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Intuitionistic fuzzy superfluous submodule

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Abstract: In this paper, we introduce the notion of intuitionistic fuzzy superfluous (or small) submodule of a module and study some of their properties. We establish the condition of an intuitionistic fuzzy submodule to be an intuitionistic fuzzy superfluous submodule. A relationship between superfluous submodule and the intuitionistic fuzzy superfluous submodule is derived. We also study the nature of intuitionistic fuzzy superfluous submodules under intuitionistic fuzzy direct sum. A relation regarding intuitionistic fuzzy superfluous submodule and the superfluous submodules does not hold in case of intuitionistic fuzzy superfluous submodules.

Keywords: Intuitionistic fuzzy superfluous submodules, Intuitionistic fuzzy indecomposable modules, Intuitionistic fuzzy direct sum, Intuitionistic fuzzy radical. **AMS Classification:** 03F55, 16D10.

1 Introduction

The concept of fuzzy subset of a non-empty set was introduced by Zadeh [21] in 1965. Rosenfeld [16] applied the concept of fuzzy sets to the theory of groups and defined the notion of fuzzy subgroups of a group. After this, many papers concerning various fuzzy algebraic structures have appeared in the literature. The concept of fuzzy modules was introduced by Negoita and

Ralescu in [14]. Since then, different types of fuzzy submodules were investigated in the last three decades. The notion of fuzzy superfluous submodule of a module was introduced by Basnet et al. in [5] which was further extended by Rahman et al. in [15].

As an important extension of fuzzy set theory, Atanassov [2], [3] and [4] introduced and developed the theory of intuitionistic fuzzy sets. Using the Atanassov's idea. Biswas [7] established the intuitionistic fuzzification of the concept of subgroup of a group and introduced the notion of intuitionistic fuzzy subgroups. Later on many mathematicians worked on it and introduced the notion of intuitionistic fuzzy subring, intuitionistic fuzzy submodule etc., see [10, 11, 12, 17, 18]. In this paper, we define the notion of intuitionistic fuzzy submodule and discuss its properties. A relationship between superfluous submodule and the intuitionistic fuzzy submodule has been obtained. We examine the nature of intuitionistic fuzzy superfluous submodule submodule and the superfluous submodule submodule submodule and the superfluous submodule submodule submodule and the superfluous submodules does not hold in case of intuitionistic fuzzy superfluous submodules.

2 Preliminaries

Throughout the paper M will always denote a left module over a ring R.

Definition 2.1. ([1, 8]) A submodule S of a left module M over a ring R is said to be a superfluous submodule of M if for any submodule K of M, $S + K = M \Rightarrow K = M$. In symbol, $S \ll M$ i.e., S is a superfluous submodule of M.

Proposition 2.2. ([1, 8]) If K and N are submodules of M such that $K \subseteq N$ and H is any submodule of M, then

(i) $N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$

(ii) $H + K \ll M$ if and only if $H \ll M$ and $K \ll M$.

Proposition 2.3. ([9]) If $K \ll M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll N$.

Definition 2.4. ([1]) A nonzero module M is said to be indecomposable if $\{0\}$ and M are the only direct summands of M.

Proposition 2.5. ([1]) If $K \ll M$ and M/K is indecomposable, then M is indecomposable.

Definition 2.6. ([2, 3]) Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

Remark 2.7.

(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., when $\nu_A(x) = 1 - \mu_A(x) = \mu_A^c(x)$. Then, A is called a fuzzy set.

(*ii*) We denote the IFS $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ by $A = (\mu_A, \nu_A)$.

Definition 2.8. ([3]) Let A and B be IFSs of the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x) \rangle | x \in X\}$. Then,

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$.

(*ii*) A = B if and only if $A \subseteq B$ and $B \subseteq A$.

(iii)
$$A^c = \{ \langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X \}.$$

(iv) $A \cap B = \{ \langle x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x) \rangle \mid x \in X \}.$

(v)
$$A \cup B = \{ \langle x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x) \rangle \mid x \in X \}.$$

Definition 2.9. ([12, 19]) Let M be a module over a ring R. An IFS $A = (\mu_A, \nu_A)$ of M is called an intuitionistic fuzzy (left) submodule (IFSM) if

(*i*) $\mu_A(0) = 1$, $\nu_A(0) = 0$;

(ii)
$$\mu_A(x+y) \ge \mu_A(x) \land \mu_A(y)$$
 and $\nu_A(x+y) \le \nu_A(x) \lor \nu_A(y)$, $\forall x, y \in M$;

(iii) $\mu_A(rx) \ge \mu_A(x)$ and $\nu_A(rx) \le \nu_A(x), \forall x \in M, r \in R$.

