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INTUITIONISTIC FUZZY GRAPHS FROM α-, β- AND (α, β)- LEVELS

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Some months ago, following [1], we introduced in [2] the concept "an Intuitionistic Fuzzy Graph" (IFGs), which is based on the theory of the Intuitionistic Fuzzy Sets (IFSs) [3]. In [4] it was shown that the IFGs can be described by means of the Index Matrix (IMs; [5]). Here we shall make the next step on the research of the IFGs.

Let the oriented graph $G = (V, A)$ be given (see, e.g. [6]), where V is a set of vertices and A is a set of arcs. Every graph arc connects two graph vertices. Therefore, $A \subset V \times V$ and hence A can be described as a (1,0)-matrix. Such an interpretation of graphs is well known - it is used, for example, in [6]. There, the rows and the columns of the matrices used are indexed by the identifiers of the vertices, or of the vertices and of the arcs. The graph G has the following matrix:

$$A = \begin{array}{c|cccc} & v_1 & v_2 & \dots & v_n \\ \hline v_1 & a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ v_2 & a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \cdot & \cdot & \cdot & \\ v_n & a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{array}$$

where $a_{i,j} \in \{0, 1\}$ $(1 \le i, j \le n)$, $V = \{v_1, v_2, \dots, v_n\}$, in brief we can write: $G = [V, V, \{a_{i,j}\}]$.

Following [7], we shall make the next step, replacing this matrix with an IM. Practically, the semantic form of the matrix may be the same, but now we can use the IM-apparatus described above. On the other hand, we can modify the above matrix form to the following $G = [V_I \cup \bar{V}, \bar{V} \cup V_O, \{a_{i,j}\}]$, where V_I, V_O and $\bar{V}$ are respectively the sets of the graph input, ouitput and internal vertices. From each vertex of the first type there starts at least one arc, but none enters; in each vertex of the second type at least one arc enters, but none starts; every vertex of the third type has at least one arc ending in it and at least one arc going out from it. In this case for the vertives $v_p \in V_I$ and $v_q \in V_O$, if there exists an arc, $a_{p,q} = 1$ and if there is no arc $a_{p,q} = 0$, but always $a_{q,p} = 0$.

Obviously, first, the graph matrix (in the sense of IM) now will be of a smaller dimension than the ordinary graph matrix, and second, it can be nonsquare, unlike the ordinary graph matrices.

As in the ordinary case, if $a_{p,p} = 1$ for the vertex $v_p \in \bar{V}$, then this vertex has a loop.

Following [8], we shall define five cases of Cartesian products of two IFSs. Let E_1 and E_2 be two universes and let

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle / x \in E_1\},$$

$$B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle / y \in E_2\},$$

be two IFSs over them. We shall define:

$A\ o_1\ B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \gamma_A(x).\gamma_B(y)\rangle / x\in E_1\ \&\ y\in E_2\}$,

$A\ o_2\ B = \{\langle\langle x, y\rangle, \mu_A(x)+\mu_B(y)-\mu_A(x).\mu_B(y), \gamma_A(x).\gamma_B(y)\rangle / x\in E_1 \& y\in E_2\}$,

$A\ o_3\ B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \gamma_A(x)+\gamma_B(y)-\gamma_A(x).\gamma_B(y)\rangle / x\in E_1 \& y\in E_2\}$,

$A\ o_4\ B = \{\langle\langle x, y\rangle, \min(\mu_A(x),\mu_B(y)), \max(\gamma_A(x),\gamma_B(y))\rangle / x\in E_1 \& y\in E_2\}$,

$A\ o_5\ B = \{\langle\langle x, y\rangle, \max(\mu_A(x),\mu_B(y)), \min(\gamma_A(x).\gamma_B(y))\rangle / x\in E_1 \& y\in E_2\}$.

Following [4] we shall describe the IM-representation of a given IFG.

