Notes on Intuitionistic Fuzzy Sets
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Development of intuitionistic fuzzy cost efficiency model in data envelopment analysis

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Abstract: Data envelopment analysis (DEA) is a non-parametric linear programming (LP) based technique to measure the relative efficiencies of decision-making units (DMUs). The conventional DEA models assume that input-output data is crisp, which may not always be feasible in practical situations due to the presence of ambiguity and imprecision. Therefore, to handle uncertain and imprecise data, the concept of intuitionistic fuzzy sets (IFS) has been introduced. In this study, the relative efficiencies of DMUs with uncertain data will be determined. For this reason, the conventional cost efficiency (CE) model of DEA is extended to IF environment. Also, the lower and upper-cost efficiency models are developed using α-cut and β-cut approach. The data for inputs, outputs and input prices are considered intuitionistic fuzzy numbers (IFNs), in particular triangular intuitionistic fuzzy numbers (TIFNs). To demonstrate the practical application of the proposed intuitionistic fuzzy cost efficiency models (IFCEMs), a numerical example is presented.
Keywords: α- and β-cuts, Cost efficiency, Data envelopment analysis, Performance measurement, Intuitionistic fuzzy set.
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1 Introduction

Charnes et al. [7] introduced the CCR-DEA model, which is a linear programming technique designed to assess the relative efficiencies of DMUs. These DMUs can include various entities such as hospitals, banks, educational institutes, police stations, libraries, and transportation services. The DEA models are capable of measuring the relative efficiencies of DMUs with multiple inputs and outputs. This enables the identification of both efficient and inefficient units, which allows for the adjustment of input and output quantities to improve the inefficiency of a DMU. Banker et al. [6] extended this concept to include a convexity constraint and discussed the nature of the return to scale (RTS) for DMUs, which can be constant, decreasing, or increasing.

In DEA, the efficiency of decision-making units is evaluated using crisp data, which may not be applicable in real-life problems where uncertainty is prevalent. To handle uncertainty, the theories of fuzzy and intuitionistic fuzzy sets have been developed. Zadeh [16] introduced fuzzy set (FS) theory. He originated the concept of membership function. For each element, the membership function defines a value between 0 to 1 to define its level of presence in the set.

Additionally, the intuitionistic fuzzy set (IFS) theory proposed by Atanassov [4] addresses acceptance, rejection and hesitation in decision-making and considers both membership and non-membership values of an element, making it a more practical approach for real-life problems.

DMU's main objective is to maximize profit with minimum use of resources or with minimum expenses. Cost efficiency minimizes the use of input quantities while producing the same level of outputs. Farrell [8] gave the concept of cost efficiency and Tone [14] presented the CE model by evaluating DMUs in a cost-based production possibility set. In the CE model, precise data of input-output and price of input entities should be known. But in real-life, data is imprecise and fuzzy. Kao and Liu [10] proposed fuzzy efficiency measurement models in DEA. Jahanshahloo et al. [9] took imprecise data for input-output and precise data for input price and developed a fuzzy CE model. Bagherzadeh Valami [5] did the opposite, i.e., precise data for input and output and imprecise data for the price (Triangular fuzzy number).

Puri and Yadav [12] gave a fully fuzzified CE model using TFNs. Aghayi [1] considered three cases for CE measurement, (i) Fuzzy input and output and crisp input price vector (ii) Crisp input and output and fuzzy input price vector (iii) Fuzzy input, output, and input price vector using triangular and trapezoidal fuzzy numbers. The authors demonstrate the effectiveness of their approach using data from a real-world case study.

