segment AB, we go through points for which the membership function (or equivalently - population voting for) decreases (from 1 at point A to 0 at point B), the non-membership function (or equivalenty - population voting against) increases (from 0 at point A to 1 at point B). For the midpoint G (Figure 2) the values of both the membership and non-membership functions are equal 0.5. So, the midpoint G has the degree of fuzziness equal 100% (we do not know if the elements represented by point G belong or if they do not belong to our set). On the segment AG the degree of fuzziness grows (from 0% at A to 100% at G). The same situation occurs on the segment BG. The degree of fuzziness is equal 0% at B, grows towards G (here it is equal to 100%).

An intuitionistic fuzzy set or a mass assignment is represented by the triangle ABD and its interior. All points which are above the segment AB represent elements with a hesition margin margin (or equivalently - the proportion abstaining) greater than 0. The most undefinded is point D. As the hesitation margin for D is equal 1, we can not tell if the elements represented by this point belong or do not belong to the set. The distance from D to A (full belonging) is equal to the distance to B (full non-belonging). So, the degree of fuzziness for D is equal 100%. But the same situation occurs for all elements x_i represented by the segment DG. For DG we have $\mu_{DG}(x_i) = \nu_{DG}(x_i)$, $\pi_{DG}(x_i) \geq 0$

(equality only for point G), and certainly $\mu_{DG}(x_i) + \nu_{DG}(x_i) + \pi_{DG}(x_i) = 1$. In terms of the support pairs for DG we have $n_{DG}(x_i) = 1 - p_{DG}(x_i)$, $p_{DG}(x_i) - n_{DG}(x_i) \geq 0$ (equality only for point G). For every $x_i \in DG$ we have: $distance(A, x_i)$=$distance(B, x_i)$.

This geometric representation motivates a ratio-based measure of fuzziness (a similar approach was proposed in (Kosko [19]) to calculate the entropy of fuzzy sets).

A ratio-based measure of fuzziness i.e., entropy of an intuitionistic fuzzy element or a support pair represented by point X (belonging to triangle ABD) is given in the following way:

Definition 3

$$E(X) = \frac{a}{b} \tag{27}$$

where a is a distance(X, X_{near}) *from X to the nearer point X_{near} among A and B, and b is the* distance(X, X_{far}) *from X to the farer point X_{far} among A and B.*

The geometric interpretation confirms that (27) satisfies axioms (21)–(24) and (25)–(26).

An interpretation of entropy (27) can be as follow. This entropy measures the whole missing information which may be necessary to have no doubts when classifying the point X (representing an element from an intuitionistic fuzzy set or a support pair) to the area of consideration, i.e. to say say that an element/support pair represented by X fully belongs (point A) or fully does not belong (point B) to our set.

Formula (27) describes the degree of fuzziness for a single element belonging to an intuitionistic fuzzy set or for a single support pair. For k elements/support pairs we have

$$E = \frac{1}{k} \sum_{i=1}^{k} E(X_i) \tag{28}$$

Fortunately enough, while applying the Hamming distances in (27), the entropy of intuitionistic fuzzy sets is the ratio of the biggest cardinalities ($\max \sum Counts$) involving only X and X^c. The following theorem was proven (Szmidt [20], Szmidt and Kacprzyk [24]).

Theorem 1 A generalized entropy measure of an intuitionstic fuzzy set of k elements is

$$E = \frac{1}{k} \sum_{i=1}^{k} \left(\frac{\max Count(X_i \cap X_i^c)}{\max Count(X_i \cup X_i^c)} \right) \tag{29}$$

where (Atanassov [1], [2]):

$$X_i \cap X_i^c = \left\langle \min(\mu_{X_i}, \mu_{X_i^c}), \quad \max(\nu_{X_i}, \nu_{X_i^c}) \right\rangle$$

$$X_i \cup X_i^c = \left\langle \max(\mu_{X_i}, \mu_{X_i^c}), \quad \min(\nu_{X_i}, \nu_{X_i^c}) \right\rangle$$

The same Theorem 1, i.e. (29) is valid for an mass assignment with k support pairs where

$$X_i \cap X_i^c = \left\langle \min(n_{X_i}, n_{X_i^c}), \quad \max(1 - p_{X_i}, 1 - p_{X_i^c}) \right\rangle$$

$$X_i \cup X_i^c = \left\langle \max(n_{X_i}, n_{X_i^c}), \quad \min(1 - p_{X_i}, 1 - p_{X_i^c}) \right\rangle$$

where a complement X_i^c represents a support pair $(1 - p_i, n_i)$, i.e. the coordinates of X_i^c are the following:

$$X_i^c = (1 - p_i, n_i, p_i - n_i)$$

Example 1 Let us calculate the entropy for an element/suport pair represented by point X_1. The coordinates of X_1 expressed in terms of intuitionistic fuzzy sets are $X_1 = (\mu, \nu, \pi)$, and in terms of mass assignment are equal to $(n, 1 - p, p - n)$. Let the coordinates are the following

$$X_1 = (\frac{3}{4}, \frac{1}{6}, \frac{1}{12}) \tag{30}$$

Thus from (10), (11)

$$d_{IFS}(A, X_1) = d_{MASS}(A, X_1) \left|1 - \frac{3}{4}\right| + \left|0 - \frac{1}{6}\right| + \left|0 - \frac{1}{12}\right| = \frac{1}{2}$$

$$d_{IFS}(B, X_1) = d_{MASS}(B, X_1) \left|0 - \frac{3}{4}\right| + \left|1 - \frac{1}{6}\right| + \left|0 - \frac{1}{12}\right| = \frac{5}{3}$$