If we replace the condition (iii) with (iii)' $\mu_A(xr) \ge \mu_A(x)$ and $\nu_A(xr) \le \nu_A(x)$, $\forall x \in M, r \in R$, then A is called an intuitionistic fuzzy (right) module of M. When R is a commutative ring, then these two types of intuitionistic fuzzy modules coincides.

Theorem 2.10. ([19]) Let $A = (\mu_A, \nu_A)$ be an IFS of a *R*-module *M*. Then, *A* is an IFSM of *M* if and only if the following conditions are satisfied:

(*i*)
$$\mu_A(0) = 1$$
, $\nu_A(0) = 0$;
(*ii*) $\mu_A(rx + sy) \ge \mu_A(x) \land \mu_A(y)$ and $\nu_A(rx + sy) \le \nu_A(x) \lor \nu_A(y)$, $\forall x, y \in M, r, s \in R$.

Definition 2.11. ([19]) Let A be an intuitionistic fuzzy set of a non-empty set M. Then, (α, β) -cut of A is a crisp subset $C_{\alpha,\beta}(A)$ of the IFS A is given by

$$C_{\alpha,\beta}(A) = \{x \in M : \mu_A(x) \ge \alpha, \nu_A(x) \le \beta\}$$
, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$.

Theorem 2.12. ([19]) If $A = (\mu_A, \nu_A)$ be IFS of a *R*-module *M*, then *A* is an IFSM of *M* if and only if $C_{\alpha,\beta}(A)$ is a submodule of *M*, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, where $\mu_A(0) \geq \alpha$, $\nu_A(0) \leq \beta$.

Theorem 2.13. ([20]) Consider a maximal chain of submodules of a R-module M

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M,$$

where \subset denotes proper inclusion. Then, there exists an intuitionistic fuzzy module A of M given by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \text{if } x \in M_2 \backslash M_1 \\ \cdots \\ \alpha_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases}; \nu_A(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1 \backslash M_0 \\ \beta_2 & \text{if } x \in M_2 \backslash M_1 \\ \cdots \\ \beta_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases}$$

where $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_n$ and $0 = \beta_0 \le \beta_1 \le \cdots \le \beta_n$ and the pair (α_i, β_i) are called double pins and the set $\wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ is called the set of double pinned flags for the IFSM A of M.

Remark 2.14. ([20]) The converse of the above theorem is also true, i.e., any intuitionistic fuzzy module A of M can be expressed in the above form.

Definition 2.15. ([6]) If $A = (\mu_A, \nu_A)$ is an IFSM of a *R*-module *M*, then we denote

$$A_* = \{x \in M : \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}.$$

Clearly, $A_* = C_{1,0}(A)$.

Proposition 2.16. ([6]) If $A = (\mu_A, \nu_A)$ is an IFSM of a *R*-module *M*, then A_* is a submodule of *M*.

Definition 2.17. ([6]) we define two IFS Ω and $\Omega(M)$ of M as

$$\Omega(x) = \begin{cases} (1,0), & \text{if } x = 0\\ (0,1), & \text{if } x \neq 0 \end{cases} \text{ and } \Omega(M)(x) = (1,0), \forall x \in M.$$

Then, the IFS Ω and $\Omega(M)$ of M are IFSMs of M. These we call trivial IFSMs. Any IFSM other than these two is called proper IFSM of M.

Proposition 2.18. If $A = (\mu_A, \nu_A)$ is an IFSM of M, then $A_* = M$ if and only if $A = \Omega(M)$. Also, $A \subseteq B$ implies that $A_* \subseteq B_*$.

Proof. Now, $A_* = M$ if and only if $\mu_A(x) = 1$ and $\nu_A(x) = 0$ for all $x \in M$, i.e., if and only if $A = \Omega(M)$.

Moreover, if $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in M$.

Let $x \in A_*$ implies $1 = \mu_A(x) \le \mu_B(x)$ and $0 = \nu_A(x) \ge \nu_B(x)$, i.e., $\mu_B(x) = 1$ and $\nu_B(x) = 0$, which implies that $x \in B_*$.

Hence, $A_* \subseteq B_*$.

Proposition 2.19. If $A = (\mu_A, \nu_A)$ and $B = (\nu_A, \nu_B)$ are two IFSMs of a *R*-module *M*, then $(A \cap B)_* = A_* \cap B_*$ and $(A \cup B)_* = A_* \cup B_*$. These results can be extended to infinite intersection and unions. Further, if *A* and *B* have finite pinned flag sets, then $(A + B)_* = A_* + B_*$ where the sum of two IFSMs is $A + B = (\mu_{A+B}, \nu_{A+B})$ and is defined as $\mu_{A+B}(x) = Sup_{x=a+b}\{min\{\mu_A(a), \mu_B(b)\}\}$ and $\nu_{A+B}(x) = Inf_{x=a+b}\{max\{\nu_A(a), \nu_B(b)\}\}; x \in M$.