Venkatesh and Kushwaha [15] investigated the efficiency of State Transport Undertakings (STUs) in India over a 10-year period by assessing short-term and long-term efficiencies using the cost variant of DEA with variable returns to scale. The study found that the variable cost efficiency of STUs can be evaluated in the short run when some inputs cannot be varied. They also concluded that some STUs perform poorly in the short run with low fleet size and are cost-inefficient in the long run. Pourmahmoud and Sharak [11] proposed a method for evaluating fuzzy cost efficiency using the α-level approach with triangular fuzzy numbers and developed a new method for ranking DMUs based on fuzzy cost efficiency. Sarab et al. [13] developed an algorithm to maintain cost efficiency despite any variation in input prices using a two-step method applied to Persian sugar mills in 2014. The first step involves locating the convex cone formed by the intersection of convex cones created by comparing the new input prices of efficient DMUs and the adjacent extreme efficient DMUs in the first quadrant, while the second step involves obtaining the subset that preserves unit efficiency in the presence of annual inflation.

Here in this study, we considered all input-output data and costs as intuitionistic fuzzy (IF) numbers, particularly triangular intuitionistic fuzzy numbers (TIFNs). Since IF is the most generalized form of the crisp data, so the proposed intuitionistic fuzzy cost efficiency models (IFCEMs) can be considered the most generalized models for estimating the cost-efficiency performance of DMUs. Due to the generalized nature of the proposed IFCEMs, these can be used for all types of data: crisp, fuzzy, and IF.

The present study involves the development of a CE model employing IF data. By means of the α and β-cut methods, the issue at hand is reformulated into a linear programming problem. A numerical example is illustrated by taking triangular IF data and evaluating their lower and upper CEs for α-cut and β-cut separately.

This paper is structured as follows: An overview of the fundamental concept of IF is provided in Section 2. In Section 3, the cost efficiency model in conventional DEA is presented. Proposed Intuitionistic fuzzy cost-efficiency models are presented in Section 4. Section 5 presents the methodology for solving IFCEMs. Section 6 illustrates a numerical example and results (lower and upper-cost efficiencies) are shown in Tables 2 and 3. Section 7, contains the conclusion of the study.

2 Preliminaries

Definition 1. (Intuitionistic fuzzy set, IFS) [4] *The Intuitionistic Fuzzy Set (IFS) $\tilde{F}^I$ is expressed as $\tilde{F}^I = \{w, \mu_{\tilde{F}^I}(w), \nu_{\tilde{F}^I}(w) : w \in \Omega\}$, where the domain of discussion is denoted by Ω. The membership function $\mu_{\tilde{F}^I} : \Omega \to [0,1]$ and non-membership function $\nu_{\tilde{F}^I} : \Omega \to [0,1]$ are the defining components of $\tilde{F}^I$.*

In the context of IFS, the degree to which w belongs to $\tilde{F}^I$ is known as the membership function and is represented by $\mu_{\tilde{F}^I}(w)$. Similarly, the degree to which w does not belong to $\tilde{F}^I$ is called the non-membership function and is represented by $\nu_{\tilde{F}^I}(w)$. These functions are subject to the constraint that their sum must be between 0 and 1, i.e., $0 \leq \mu_{\tilde{F}^I}(w) + \nu_{\tilde{F}^I}(w) \leq 1$.

The degree of hesitation or indeterminacy regarding the inclusion of an element w in the IFS $\tilde{F}^I$ is referred to as the hesitation degree and denoted by $\pi_{\tilde{F}^I}(w)$. This degree is precisely defined as follows:

$$(\forall\ w \in \Omega)\pi_{\tilde{F}^I}(w) = 1 - \mu_{\tilde{F}^I}(w) - \nu_{\tilde{F}^I}(w),$$
$$0 \leq \pi_{\tilde{F}^I}(w) \leq 1.$$

Definition 2. (Convex IFS) [3] *An IFS $\tilde{F}^I = \{(w, \mu_{\tilde{F}^I}(w), \nu_{\tilde{F}^I}(w)) : w \in \Omega\}$ is a convex IFS if*

(i) $\mu_{\tilde{F}^I}(\lambda_1 w_1 + \lambda_2 w_2) \geq \min(\mu_{\tilde{F}^I}(w_1), \mu_{\tilde{F}^I}(w_2))$, $\forall w_1, w_2 \in \Omega$, *where* $\lambda_1 + \lambda_2 = 1$ *and* $\lambda_1,\ \lambda_2 \geq 0$, *i.e.,* $\mu_{\tilde{F}^I}$ *is quasi-concave over* Ω.