As $d_{IFS}(A, X_1) = d_{MASS}(A, X_1)$, we will denote the both distances as $d(A, X_1)$.

From (27)

$$E(X_1) = \frac{d(A, X_1)}{d(B, X_1)} = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} \tag{31}$$

We can obtain the same result using formula (29) and having in mind that

$$X_1^c = (\frac{1}{6}, \frac{3}{4}, \frac{1}{12})$$

and

$$X_1 \cap X_1^c = (\frac{1}{6}, \frac{3}{4}, \frac{1}{12}) = X_1^c$$

$$\max Count(X_1 \cap X_1^c) = \frac{1}{6} + \frac{1}{12} = \frac{3}{12}$$

$$X_1 \cup X_1^c = (\frac{3}{4}, \frac{1}{6}, \frac{1}{12}) = X_1$$

$$\max Count(X_1 \cup X_1^c) = \frac{3}{4} + \frac{1}{12} = \frac{10}{12}$$

so that

$$E(X_1) = \frac{\max Count(X_1 \cap X_1^c)}{\max Count(X_1 \cup X_1^c)} = \frac{3}{10} \quad (32)$$

i.e. the same value as (31).

Let us consider another element represented by X_2 with the coordinates

$$X_2 = (\frac{1}{2}, 0, \frac{1}{2}) \quad (33)$$

From (10), (11), (27) we have

$$E(X_2) = \frac{d(A, X_2)}{d(B, X_2)} = \frac{\left|1 - \frac{1}{2}\right| + |0 - 0| + \left|0 - \frac{1}{2}\right|}{\left|0 - \frac{1}{2}\right| + |1 - 0| + \left|0 - \frac{1}{2}\right|} = \frac{1}{2} \quad (34)$$

or having in mind that $X_2^c = (0, \frac{1}{2}, \frac{1}{2})$, we obtain

$$X_2 \cap X_2^c = (0, \frac{1}{2}, \frac{1}{2}) = X_2^c$$

$$X_2 \cup X_2^c = (\frac{1}{2}, 0, \frac{1}{2}) = X_2$$

and

$$E(X_2) = \frac{\max Count(X_2 \cap X_2^c)}{\max Count(X_2 \cup X_2^c)} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \quad (35)$$

i.e. the same value as (34).

For another element represented by X_3 with the coordinates $X_3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, we obtain due to (27)

$$E(X_3) = \frac{d(A, X_3)}{d(B, X_3)} = \frac{\left|1 - \frac{1}{2}\right| + \left|0 - \frac{1}{4}\right| + \left|0 - \frac{1}{4}\right|}{\left|0 - \frac{1}{2}\right| + \left|1 - \frac{1}{4}\right| + \left|0 - \frac{1}{4}\right|} = \frac{1}{\frac{3}{2}} = \frac{2}{3} \quad (36)$$

or, taking into account that $X_3^c = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$,

$$X_3 \cap X_3^c = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}) = X_3^c$$

$$X_3 \cup X_3^c = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) = X_3$$

we obtain from (29)

$$E(X_3) = \frac{\max Count(X_3 \cap X_3^c)}{\max Count(X_3 \cup X_3^c)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} \quad (37)$$

i.e. the same value as (36).

It is worth noticing that despite of the fact that the lack of knowledge concerning X_2, i.e. $\pi_{X_2} = p_{X_2} - n_{X_2} = 0.5$ is greater than that for X_3 (i.e., $\pi_{X_2} = p_{X_2} - n_{X_2} = 0.25$) , the entropy of X_2 is less than the entropy of X_3. It can be explained simply via axiom (23). This case is also a good illustration of the nature of entropy. For point X_2, in the best case which can be achieved is a crisp point, i.e.

$$(\mu_{X_2} + \pi_{X_2}, \nu_{F_2}) = (p_{X_2}, 1 - p_{X_2}) = (1, 0) \tag{38}$$

while for X_3, the best what can be attained is

$$(\mu_{X_3} + \pi_{X_3}, \nu_{X_3}) = (p_{X_3}, 1 - p_{X_3}) = (\frac{3}{4}, \frac{1}{4}) \tag{39}$$

Formula (38) means that in the best case X_2 can attain a crisp point A (Figure 2), whereas X_3 (39) will never do this. So, a (degree of) fuzziness is bigger for X_3 than for X_2.