Let E_1 and E_2 be two sets, let everywhere below $x \in E_1$ and $y \in E_2$ and let operation $o \in \{o_1, \ldots, o_5\}$. Therefore $\langle x, y\rangle \in E_1\ o\ E_2$. The set $G^* = \{\langle\langle x, y\rangle, \mu_G(x, y), \gamma_G(x, y)\rangle / \langle x, y\rangle \in E_1\ o\ E_2\}$ is called an o-IFG (shortly, IFG), if the functions $\mu_G: E_1\ o\ E_2 \to [0, 1]$ and $\gamma_G: E_1 \times E_2 \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $\langle x, y\rangle \in E_1\ o\ E_2$ to the set G, which is a subset of $E_1\ o\ E_2$, respectively, and for every $\langle x, y\rangle \in E_1\ o\ E_2$:

$$0 \le \mu_G(x, y) + \gamma_G(x, y) \le 1$$

(see [2]). For simplicity below, we shall write G instead of G^*.

We must note immediately, that the μ- and γ-values of every element $\langle x, y\rangle \in E_1\ o\ E_2$ are given (abstractly defined or measured by some means) directly. In the particular case, the μ- and γ-components of a given IFG can be obtained by some ways using the above Cartesian product definitions.

On the basis of the above definitions, we can show the form of the IM-representation of a given IFG. It is $G = [V, V, A]$, where $V = \{v_1, v_2, \ldots, v_n\}$ and

$$A = \begin{array}{c|cccc} & v_1 & v_2 & \ldots & v_n \\ \hline v_1 & \langle\mu(v_1,v_1), \gamma(v_1,v_1)\rangle & \langle\mu(v_1,v_2), \gamma(v_1,v_2)\rangle & \ldots & \langle\mu(v_1,v_n), \gamma(v_1,v_n)\rangle \\ v_2 & \langle\mu(v_2,v_1), \gamma(v_2,v_1)\rangle & \langle\mu(v_2,v_2), \gamma(v_2,v_2)\rangle & \ldots & \langle\mu(v_2,v_n), \gamma(v_2,v_n)\rangle \\ \vdots & . & . & . & \\ v_n & \langle\mu(v_n,v_1), \gamma(v_n,v_1)\rangle & \langle\mu(v_n,v_2), \gamma(v_n,v_2)\rangle & \ldots & \langle\mu(v_n,v_n), \gamma(v_n,v_n)\rangle \end{array}$$

where $\langle\mu(v_i, v_j), \gamma(v_i, v_j)\rangle \in [0, 1] \times [0, 1]$, i.e., the arc between vertices v_i and v_j is indexed by $\langle\mu(v_i,v_j), \gamma(v_i,v_j)\rangle$.

Obviously, this IFG representation can be transformed to the form $G = [V_I \cup \bar{V}, \bar{V} \cup V_O, A]$, too.

Following [9], we shall define for the given $\alpha, \beta \in [0, 1]$ and for the given IFG $G = [V, V, A]$:

- the IFG $N_\alpha(G) = [V', V'', A']$ for which the arc between the vertices v_i and v_j is indexed by $\langle a_{i,j}, b_{i,j}\rangle$, where:

$$a_{i,j} = \begin{cases} \mu(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \\ 0 & \text{, otherwise} \end{cases}$$

$$b_{i,j} = \begin{cases} \gamma(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \\ 1 & \text{, otherwise} \end{cases} ;$$

- the IFG $N^\beta(G) = [V', V'', A']$ for which the arc between the vertices v_i and v_j is indexed by $\langle a_{i,j}, b_{i,j}\rangle$, where:

$$a_{i,j} = \begin{cases} \mu(v_i, v_j), & \text{if } \gamma(v_i, v_j) \leq \beta \\ 0 & \text{, otherwise} \end{cases}$$

$$b_{i,j} = \begin{cases} 1 & \text{, otherwise} \\ \gamma(v_i, v_j), & \text{if } \gamma(v_i, v_j) \leq \beta \end{cases} ;$$

- the IFG $N_{\alpha,\beta}(G) = [V', V', A']$ for which the arc between the vertices v_i and v_j is indexed by $\langle a_{i,j}, b_{i,j}\rangle$, where:

$$a_{i,j} = \begin{cases} \mu(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \text{ and } \gamma(v_i, v_j) \leq \beta \\ 0 & \text{, otherwise} \end{cases}$$

$$b_{i,j} = \begin{cases} \gamma(v_i, v_j), & \text{if } \gamma(v_i, v_j) \leq \beta \text{ and } \gamma(v_i, v_j) \leq \beta \\ 1 & \text{, otherwise} \end{cases}$$

and $v_i \in V'$, $v_j \in V''$, iff $v_i, v_j \in V$ and if in the first and in the third case $a_{i,j} \geq \alpha$ and if in the second and in the third case $b_{i,j} \leq \beta$.