(ii) $\nu_{\tilde{F}^I}(\lambda_1 w_1 + \lambda_2 w_2) \leq \max(\nu_{\tilde{F}^I}(w_1), \nu_{\tilde{P}^I}(w_2))$, $\forall w_1, w_2 \in \Omega$, *where* $\lambda_1 + \lambda_2 = 1$ *and* $\lambda_1,\ \lambda_2 \geq 0$, *i.e.,* $\nu_{\tilde{F}^I}$ *is quasi-convex over* Ω.

Definition 3. (Intuitionistic fuzzy number, IFN) [3] *An Intuitionistic Fuzzy Set (IFS)* $\tilde{F}^I = \{(w, \mu_{\tilde{F}^I}(w), \nu_{\tilde{F}^I}(w)) : w \in \mathbb{R}\}$ *is called an IFN* (i) *if it is a convex set,* (ii) *there exists unique* $w_0 \in \mathbb{R}$ *such that* $\mu_{\tilde{F}^I}(w_0) = 1$*, and there exists* $w_1 \in \mathbb{R}$ *such that* $\nu_{\tilde{F}^I}(w_1) = 1$*,* $w_0 \neq w_1$*.* w_0 *is called the mean value of* $\tilde{F}^I$*.*

Definition 4. (Triangular intuitionistic fuzzy number, TIFN) [3] *The intuitionistic fuzzy number* $\tilde{F}^I = \{(w, \mu^I_{\tilde{F}}(w), \nu^I_{\tilde{F}}(w)) : w \in \mathbb{R}\}$ *is called TIFN if*

$$\mu_{\tilde{F}^I}(w) = \begin{cases} \dfrac{w - f^L}{f^M - f^L}, & f^L < w \leq f^M, \\ \dfrac{f^U - w}{f^U - f^M}, & f^M \leq w < f^U, \\ 0 & , otherwise. \end{cases} \qquad \nu_{\tilde{F}^I}(w) = \begin{cases} \dfrac{f^M - w}{f^M - f'^L}, & f'^L < w \leq f^M, \\ \dfrac{w - f^M}{f'^U - f^M}, & f^M \leq w < f'^U, \\ 1 & , otherwise. \end{cases}$$

where $f^L, f^M, f^U, f'^L, f'^U \in \mathbb{R}$ *such that* $f'^L \leq f^L < f^M < f^U \leq f'^U$*. The TIFN as defined above is represented by* $\tilde{F}^I = (f^L, f^M, f^U; f'^L, f^M, f'^U)$*.*

The graphical portrayal is given in Figure 1.