Proof. The first two proofs are trivial. Also for the last part we have $A, B \subseteq A + B$, so $A_*, B_* \subseteq (A + B)_*$ and hence $A_* + B_* \subseteq (A + B)_*$. Next, if $x \in (A + B)_*$, then $\mu_{A+B}(x) = 1$ and $\nu_{A+B}(x) = 0$. Now $Sup_{x=a+b}\{min\{\mu_A(a), \mu_B(b)\}\} = 1$ and $Inf_{x=a+b}\{max\{\nu_A(a), \nu_B(b)\}\} = 0$ $\Leftrightarrow min\{\mu_A(a), \mu_B(b)\} = 1$ and $max\{\nu_A(a), \nu_B(b)\} = 0$ $\Leftrightarrow \mu_A(a) = 1, \mu_B(b) = 1$ and $\nu_A(a) = 0, \nu_B(b) = 0$ $\Leftrightarrow \mu_A(a) = 1$ and $\nu_A(a) = 0$ and $\mu_B(b) = 1$ and $\nu_B(b) = 0$ $\Leftrightarrow a \in A_*$ and $b \in B_*$ $\Leftrightarrow a + b \in A_* + B_*$. Hence, $(A + B)_* = A_* + B_*$. □

Definition 2.20. ([13]) Let $A_i = (\mu_i, \nu_i), (i \in J, |J| > 1)$, be a family of IFSMs of M. Then,

$$\sum_{i \in J} A_i = \{ \langle x, \mu_{\sum_{i \in J} A_i}(x), \nu_{\sum_{i \in J} A_i}(x) \rangle \mid x \in M \},\$$

where

$$\mu_{\sum_{i\in J}A_i}(x) = \bigvee\{\land(\mu_{A_i}(x_i): x_i\in M, i\in J, \sum_{i\in J}x_i=x)\}, \forall x\in M,$$

and

$$\nu_{\sum_{i \in J} A_i}(x) = \wedge \{ \lor (\nu_{A_i}(x_i) : x_i \in M, i \in J, \sum_{i \in J} x_i = x) \},\$$

where in $\sum_{i \in J} x_i$ almost finitely many x_i 's are not equal to zero. Then, $\sum_{i \in J} A_i$ is called a weak sum of A_i .

Theorem 2.21. ([13]) Let $A_i = (\mu_i, \nu_i), (i \in J, |J| > 1)$, be a family of IFSMs of a *R*-module *M*. Then, $\sum_{i \in J} A_i$ is an IFSM of *M* and $\sum_{i \in J} (A_i)_* \subseteq (\sum_{i \in J} A_i)_*$.

3 Intuitionistic fuzzy superfluous module

Definition 3.1. An IFSM A of a R-module M is said to be an intuitionistic fuzzy superfluous submodule (IFSSM) of M (or of $\Omega(M)$) if for any IFSM B of M, $A+B = \Omega(M) \Rightarrow B = \Omega(M)$ or equivalently $B \neq \Omega(M) \Rightarrow A + B \neq \Omega(M)$. We denote it by $A \ll_{IF} \Omega(M)$ or $A \ll_{IF} M$.

Remark 3.2. From the definition, we observe that $A \ll_{IF} \Omega(M)$ if and only if $A + B \neq \Omega(M)$ for every proper IFSM *B* of *M*.

Proposition 3.3. If $A \ll_{IF} \Omega(M)$ if and only if $A_* \ll M$ i.e., A is an intuitionistic fuzzy superfluous submodule of M if and only if A_* is superfluous submodule of M. *Proof.* Firstly, let $A \ll_{IF} \Omega(M)$. Let N be any submodule of M such that $A_* + N = M$. Let χ_N be the characteristic intuitionistic function of N defined by $\chi_N(x) = (\mu_{\chi_N}(x), \nu_{\chi_N}(x))$, where

$$\mu_{\chi_N}(x) = \begin{cases} 1, & \text{if } x \in N \\ 0, & \text{if } x \notin N \end{cases}; \quad \nu_{\chi_N}(x) = \begin{cases} 0, & \text{if } x \in N \\ 1, & \text{if } x \notin N \end{cases}.$$

Clearly, χ_N is an IFSM of M such that $(\chi_N)_* = N$. Then,

$$A_* + (\chi_N)_* = M \Rightarrow (A + \chi_N)_* = M \Rightarrow A + \chi_N = \Omega(M) \Rightarrow \chi_N = \Omega(M)$$

and so $N = (\chi_N)_* = M$. Thus, $A_* \ll M$.

