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The Inclusion-Exclusion Principle on some algebraic structures

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Abstract

This paper is a remark to the generalization of Grzegorzewski theorem [2]. He has proved the method of Inclusion- Exclusion Principle for a special case of IF-events. We apply it on different structures: commutative and associative algebraic systems and lattices.

Definition

Have an algebraic system with 2 binary operations: $(M, +, .)$, and a mapping $m : M \to < 0, 1 >$ satisfying the following conditions:

1. The operations $+, .$ are commutative and associative
2. The distributive law holds for every $a, b, c \in M$
$$(a + b).c = a.c + b.c$$
3. for every $c \in M$
$$c.c = c$$
4. for every $a, b \in M$
$$m(a + b) = m(a) + m(b) - m(a.b)$$

Example 1 *Let $a, b, c \in M$. Then, $m\left((a + b).c\right) = m\left(a.c + b.c\right)$, hence*

$$m(a + b + c) = m(a + b) + m(c) - m((a + b).c) =$$
$$= m(a) + m(b) + m(c) - m(a.b) - m(a.c) - m(b.c) + m(a.c.b.c) =$$
$$= m(a) + m(b) + m(c) - m(a.b) - m(a.c) - m(b.c) + m(a.b.c).$$

Proposition Let $a_1, ..., a_n \in M$. Then

$$m(a_1 + ... + a_n) = \sum_{i=1}^{n} m(a_i) - \sum_{i<j}^{n} m(a_i.a_j) + \sum_{i<j<k}^{n} m(a_i.a_j.a_k) - ... + (-1)^{n+1} m\left(\prod_{i=1}^{n} a_i\right)$$

Proof. We shall use induction. The assertion holds for $n = 2$.
Let the assertion holds for $n \in N$:

$$m(a_1 + ... + a_n) = \sum_{i=1}^{n} m(a_i) - \sum_{i<j}^{n} m(a_i.a_j) + \sum_{i<j<k}^{n} m(a_i.a_j.a_k) - ... + (-1)^{n+1} m\left(\prod_{i=1}^{n} a_i\right)$$

Then

$$m(a_1 + ... + a_{n+1}) = \sum_{i=1}^{n+1} m(a_i) - \sum_{i<j}^{n+1} m(a_i.a_j) + \sum_{i<j<k}^{n+1} m(a_i.a_j.a_k) - ... + (-1)^{n+2} m\left(\prod_{i=1}^{n+1} a_i\right)$$

$$m(a_1 + ... + a_n + a_{n+1}) = m(a_1 + ... + a_n) + m(a_{n+1}) - m((a_1 + ... + a_n).a_{n+1}) =$$

$$= m(a_1 + ... + a_n) + m(a_{n+1}) - m(a_1.a_{n+1} + ... + a_n.a_{n+1}) =$$

$$= \sum_{i=1}^{n} m(a_i) - \sum_{i<j}^{n} m(a_i.a_j) + \sum_{i<j<k}^{n} m(a_i.a_j.a_k) - ... + (-1)^{n+1} m\left(\prod_{i=1}^{n} a_i\right) + m(a_{n+1}) -$$

$$-\left(\sum_{i=1}^{n} m(a_i.a_{n+1}) - \sum_{i<j}^{n} m(a_i.a_j.a_{n+1}) + ... + (-1)^{n} \sum_{i=1}^{n} m\left(\prod_{k \neq i}^{n} a_k\right).a_{n+1} +\right.$$

$$\left. +(-1)^{n+1} m\left(\left(\prod_{i=1}^{n} a_i\right).a_{n+1}\right)\right) =$$

$$= \sum_{i=1}^{n+1} m(a_i) - \sum_{i<j}^{n+1} m(a_i.a_j) + \sum_{i<j<k}^{n+1} m(a_i.a_j.a_k) +$$

$$... + (-1)^{n+1} \sum_{i=1}^{n+1} m\left(\prod_{k=1, k \neq i}^{n+1} a_k\right) + (-1)^{n+2} m\left(\prod_{i=1}^{n+1} a_i\right).$$

Example 2 *Let $(L, \vee, \wedge)$ be a distributive lattice, and $m : L \rightarrow < 0, 1 >$ be such that*

$$m(a \vee b) + m(a \wedge b) = m(a) + m(b)$$

for any $a, b \in L$. Put

$$+ = \vee, . = \wedge.$$

Then all assumptions of Proposition are satisfied.

Example 3 *Let F be the family of all IF subsets of Ω, i.e. $A = (\mu_A, \nu_A)$,*

$$\mu_A, \nu_A : \Omega \to < 0, 1 >, \mu_A + \nu_A \leq 1.$$

Define

$$A \vee B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$$

$$A \wedge B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

Let $m : F \to < 0, 1 >$ be a Gödel state, i.e.

$$m(A \vee B) + m(A \wedge B) = m(A) + m(B).$$

Then again all assumptions of Proposition are satisfied.

References

[1] Atanassov, K.,1986, Intuitionistic Fuzzy Sets: Theory and Applications, Physica- Verlag.

[2] Grzegorzewski, P., 2010, (To appear), The Inclusion-Exclusion Principle for IF - Events.