

SOME OPERATORS ON INTUITIONISTIC FUZZY SETS

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Certain new operators over Intuitionistic Fuzzy Sets (IFSs) will be introduced. Basing on the notations and definitions from [1], we begin with the definition of the concept of the IFS and the definitions of some of the operations and operators over IFSs (see, e.g. [2-4]).

Let a set E be fixed. An IFS A^* in E is an object of the following form:

$$A^* = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in E\},$$

where the functions $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

If

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x),$$

then $\pi_A(x)$ is the degree of non-determinacy of the element $x \in E$ to the set A . In the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in E$.

For simplicity below we shall write A instead of A^* .

The first geometrical interpretation of the IFSs is introduced on Fig. 1.

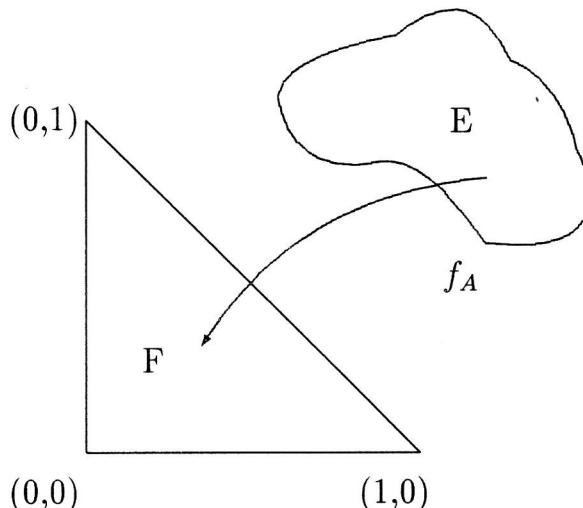


Fig. 1.

For every two IFSs A and B , the following relations and operations can be defined (everywhere below "iff" will stand for "if and only if"):

$$\begin{aligned}
A \subset B &\text{ iff } (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x)); \\
A \supset B &\text{ iff } B \subset A; \\
A = B &\text{ iff } (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)); \\
\neg A &= \{\langle x, \nu_A(x), \mu_A(x) \rangle | x \in E\}; \\
A \cap B &= \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle | x \in E\}; \\
A \cup B &= \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle | x \in E\}; \\
A + B &= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x) \rangle | x \in E\}; \\
A \cdot B &= \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x) \rangle | x \in E\}; \\
A @ B &= \{\langle x, \left(\frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2}\right) \rangle | x \in E\}; \\
A \$ B &= \{\langle x, \sqrt{\mu_A(x).\mu_B(x)}, \sqrt{\nu_A(x).\nu_B(x)} \rangle | x \in E\};
\end{aligned}$$

Let A be an IFS and let $\alpha, \beta \in [0, 1]$. The following operators are defined:

$$\begin{aligned}
\square A &= \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle | x \in E\}; \\
\Diamond A &= \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle | x \in E\}; \\
D_\alpha(A) &= \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1 - \alpha).\pi_A(x) \rangle | x \in E\}; \\
F_{\alpha,\beta}(A) &= \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x) \rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1; \\
G_{\alpha,\beta}(A) &= \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x) \rangle | x \in E\}. \\
H_{\alpha,\beta}(A) &= \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x) \rangle | x \in E\}, \\
H_{\alpha,\beta}^*(A) &= \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x)) \rangle | x \in E\}, \\
J_{\alpha,\beta}(A) &= \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x) \rangle | x \in E\}, \\
J_{\alpha,\beta}^*(A) &= \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x) \rangle | x \in E\},
\end{aligned}$$

$$C(A) = \{\langle x, K, L \rangle | x \in E\},$$

where

$$K = \max_{y \in E} \mu_A(y), L = \min_{y \in E} \nu_A(y)$$

and

$$I(A) = \{\langle x, k, l \rangle | x \in E\},$$

where

$$k = \min_{y \in E} \mu_A(y), l = \max_{y \in E} \nu_A(y).$$

First, we shall introduce the operators of modal type, which are similar to the operators from the intuitionistic fuzzy modal logic studied in [5]. They are the following (A is an IFS):

$$\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{(\nu_A(x) + 1)}{2} \rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \frac{(\mu_A(x) + 1)}{2}, \frac{\nu_A(x)}{2} \rangle | x \in E\}.$$

The following assertions hold of the new operators.