Conversely, let $A_* \ll M$ and B be any IFSM of M such that $A + B = \Omega(M)$. Then, $(A + B)_* = M \Rightarrow A_* + B_* = M \Rightarrow B_* = M$ [As $A_* \ll M$] $\Rightarrow B = \Omega(M)$.

Hence, A is an IFSSM of M, i.e., $A \ll_{IF} \Omega(M)$.

Example 3.4. Consider the Z-module Z_8 and an IFS $A = (\mu_A, \nu_A)$ of Z_8 defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0, 2, 4, 6\\ 0.7, & \text{if } x = 1, 7\\ 0.5, & \text{if } x = 3, 5 \end{cases}; \nu_A(x) = \begin{cases} 0, & \text{if } x = 0, 2, 4, 6\\ 0.2, & \text{if } x = 1, 7\\ 0.3, & \text{if } x = 3, 5. \end{cases}$$

Then, A is an IFSM of Z_8 and $A_* = \{0, 2, 4, 6, 8\}$. Since A_* is a superfluous submodule of Z_8 and by Proposition 3.3, A is an IFSSM of Z_8 .

Proposition 3.5. If A and B be any two IFSMs of M. Then, $A + B \ll_{IF} \Omega(M)$ if and only if $A \ll_{IF} \Omega(M)$ and $B \ll_{IF} \Omega(M)$.

Proof. Firstly, let $A \ll_{IF} \Omega(M)$ and $B \ll_{IF} \Omega(M)$. Let *C* be any IFSM of *M* such that $(A + B) + C = \Omega(M)$. Then, we get $A + (B + C) = \Omega(M) \Rightarrow B + C = \Omega(M)$ [Since $A \ll_{IF} \Omega(M)$] ⇒ $C = \Omega(M)$ [Since $B \ll_{IF} \Omega(M)$]. This shows that $A + B \ll_{IF} \Omega(M)$. Conversely, let $A + B \ll_{IF} \Omega(M)$. Let *D* be any IFSM of *M* such that $A + D = \Omega(M)$. Now, $\Omega(M) = A + D \subseteq (A + B) + D$ [Since $A \subseteq A + B$] ⇒ $\Omega(M) = (A + B) + D \Rightarrow \Omega(M) = D$ [Since $A + B \ll_{IF} \Omega(M)$]. This shows that $A \ll_{IF} \Omega(M)$.

Theorem 3.6. Let M be a module and N be a submodule of M. Then, $N \ll M$ if and only if $\chi_N \ll_{IF} \Omega(M)$.

Proof. Let $N \ll M$. We assume that χ_N is not an IFSSM of M. Then, there exists an IFSM B such that $B \neq \Omega(M) \Rightarrow \chi_N + B = \Omega(M)$. Let $x \in M$, then

$$\Omega(M)(x) = (1,0) = (\chi_N + B)(x)$$

$$= (Sup_{x=a+b}\{min\{\mu_{\chi_N}(a), \mu_B(b)\}\}, Inf_{x=a+b}\{max\{\nu_{\chi_N}(a), \nu_B(b)\}\})$$

So, $\exists s a_0, b_0, c_0, d_0 \in M$ such that $x = a_0 + b_0 = c_0 + d_0$ and $\min\{\mu_{\chi_N}(a_0), \mu_B(b_0)\} = 1, \max\{\nu_{\chi_N}(c_0), \nu_B(d_0)\} = 0$ $\Rightarrow \mu_{\chi_N}(a_0) = \mu_B(b_0) = 1, \nu_{\chi_N}(c_0) = \nu_B(d_0) = 0$ $\Rightarrow \nu_{\chi_N}(a_0) \le 1 - \mu_{\chi_N}(a_0) = 1 - 1 = 0 \Rightarrow \nu_{\chi_N}(a_0) = 0.$ Similarly, we get $\nu_B(b_0) = 0$. So $\mu_{\chi_N}(a_0) = 1$, $\nu_{\chi_N}(a_0) = 0 \Rightarrow a_0 \in N$ and $\mu_B(b_0) = 1$, $\nu_B(b_0) = 0 \Rightarrow b_0 \in B_*$. Thus, $x = a_0 + b_0 \in N + B_*$. Since x is an arbitrary, this implies $M = N + B_*$. As $N \ll M$. So, we must have $M = B_*$ and this implies $B = \Omega(M)$, a contradiction. Therefore, $\chi_N \ll_{IF} \Omega(M)$. Conversely, we assume $\chi_N \ll_{IF} \Omega(M)$. If possible let N is not superfluous submodule of M. Thus, \exists 's a submodule T of M such that $T \neq M$ implies N + T = M. Let $y \in M$ such that y = n + t, where $n \in N$ and $t \in T$. Now, $(\chi_N + \chi_T)(y) = (Sup_{y=p+q}\{min\{\mu_{\chi_N}(p), \mu_{\chi_T}(q)\}\}, Inf_{y=p+q}\{max\{\nu_{\chi_N}(p), \nu_{\chi_T}(q)\}\})$ But, $Sup_{y=p+q}\{min\{\mu_{\chi_N}(p), \mu_{\chi_T}(q)\}\} \ge min\{\mu_{\chi_N}(n), \mu_{\chi_T}(t)\} = 1$ and $Inf_{y=p+q}\{max\{\nu_{\chi_N}(p), \nu_{\chi_T}(q)\}\} \le max\{\nu_{\chi_N}(p), \nu_{\chi_T}(q)\} = 0$ $\Rightarrow (\chi_N + \chi_T)(y) = (1, 0)$. But y is an arbitrary element of M. So, $\chi_N + \chi_T = \Omega(M)$. As $\chi_N \ll_{IF} \Omega(M)$ therefore, $\chi_T = \Omega(M)$, i.e., T = M, which is not possible. Hence, $N \ll M$.