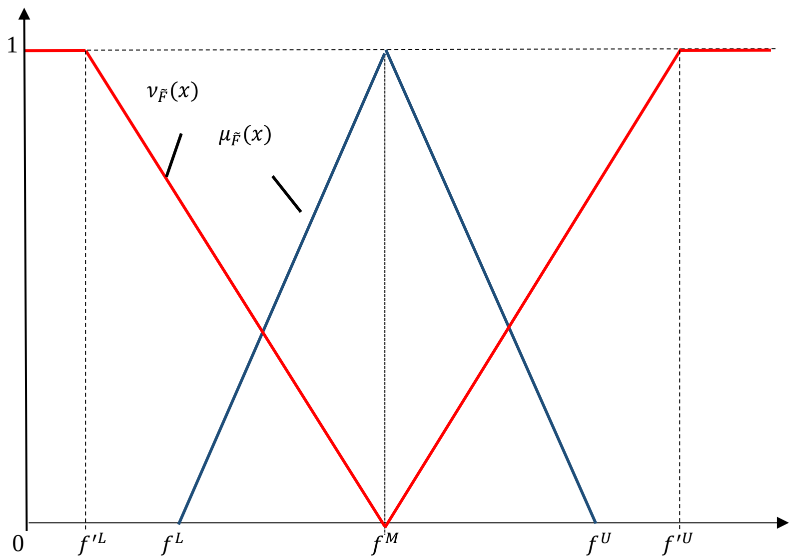

Figure 1. TIFN $\tilde{F}^I = (f^L, f^M, f^U; f'^L, f^M, f'^U)$

Definition 5. (Arithmetic operations on TIFNs) [2] *Let* $\tilde{F}^I = (f^L, f^M, f^U; f'^L, f^M, f'^U)$ *and* $\tilde{G}^I = (g^L, g^M, g^U; g'^L, g^M, g'^U)$ *be two TIFNs. The arithmetic operations on TIFNs are defined as follows:*

(i) ***Addition****:* $\tilde{F}^I \oplus \tilde{G}^I = (f^L + g^L, f^M + g^M, f^U + g^U; f'^L + g'^L, f^M + g^M, f'^U + g'^U)$.

(ii) ***Subtraction****:* $\tilde{F}^I \ominus \tilde{G}^I = (f^L - g^U, f^M - g^M, f^U - g^L; f'^L - g'^U, f^M - g^M, f'^U - g'^L)$.

(iii) ***Multiplication****:* $\tilde{F}^I \otimes \tilde{G}^I \approx (f^L g^L, f^M g^M, f^U g^U; f'^L g'^L, f^M g^M, f'^U g'^U)$, *where* $f'^L, g'^L > 0$.

(iv) ***Division:***

$$\frac{\tilde{F}^I}{\tilde{G}^I} \approx \left(\frac{f^L}{g^U}, \frac{f^M}{g^M}, \frac{f^U}{g^L}; \frac{f'^L}{g'^U}, \frac{f^M}{g^M}, \frac{f'^U}{g'^L}\right), \textit{ where } f'^L, g'^L > 0.$$

(v) ***Scalar multiplication**: If* $\lambda \in \mathbb{R}$, *then*

$$\lambda\tilde{F}^I = \begin{cases} (\lambda f^L, \lambda f^M, \lambda f^U; \lambda f'^L, \lambda f^M, \lambda f'^U), & \textit{for } \lambda \geq 0, \\ (\lambda f^U, \lambda f^M, \lambda f^L; \lambda f'^U, \lambda f^M, \lambda f'^L), & \textit{for } \lambda < 0. \end{cases}$$

3 Cost efficiency (CE)

The cost efficiency (CE) of a DMU is defined as the quotient of its minimum cost to its actual cost. Thus,

$$\text{CE} = \frac{\text{minimal cost}}{\text{actual cost}}.$$

CE-DEA models [8] estimate the cost-minimizing level of input quantities while producing the same level of outputs.

Suppose that we have n DMUs and we have to evaluate their performance efficiencies. Let each DMU have m inputs and s outputs. Then, for DMU$_k$, $k = 1, 2, \ldots, n$, the CE is defined as follows:

$$\text{CE}_k = \frac{\sum_{i=1}^m c_{ik} x_i}{\sum_{i=1}^m c_{ik} x_{ik}}$$

Now we introduce CE-DEA Model [8].

Model 1. (CE-DEA Model): For DMU$_k$,

$$\min \text{CE}_k = \frac{\sum_{i=1}^m c_{ik} x_i}{\sum_{i=1}^m c_{ik} x_{ik}}$$

subject to

$$\begin{aligned} \sum_{j=1}^n \lambda_j x_{ij} &\leq x_i, & i &= 1, 2, \ldots, m, \\ \sum_{j=1}^n \lambda_j y_{rj} &\geq y_{rk}, & r &= 1, 2, \ldots, s, \\ \lambda_j,\ x_i &\geq 0 \end{aligned}$$

where λ_j and x_i are variables, x_i = unknown quantity of the i-th input, c_{ik} = price of the i-th input for DMU$_k$, x_{ik} = actual value of the i-th input for DMU$_k$, x_{ij} = actual value of the i-th input for DMU$_j$, y_{rj} = actual value of the r-th output for DMU$_j$.