THEOREM 1: For every IFS A :

- (a) $\boxplus A \subset A \subset \boxtimes A$,
- (b) $\neg \boxplus \neg A = \boxtimes A$,
- (c) $\boxplus \boxplus A \subset \boxplus A$,
- (d) $\boxtimes \boxtimes A \supset \boxtimes A$,
- (e) $\boxplus \boxtimes A = \boxtimes \boxplus A$,
- (f) $\boxplus \square A = \square \boxplus A$,
- (g) $\boxtimes \square A = \square \boxtimes A$,
- (h) $\boxplus \diamond A = \diamond \boxplus A$,
- (i) $\boxtimes \diamond A = \diamond \boxtimes A$,

THEOREM 2: For every two IFSs A and B :

- (a) $\boxplus(A \cap B) = \boxplus A \cap \boxplus B$,
- (b) $\boxtimes(A \cap B) = \boxtimes A \cap \boxtimes B$,
- (c) $\boxplus(A \cup B) = \boxplus A \cup \boxplus B$,
- (d) $\boxtimes(A \cup B) = \boxtimes A \cup \boxtimes B$,
- (e) $\boxplus(A + B) \subset \boxplus A + \boxplus B$,
- (f) $\boxtimes(A + B) \supset \boxtimes A + \boxtimes B$,
- (g) $\boxplus(A \cdot B) \supset A \cdot \boxplus B$,
- (h) $\boxtimes(A \cdot B) \subset \boxtimes A \cdot \boxtimes B$,
- (i) $\boxplus(A @ B) = \boxplus A @ \boxplus B$,
- (j) $\boxtimes(A @ B) = \boxtimes A @ \boxtimes B$,
- (k) $\boxplus(A \$ B) \supset \boxplus A \$ \boxplus B$,
- (l) $\boxtimes(A \$ B) \subset \boxtimes A \$ \boxtimes B$.

THEOREM 3: For every IFS A :

- (a) $\boxplus C(A) = C(\boxplus A)$,
- (b) $\boxtimes C(A) = C(\boxtimes A)$,
- (c) $\boxplus I(A) = I(\boxplus A)$,
- (d) $\boxtimes I(A) = I(\boxtimes A)$.

Now we shall generalize the two operators introduced above. So far the new operators have no analogues in the intuitionistic fuzzy logic (this will be subject of another research by the author).

Let $\alpha \in [0, 1]$ and let A be an IFS. Then we can define:

$$\boxplus_\alpha A = \{\langle x, \alpha \cdot \mu_A(x), \alpha \cdot \nu_A(x) + 1 - \alpha \rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \alpha \cdot \mu_A(x) + 1 - \alpha, \alpha \cdot \nu_A(x) \rangle | x \in E\}.$$

Obviously,

$$0 \leq \alpha \cdot \mu_A(x) + \alpha \cdot \nu_A(x) + 1 - \alpha = 1 - \alpha \cdot (1 - \mu_A(x) - \alpha \cdot \nu_A(x)) \leq 1.$$

For every IFS A :

$$\boxplus_{0.5} A = \boxplus A,$$

$$\boxtimes_{0.5} A = \boxtimes A.$$

Therefore, the new operators “ \boxplus_α ” and “ \boxtimes_α ” are generalizations of the first ones. Their graphical interpretations are given on Fig. 2 and Fig. 3, respectively.

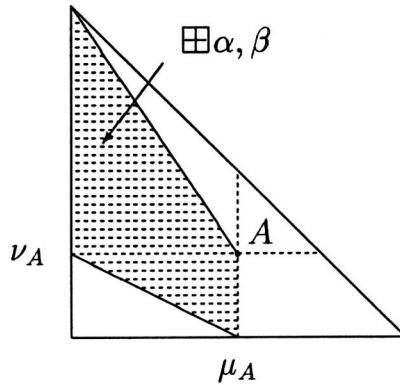


Fig. 2.

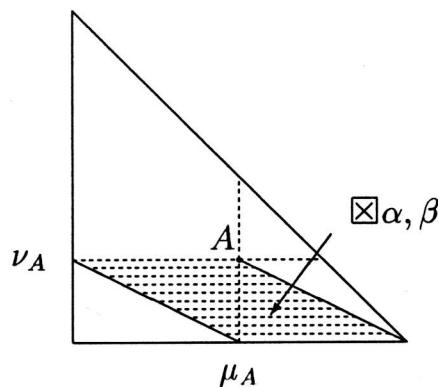


Fig. 3.

The following assertions (which are direct generalizations of the above ones) are valid for the new operators.

THEOREM 4: For every IFS A and for every $\alpha \in [0, 1]$:

- (a) $\boxplus_\alpha A \subset A \subset \boxtimes_\alpha A$,
- (b) $\neg \boxplus_\alpha \neg A = \boxtimes_\alpha A$,
- (c) $\boxplus_\alpha \boxplus_\alpha A \subset \boxplus_\alpha A$,
- (d) $\boxtimes_\alpha \boxtimes_\alpha A \supset \boxtimes_\alpha A$,
- (e) $\boxplus_\alpha \boxtimes_\alpha A = \boxtimes_\alpha \boxplus_\alpha A$,
- (f) $\boxplus_\alpha \square A = \square \boxplus_\alpha A$,
- (g) $\boxtimes_\alpha \square A = \square \boxtimes_\alpha A$,
- (h) $\boxplus_\alpha \diamond A = \diamond \boxplus_\alpha A$,
- (i) $\boxtimes_\alpha \diamond A = \diamond \boxtimes_\alpha A$,