Proposition 3.7. Let $\phi : M \to N$ be a module epimorphism and $A \ll_{IF} \Omega(M)$, then $\phi(A) \ll_{IF} \Omega(N)$.

Proof. First we prove $\phi(A_*) = \phi(A)_*$. Let $y \in \phi(A)_*$, then $\phi(A)(y) = (1, 0)$. $(Sup{\mu_A(x) : x \in \phi^{-1}(y)}, Inf{\nu_A(x) : x \in \phi^{-1}(y)}) = (1, 0)$. ∴ ∃'s $x_1, x_2 \in M$ such that $\phi(x_1) = \phi(x_2) = y$ and $\mu_A(x_1) = 1, \nu_A(x_2) = 0$. Now, $\nu_A(x_1) \leq 1 - \mu_A(x_1) = 1 - 1 = 0 \Rightarrow \nu_A(x_1) = 0$ and so, $x_1 \in A_*$. $\Rightarrow y = \phi(x_1) \in \phi(A_*)$. So $\phi(A)_* \subseteq \phi(A_*)$. Conversely, let $y \in \phi(A_*)$, then $y = \phi(x)$, for some $x \in A_*$ $\Rightarrow \mu_A(x) = 1, \nu_A(x) = 0$, where $x \in \phi^{-1}(y)$ $\Rightarrow Sup{\mu_A(x) : x \in \phi^{-1}(y)} \ge \mu_A(x) = 1; Inf{\nu_A(x) : x \in \phi^{-1}(y)} \le \nu_A(x) = 0$. $\Rightarrow (Sup{\mu_A(x) : x \in \phi^{-1}(y)}, Inf{\nu_A(x) : x \in \phi^{-1}(y)}) = (1, 0)$. $\phi(A)(y) = (1, 0) \Rightarrow y \in \phi(A)_*$. So $\phi(A_*) \subseteq \phi(A)_*$. Thus, $\phi(A)_* = \phi(A_*)$. Now, $A \ll_{IF} \Omega(M) \Rightarrow A_* \ll M$, so by Proposition (2.3) $\phi(A_*) \ll N$ i.e., $\phi(A)_* \ll N$ and so $\phi(A) \ll_{IF} \Omega(N)$ [by Proposition (3.3)].

Definition 3.8. Let A and B be two IFSMs of M such that $A \subseteq B$. Then, A_* is a submodule of B_* . An intuitionistic fuzzy subset $B/A = (\mu_{B/A}, \nu_{B/A})$ of M/A_* defined by

$$\mu_{B/A}(x+A_*) = \lor \{\mu_B(x+y) | y \in A_*\},\$$
$$\nu_{B/A}(x+A_*) = \land \{\nu_B(x+y) | y \in A_*\}$$

is an IFSM of M/A_* .

Proposition 3.9. If A and B are two IFSMs of M such that $A \subseteq B$. Then, $(B/A)_* = B_*/A_*$

Proof. Let $x + A_* \in (B/A)_*$ be any element, then $\mu_{B/A}(x + A_*) = 1, \nu_{B/A}(x + A_*) = 0$ $\Rightarrow \vee \{\mu_B(x + y) | y \in A_*\} = 1, \land \{\nu_B(x + y) | y \in A_*\} = 0.$ Since *A* has finite double pinned flags set, so there exist $y_1, y_2 \in A_*$ such that $\mu_B(x + y_1) = 1$ and $\nu_B(x + y_2) = 0.$ Now, $\nu_B(x + y_1) \leq 1 - \mu_B(x + y_1) = 1 - 1 = 0$ implies that $\nu_B(x + y_1) = 0.$ So, $x + y_1 \in B_*$. Also, $y_1 \in A_* \subseteq B_*$, so $x \in B_*$. Hence, $x + A_* \in B_*/A_*$. Conversely, let $x + A_* \in B_*/A_*$, then $x \in B_*$, which implies that $\mu_B(x) = 1$ and $\nu_B(x) = 0.$ $\Rightarrow \vee \{\mu_B(x + y) | y \in A_*\} \geq \mu_B(x) = 1$ and $\wedge \{\nu_B(x + y) | y \in A_*\} \leq \nu_B(x) = 0$ $\Rightarrow \mu_{B/A}(x + A_*) = 1$ and $\nu_{B/A}(x + A_*) = 0$ $\Rightarrow x + A_* \in (B/A)_*.$ Hence, the result proved. □