From (28) we can calculate entropy of an intuitionistic fuzzy set $Z \subseteq X = \{X_1, X_2, X_3\}$. Taking into account (31), (35), and (36) we have

$$E(Z) = \frac{1}{3}\{E(X_1) + E(X_2) + E(X_3)\} = \frac{1}{3}(\frac{3}{10} + \frac{1}{2} + \frac{2}{3}) = 0.49 \tag{40}$$

□

6 Concluding remarks

We reminded the parallels of intuitionistic fuzzy sets and mass assignment theory. Next, we formulated the counterparts of De Luca and Termini axioms for both theories. Starting from the same geometrical interpretation for both theories, we introduced a common measure of non-probabilistic entropy. Two ways of calculating the measure of entropy were proposed.

References

[1] Atanassov K. (1986), Intuitionistic Fuzzy Sets. *Fuzzy Sets and Systems*, 20, 87–96.

[2] Atanassov K. (1999), Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[3] Baldwin. J.F. (1991), Combining Evidences for Evidential Reasoning. International Journal of Intelligent Systems, 6, 569–616.

[4] Baldwin J.F. (1992), Fuzzy and Probabilistic Uncertainties in Encyclopaedia of AI (ed. S.A. Shapiro), John Wiley, 528–537.

[5] Baldwin J.F. (1994), Mass assignments and fuzzy sets for fuzzy databases. In. Advances in the Dempster-Shafer theory of evidence. Ed. R. Yager at al. John Wiley, 577–594.

[6] Baldwin J.F., Pilsworth B.W. (1990), Semantic Unification with Fuzzy Concepts in Fril. IPMU'90, Paris.

[7] Baldwin J.F., T.P. Martin, B.W. Pilsworth (1995) FRIL – Fuzzy and Evidential Reasoning in Artificial Intelligence. John Wiley.

[8] Baldwin J.F., Lawry J., Martin T.P.(1995a), A Mass Assignment Theory of the Probability of Fuzzy Events. ITRC Report 229, University of Bristol, UK.

[9] Baldwin J.F., Coyne M.R., Martin T.P.(1995b), Intelligent Reasoning Using General Knowledge to Update Specific Information: A Database Approach. Journal of Intelligent Information Systems, 4, 281–304.

[10] Baldwin J.F., T.P. Martin T.P. (1995c), Extracting Knowledge from Incomplete Databases using the Fril Data Browser. EUFIT'95, Aachen, 111–115.

[11] Baldwin J.F., T.P. Martin (1996), FRIL as an Implementation Language for Fuzzy Information Systems. IPMU'96, Granada, 289–294.

[12] Ban A. (2000) Measurable entropy on intuitionistic fuzzy dynamical system. *Notes on IFS*, 6, No.4, 35–47.

[13] Burillo P. and Bustince H. (1996) - Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. Fuzzy Sets and Systems, 78, 305-316.

[14] Cornelis Ch. and Kerre E. (2003). Inclusion measures in intuitionistic fuzzy set theory. Proc. ECSQARU-2003. Lecture Notes in AI, 2711, 345–356.

[15] De Luca, A. and Termini, S. (1972). A definition of a non-probabilistic entropy in the setting of fuzzy sets theory. *Inform. And Control*, 20, 301–312.

[16] Jaynes E.T. (1979) - Where do we stand on maximum entropy? In: The Maximum Entropy Formalism, ed. By Levine and Tribus, MIT Press, Cambridge Mass.

[17] Kaufmann A. (1975) - Introduction to the Theory of Fuzzy Subsets - Vol.1: Fundamental Theoretical Elements. Academic Press, New York.

[18] Klir G.J. and Wierman M.J. (1997) - Uncertainty - Based Information Elements of Generalized Information Theory. Lecture Notes in Fuzzy Mathematics and Computer Science, 3, Creighton University, Omaha, Nebraska 68178 USA.

[19] Kosko B. (1997) - Fuzzy Engineering. Prentice-Hall.

[20] Szmidt E. (2000) Applications of Intuitionistic Fuzzy Sets in Decision Making. (D.Sc. dissertation) Techn. Univ., Sofia, 2000.

[21] Szmidt E. and Baldwin J. (2003) New Similarity Measure for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory. *Notes on IFS*, 9, No.3, 60–76.

[22] Szmidt E. and Kacprzyk J. (1999). Probability of Intuitionistic Fuzzy Events and their Applications in Decision Making. Proc. of EUSFLAT-ESTYLF Conf. 1999. Palma de Mallorca, 457–460.

[23] Szmidt E. and Kacprzyk J. (2000) Distances between intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 114, No.3, 505 – 518.

[24] Szmidt E., Kacprzyk J. (2001) Entropy for intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 118, No. 3, 467–477.

[25] Yager R.R. (1979) - On the measure of fuziness and negation. Part I: Membership in the unit interval. Internat. J. Gen. Systems, Vol. 5, 189-200.

[26] L.A. Zadeh (1965) Fuzzy sets. *Information and Control*, 8, 338–353.