Definition 6. (Cost efficiency) [8] *Let* (x_i^*, λ_j^*), $i = 1, 2, ..., m$; $j = 1, 2, ..., n$ *be the optimal value of* (x_i, λ_j) *and CE$_k^*$ be the optimal value of CE$_k$. Then DMU$_k$ is said to be cost-efficient if CE$_k^*$* $= 1$, *otherwise DMU$_k$ is said to be cost-inefficient.*

4 Proposed intuitionistic fuzzy cost efficiency model (IFCEM)

In this model, we take IF data for input, output and input cost. The Proposed IFCEM (Model 2) is given as follows:

Model 2.

$$\min \tilde{\mathrm{CE}}_k{}^I = \frac{\sum_{i=1}^{m} \tilde{c}_{ik}^I x_i}{\sum_{i=1}^{m} \tilde{c}_{ik}^I \tilde{x}_{ik}^I}$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^I &\leq x_i, && i = 1, 2, \ldots, m, \\ \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^I &\geq \tilde{y}_{rk}^I, && r = 1, 2, \ldots, s, \\ \lambda_j,\ x_i \geq 0. \end{aligned}$$

Assume that IF input $\tilde{x}_{ij}^I$, IF output $\tilde{y}_{rj}^I$ and IF input price $\tilde{c}_{ij}^I$ are TIFNs. Let

$$\tilde{x}_{ij}^I = (x_{ij}^L, x_{ij}^M, x_{ij}^U; x_{ij}'^L, x_{ij}^M, x_{ij}'^U),$$

$$\tilde{y}_{rj}^I = (y_{rj}^L, y_{rj}^M, y_{rj}^U; y_{rj}'^L, y_{rj}^M, y_{rj}'^U),$$

and

$$\tilde{c}_{ij}^I = (c_{ij}^L, c_{ij}^M, c_{ij}^U; c_{ij}'^L, c_{ij}^M, c_{ij}'^U).$$

Model 2 is converted to Model 3 in the following manner.

Model 3.

$$\min \tilde{\mathrm{CE}}_k{}^I = \frac{\sum_{i=1}^{m} \left(c_{ik}^L, c_{ik}^M, c_{ik}^U; c_{ik}'^L, c_{ik}'^M, c_{ik}'^U\right) (x_i, x_i, x_i; x_i, x_i, x_i)}{\sum_{i=1}^{m} (c_{ik}^L, c_{ik}^M, c_{ik}^U; c_{ik}'^L, c_{ik}'^M, c_{ik}'^U)\, (x_{ik}^L, x_{ik}^M, x_{ik}^U; x_{ik}'^L, x_{ik}'^M, x_{ik}'^U)}$$

subject to

$$\sum_{j=1}^{n} \lambda_j \left(x_{ij}^L, x_{ij}^M, x_{ij}^U; x_{ij}'^L, x_{ij}'^M, x_{ij}'^U\right) \leq (x_i, x_i, x_i; x_i, x_i, x_i), \qquad i = 1, 2, \ldots, m,$$

$$\sum_{j=1}^{n} \lambda_j \left(y_{rj}^L, y_{rj}^M, y_{rj}^U; y_{rj}'^L, y_{rj}'^M, y_{rj}'^U\right) \geq \left(y_{rk}^L, y_{rk}^M, y_{rk}^U; y_{rk}'^L, y_{rk}^M, y_{rk}'^U\right), \qquad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \geq 0.$$