Proposition 3.10. If A and B are two IFSMs of M such that $A \subseteq B$. Then, $B \ll_{IF} \Omega(M)$ if and only if $A \ll_{IF} \Omega(M)$ and $B/A \ll_{IF} \Omega(M)/A$.

Proof. Since *A* and *B* are two IFSMs of *M* such that *A* ⊆ *B*. So, *A*_{*} and *B*_{*} are submodules of *M* such that *A*_{*} ⊆ *B*_{*}. First let *B* ≪_{*IF*} Ω(*M*), then *B*_{*} ≪ *M* and so by Proposition 2.2. (i) *A*_{*} ≪ *M* and *B*_{*}/*A*_{*} ≪ *M*/*A*_{*} i.e., (*B*/*A*)_{*} ≪ *M*/*A*_{*}. Hence, by Proposition 3.3., we have *A* ≪_{*IF*} Ω(*M*) and *B*/*A* ≪_{*IF*} Ω(*M*)/*A*. Conversely, let *A* ≪_{*IF*} Ω(*M*) and *B*/*A* ≪_{*IF*} Ω(*M*)/*A*. Let *C* be an IFSM of *M* such that *B* + *C* = Ω(*M*). Now, *B*/*A* + (*C* + *A*)/*A* = Ω(*M*)/*A*. But *B*/*A* ≪_{*IF*} Ω(*M*)/*A*. Therefore, (*C* + *A*)/*A* = Ω(*M*)/*A* ⇒ *C* + *A* = Ω(*M*). But *A* ≪_{*IF*} Ω(*M*) ⇒ *C* = Ω(*M*).Thus, *B* + *C* = Ω(*M*) ⇒ *C* = Ω(*M*). Hence, *B* ≪_{*IF*} Ω(*M*).

Proposition 3.11. Let A, B, C, D are IFSMs of M such that $A \ll_{IF} \Omega(M)$ and $C \ll_{IF} \Omega(M)$ where $A \subseteq B$ and $C \subseteq D$. If $B/A \ll_{IF} \Omega(M)/A$ and $D/C \ll_{IF} \Omega(M)/C$, then $(B+D)/(A+C) \ll_{IF} \Omega(M)/(A+C)$.

Proof. Since $B/A \ll_{IF} \Omega(M)/A$ and $D/C \ll_{IF} \Omega(M)/C$ so $(B/A)_* \ll (\Omega(M)/A)_*$ and $(D/C)_* \ll (\Omega(M)/C)_*$ i.e., $B_*/A_* \ll M/A_*$ and $D_*/C_* \ll M/C_*$. But $A \ll_{IF} \Omega(M)$ and $C \ll_{IF} \Omega(M)$ implies $A_* \ll M$ and $C_* \ll M$. Then, by Proposition 2.2. (i) we get $B_* \ll M$ and $D_* \ll M$. $\Rightarrow (B_* + D_*) \ll M$ i.e., $(B + D)_* \ll M$ and so $(B + D) \ll_{IF} \Omega(M)$. Since $(A + C) \subseteq (B + D)$ and $(A + C) \ll_{IF} \Omega(M)$. So using Proposition 3.10., we get $(B + D)/(A + C) \ll_{IF} \Omega(M)/(A + C)$.

IFSSM, Intuitionistic fuzzy maximal submodule, 4 Indecomposable submodule

Definition 4.1. If A and B are two IFSMs of a R-module M, then the sum A + B is called the direct sum of A and B if $A \cap B = \Omega$, and we write it as $A \oplus B$.

Proposition 4.2. If A and B are two IFSMs of a R-module M with finite double pinned flags sets. Then, $(A \oplus B)_* = A_* \oplus B_*$.

Proof. By Proposition 2.19., we have $(A+B)_* = A_* + B_*$. Now, for $A \oplus B$ we have $A \cap B = \Omega$, which implies that $(A \cap B)_* = A_* \cap B_* = \{0\}.$

Definition 4.3. An IFSM $A \neq \Omega$ of M is said to be an intuitionistic fuzzy indecomposable if there does not exist IFSM B and $C \neq \Omega, A$ of M such that $A = B \oplus C$.