For the IMs $A = [K, L, \{\langle a'_{k_i,l_j}, a''_{k_i,l_j}\rangle\}]$, $B = [P, Q, \{\langle b'_{p_r,q_s}, b''_{p_r,q_s}\rangle\}]$, where

$$K \cap L \cap P = K \cap L \cap Q = K \cap P \cap Q = L \cap P \cap Q = \emptyset,$$

some operations and relations can be defined (see [5]). Two of these operations and an relation (in the intuityionistic fuzzy form) are the following:

$A \subset B$ iff $(K \subset P)$ & $(L \subset Q)$ & $(\forall k_i \in K)(\forall l_j \in L)(a'_{k_i,l_j} \leq a''_{k_i,l_j}$ & $b'_{k_i,l_j} \geq b''_{k_i,l_j})$

$A \cap B = [K \cap P, L \cap Q, \{\langle c'_{t_u,v_w}, c''_{t_u,v_w}\rangle\}]$, where

$$\langle c'_{t_u,v_w}, c''_{t_u,v_w}\rangle = \langle \min(a'_{k_i,l_j}, b'_{p_r,q_s}), \max(a''_{k_i,l_j}, b''_{p_r,q_s})\rangle$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ and

$A \cup B = [K \cup P, L \cup Q, \{\langle c'_{t_u,v_w}, c''_{t_u,v_w}\rangle\}]$, where

$$\langle c'_{t_u, v_w}, c''_{t_u, v_w} \rangle = \begin{cases} \langle a'_{k_i, l_j}, a''_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \\ & \in L - Q \text{ or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L \\ \langle b'_{p_r, q_s}, b''_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \\ & \in Q - L \text{ or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q \\ \langle \max(a'_{k_i, l_j}, b'_{p_r, q_s}), \min(a''_{k_i, l_j}, b''_{p_r, q_s}) \rangle & \\ \text{if } t_u = k_i = p_r \in K \cap P \text{ and } v_w = l_j = q_s \in L \cap Q & \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

In [2] are introduced the following two IFS-operators:

$\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x) \rangle / x \in E\}$,

$\Diamond A = \{\langle x, 1-\gamma_A(x), \gamma_A(x) \rangle / x \in E\}$,

and here we shall introduce two other IFS-operators (see [3]):

$C(A) = \{\langle x, \max_{x \in E} \mu_A(x), \min_{x \in E} \gamma_A(x) \rangle / x \in E\}$,

$I(A) = \{\langle x, \min_{x \in E} \mu_A(x), \max_{x \in E} \gamma_A(x) \rangle / x \in E\}$,

which can be transformed for the IFG-case directly.

The following theorems hold.

THEOREM 1: For every two IFGs A and B and for every two $\alpha, \beta \in [0, 1]$:

(a) $N_{\alpha,\beta}(A) = N_\alpha(A) \cap N^\beta(A)$,

(b) $N_{\alpha,\beta}(A \cap B) = N_{\alpha,\beta}(A) \cap N_{\alpha,\beta}(B)$,

(c) $N_{\alpha,\beta}(A \cup B) = N_{\alpha,\beta}(A) \cup N_{\alpha,\beta}(B)$.

THEOREM 2: For every IFG G and for every two $\alpha, \beta \in [0, 1]$:

(a) $\Box N_{\alpha,\beta}(G) \supset N_{\alpha,\beta}(\Box G)$,

(b) $N_{\alpha,\beta}(\Diamond G) \subset \Diamond N_{\alpha,\beta}(G)$,

(c) $C(N_{\alpha,\beta}(G)) \subset N_{\alpha,\beta}(C(G))$,

(d) $I(N_{\alpha,\beta}(G)) \supset N_{\alpha,\beta}(I(G))$.

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