THEOREM 5: For every two IFSs A and B :

- (a) $\boxplus_\alpha(A \cap B) = \boxplus_\alpha A \cap \boxplus_\alpha B$,
- (b) $\boxtimes_\alpha(A \cap B) = \boxtimes_\alpha A \cap \boxtimes_\alpha B$,
- (c) $\boxplus_\alpha(A \cup B) = \boxplus_\alpha A \cup \boxplus_\alpha B$,
- (d) $\boxtimes_\alpha(A \cup B) = \boxtimes_\alpha A \cup \boxtimes_\alpha B$,
- (e) $\boxplus_\alpha(A + B) \subset \boxplus_\alpha A + \boxplus_\alpha B$,
- (f) $\boxtimes_\alpha(A + B) \supset \boxtimes_\alpha A + \boxtimes_\alpha B$,
- (g) $\boxplus_\alpha(A \cdot B) \supset A \cdot \boxplus_\alpha B$,
- (h) $\boxtimes_\alpha(A \cdot B) \subset \boxtimes_\alpha A \cdot \boxtimes_\alpha B$,
- (i) $\boxplus_\alpha(A @ B) = \boxplus_\alpha A @ \boxplus_\alpha B$,
- (j) $\boxtimes_\alpha(A @ B) = \boxtimes_\alpha A @ \boxtimes_\alpha B$,
- (k) $\boxplus_\alpha(A \$ B) \supset \boxplus_\alpha A \$ \boxplus_\alpha B$,
- (l) $\boxtimes_\alpha(A \$ B) \subset \boxtimes_\alpha A \$ \boxtimes_\alpha B$.

THEOREM 6: For every IFS A :

- (a) $\boxplus_\alpha C(A) = C(\boxplus_\alpha A)$,
- (b) $\boxtimes_\alpha C(A) = C(\boxtimes_\alpha A)$,
- (c) $\boxplus_\alpha I(A) = I(\boxplus_\alpha A)$,
- (d) $\boxtimes_\alpha I(A) = I(\boxtimes_\alpha A)$.

Moreover, the following assertions also are valid.

THEOREM 7: For every IFS A and for every two real numbers $\alpha, \beta \in [0, 1]$:

- (a) $\boxplus_\alpha \boxplus_\beta A = \boxplus_{\text{beta}} \boxplus_\alpha A$,
- (b) $\boxtimes_\alpha \boxtimes_\beta A = \boxtimes_{\text{beta}} \boxtimes_\alpha A$,
- (c) $\boxtimes_\alpha \boxplus_\beta A \supset \boxplus_\beta \boxtimes_\alpha A$.

THEOREM 8: For every IFS A and for every three real numbers $\alpha, \beta, \gamma \in [0, 1]$:

- (a) $\boxplus_\alpha D_\beta(A) = D_\beta(\boxplus_\alpha A)$,
- (b) $\boxplus_\alpha F_{\beta, \gamma}(A) = F_{\beta, \gamma}(\boxplus_\alpha A)$, where $\beta + \gamma \leq 1$,
- (c) $\boxplus_\alpha G_{\beta, \gamma}(A) \subset G_{\beta, \gamma}(\boxplus_\alpha A)$,
- (d) $\boxplus_\alpha H_{\beta, \gamma}(A) = H_{\beta, \gamma}(\boxplus_\alpha A)$,
- (e) $\boxplus_\alpha H_{\beta, \gamma}^*(A) = H_{\beta, \gamma}^*(\boxplus_\alpha A)$,
- (f) $\boxplus_\alpha J_{\beta, \gamma}(A) = J_{\beta, \gamma}(\boxplus_\alpha A)$,
- (g) $\boxplus_\alpha J_{\beta, \gamma}^*(A) = J_{\beta, \gamma}^*(\boxplus_\alpha A)$,
- (h) $\boxtimes_\alpha D_\beta(A) = D_\beta(\boxtimes_\alpha A)$,
- (i) $\boxtimes_\alpha F_{\beta, \gamma}(A) = F_{\beta, \gamma}(\boxtimes_\alpha A)$, where $\beta + \gamma \leq 1$,
- (j) $\boxtimes_\alpha G_{\beta, \gamma}(A) \supset G_{\beta, \gamma}(\boxtimes_\alpha A)$,
- (k) $\boxtimes_\alpha H_{\beta, \gamma}(A) = H_{\beta, \gamma}(\boxtimes_\alpha A)$,
- (l) $\boxtimes_\alpha H_{\beta, \gamma}^*(A) = H_{\beta, \gamma}^*(\boxtimes_\alpha A)$,
- (m) $\boxtimes_\alpha J_{\beta, \gamma}(A) = J_{\beta, \gamma}(\boxtimes_\alpha A)$,
- (n) $\boxtimes_\alpha J_{\beta, \gamma}^*(A) = J_{\beta, \gamma}^*(\boxtimes_\alpha A)$.

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