Proposition 4.4. $\Omega(M)$ is an intuitionistic fuzzy indecomposable if and only if M is indecomposable module.

Proof. First let $\Omega(M)$ is an intuitionistic fuzzy indecomposable. Let $M = P \oplus Q$, where P and Q are non-zero proper submodules of M. Let χ_P and χ_Q be the characteristic intuitionistic fuzzy functions on P and Q, respectively.

Then, χ_P and χ_Q are IFSMs of M such that $(\chi_P)_* = P$ and $(\chi_Q)_* = Q$.

So, $M = (\chi_P)_* \oplus (\chi_Q)_* = (\chi_P \oplus \chi_Q)_* \Rightarrow \Omega(M) = \chi_P \oplus \chi_Q.$

Since P and Q are non-zero proper submodules of M so $\chi_P, \chi_Q \neq \Omega, \Omega(M)$.

This contradict that $\Omega(M)$ is indecomposable. Hence, M is indecomposable.

Conversely, let M is indecomposable. Let A and B be two IFSMs of M such that $A, B \neq \Omega, \Omega(M)$ and $\Omega(M) = A \oplus B$. Then, $M = A_* \oplus B_*$. Since $A, B \neq \Omega(M)$ so $A_*, B_* \neq M$. Also, if $A_* = \{0\}$, then $B_* = M$ and so $B = \Omega(M)$, which is not true.

Hence, $\Omega(M)$ is an intuitionistic fuzzy indecomposable.

Proposition 4.5. If $A \ll_{IF} \Omega(M)$ and $\Omega(M)/A$ is an intuitionistic fuzzy indecomposable, then $\Omega(M)$ is an intuitionistic fuzzy indecomposable.

Proof. Since $A \ll_{IF} \Omega(M)$ and $\Omega(M)/A$ is an intuitionistic fuzzy indecomposable, so $A_* \ll M$ and M/A_* is indecomposable. Therefore by Proposition 2.5., M is indecomposable and hence by Proposition 4.4., $\Omega(M)$ is an intuitionistic fuzzy indecomposable.

Definition 4.6. The sum of all IFSSMs of a *R*-module *M* is again an IFSM of *M* and it is denoted by IFsuperfl(M), i.e., IFsuperfl(M)= $\sum_{\alpha \in J} A_{\alpha}$, where the A_{α} 's are IFSSM of M.

Remark 4.7. IFsuperfl(M) is not necessary an IFSSM of M as shown by the following example.

Example 4.8. Consider the Z-module Q. Then, for any $q \in Q$, the submodule $\langle q \rangle$ generated by q is a superfluous submodule of Q. Let χ_q be the characteristic intuitionistic fuzzy function of $\langle q \rangle$,

$$\chi_q(x) = \begin{cases} (1,0), & \text{if } x \in \langle q \rangle \\ (0,1), & \text{if } x \notin \langle q \rangle \end{cases}.$$

Then, χ_q is an IFSSM of Q [by Theorem (3.6)] and $(\chi_q)_* = \langle q \rangle$ for all $q \in Q$. Now, if IFsuperfl(Q) is an IFSSM of Q, then $\sum_{q \in Q} \chi_q \subseteq IFsuperfl(Q) \subseteq \chi_Q = \Omega(Q)$. So by Proposition 3.10, $\sum_{q \in Q} \chi_q \ll_{IF} \Omega(Q)$. Thus, $(\sum_{q \in Q} \chi_q)_* \ll Q$. But $(\sum_{q \in Q} \chi_q)_* = \sum_{q \in Q} \langle q \rangle = Q$, which is not a superfluous submodule of Q. So, we get a contradiction. Hence, IFsuperfl(Q) is not an IFSSM of Q.

Definition 4.9. An IFSM A of a R-module M is said to be an intuitionistic fuzzy maximal if for any IFSM B of $M, A \subseteq B \Rightarrow$ either $B = \Omega(M)$ or $B_* = A_*$.

Proposition 4.10. A non-constant IFSM $A = (\mu_A, \nu_A)$ of a *R*-module *M* is an intuitionistic fuzzy maximal if and only if the double pinned flags set for A is $\{(1,0), (s,t)\}$, where $s, t \in (0,1)$ such that s + t < 1, i.e., A is defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in M_0 \\ s, & \text{if } x \in M \setminus M_0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in M_0 \\ t, & \text{if } x \in M \setminus M_0 \end{cases}; \forall x \in M, \end{cases}$$

where M_0 is a maximal submodule of M.

Proof. Firstly, let the IFSM $A = (\mu_A, \nu_A)$ of a *R*-module *M* is defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in M_0 \\ s, & \text{if } x \in M \setminus M_0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in M_0 \\ t, & \text{if } x \in M \setminus M_0 \end{cases}; \forall x \in M, \end{cases}$$

where M_0 is a maximal submodule of M and $s, t \in (0, 1)$ such that s + t < 1.

Clearly, $A_* = M_0$. Then, for any IFSM *B* of *M* such that $A \subseteq B \Rightarrow A_* \subseteq B_*$, i.e., $M_0 \subseteq B_* \Rightarrow$ either $B_* = M_0$ or $B_* = M$ [As M_0 is a maximal submodule of *M*] \Rightarrow either $B_* = A_*$ or $B = \Omega(M)$.

Thus, A is an intuitionistic fuzzy maximal submodule of M.

Conversely, let A be a non-constant intuitionistic fuzzy maximal submodule of M with double pinned flags set $\{(1,0), (\alpha, \beta)\}$, where $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$ defined as

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in M_0 \\ \alpha, & \text{if } x \in M \setminus M_0 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in M_0 \\ \beta, & \text{if } x \in M \setminus M_0. \end{cases}; \forall x \in M, \end{cases}$$

where M_0 is a submodule of M. Clearly, $A_* = M_0$.

We show that M_0 is a maximal submodule of M. Let B be any IFSM of M such that $A \subseteq B$. Therefore, A_* and B_* are submodules of M such that $A_* \subseteq B_* \subseteq M$.

As A is an intuitionistic fuzzy maximal submodule of M, therefore, either $B = \Omega(M)$ or $B_* = A_*$ i.e., either $B_* = M$ or $B_* = A_*$.

This implies that A_* is a maximal submodule of M. Hence, M_0 is a maximal submodule of M. This completes the proof.

Definition 4.11. The intersection of all intuitionistic fuzzy maximal submodules of M is called the intuitionistic fuzzy radical of M and it is denoted by IFrad(M).

Remark 4.12. In module theory Superfl(M) = rad(M), i.e., the sum of all superfluous submodules of M is equal to the radical of M. But it is not true in case of intuitionistic fuzzy submodules as shown by the following example.

Example 4.13. Consider the Z-module Z_8 and for each $k \in [0, 1)$, define the IFS $A_k = (\mu_{A_k}, \nu_{A_k})$, where

$$\mu_{A_k}(x) = \begin{cases} 1, & \text{if } x \in \{0, 2, 4, 6\} \\ k, & \text{if } x \in \{1, 3, 5, 7\} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in \{0, 2, 4, 6\} \\ 1 - k, & \text{if } x \in \{1, 3, 5, 7\} \end{cases}; \forall x \in Z_8.$$

Then, A_k is an IFSM of Z_8 with set of double pinned flags $\wedge(A_k) = \{(1,0), (k, 1-k)\}$. Also, $(A_k)_* = \{0, 2, 4, 6\}$ which is a maximal submodule of Z_8 .

So, A_k is an intuitionistic fuzzy maximal submodule of Z_8 and these are the only intuitionistic fuzzy maximal submodules of Z_8 .

Thus, we have $IFrad(Z_8) = \bigcap \{A_k : 0 \le k < 1\} = B(say)$. Then, $B = (\mu_B, \nu_B)$ is an IFS on Z_8 defined as

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in \{0, 2, 4, 6\} \\ 0, & \text{if } x \in \{1, 3, 5, 7\} \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in \{0, 2, 4, 6\} \\ 1, & \text{if } x \in \{1, 3, 5, 7\} \end{cases}$$

Now, if we define the IFS C on Z_8 by

$$\mu_C(x) = \begin{cases} 1, & \text{if } x \in \{0, 2, 4, 6\} \\ 0.8, & \text{if } x \in \{1, 3, 5, 7\} \end{cases}; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x \in \{0, 2, 4, 6\} \\ 0.1, & \text{if } x \in \{1, 3, 5, 7\} \end{cases},$$

then, C is an IFSM of Z_8 and $C_* = \{0, 2, 4, 6\}$, which is a superfluous submodule of Z_8 . So C is an IFSSM of Z_8 . Thus, we have $IFrad(Z_8) = B \subset C \subseteq IFsuperfl(Z_8)$. This shows that $IFsuperfl(Z_8) \neq IFrad(Z_8)$.

5 Conclusions

In this paper some aspects and properties of intuitionistic fuzzy superfluous submodules of a module have been introduced. This concept has given a new way to the study of intuitionistic fuzzy Goldie dimension. In our further study we may investigate various aspects of:

- (i) spanning dimension of intuitionitic fuzzy submodules,
- (ii) intuitionitic fuzzy lifting modules with chain condition on intuitionistic fuzzy superfluous submodules, and
- (iii) Noetherian and Artinion conditions on intuitionistic fuzzy Jacobson radical of a module.

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