5 Methodology for solving IFCEM

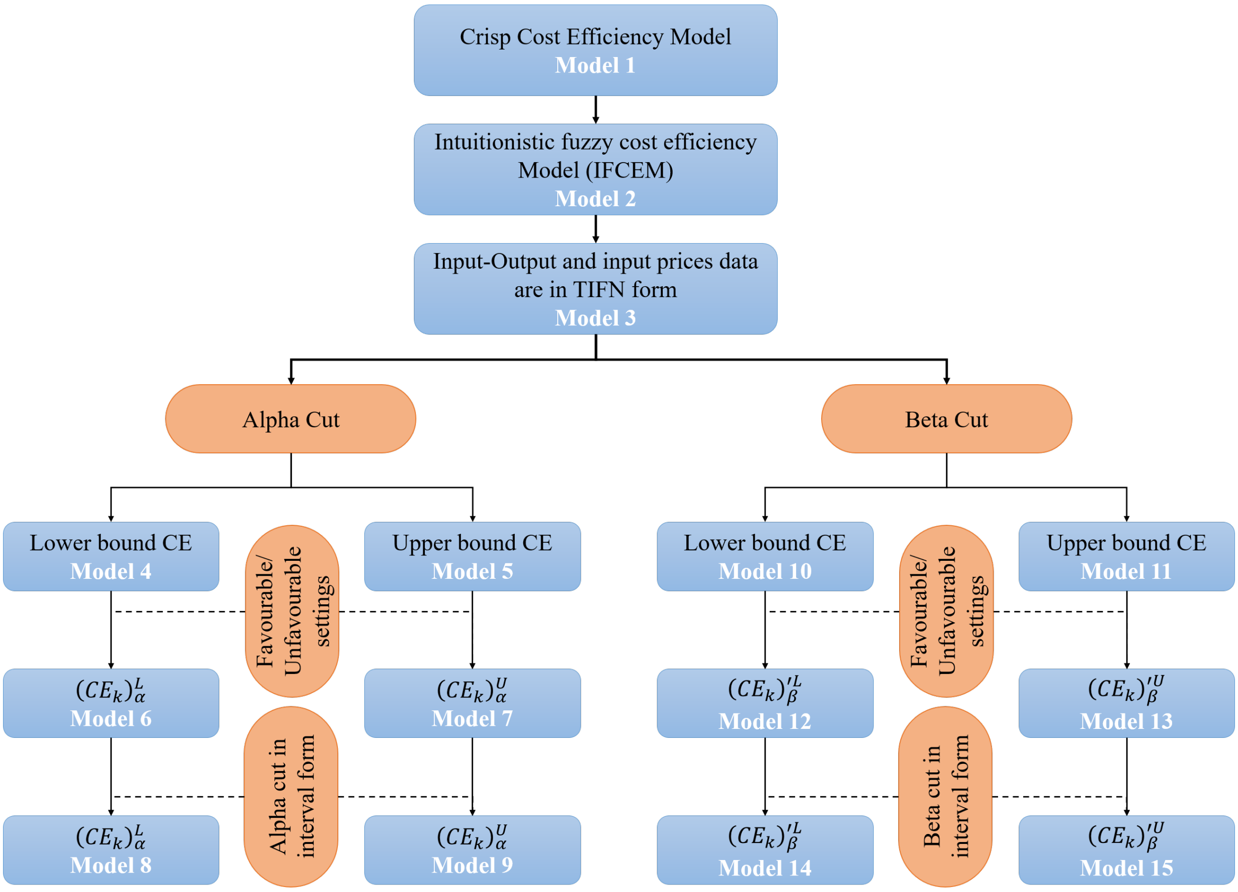

Figure 2. Development of models

Let $\mathrm{S}(\tilde{x}^I_{ij})$, $\mathrm{S}(\tilde{y}^I_{rj})$ and $\mathrm{S}(\tilde{c}^I_{ik})$ denote the supports of $\tilde{x}^I_{ij}$, $\tilde{y}^I_{rj}$ and $\tilde{c}^I_{ik}$ respectively.

$$\forall\, i,j, \quad \mathrm{S}(\tilde{x}^I_{ij}) = \left\{x_{ij} \mid \mu_{\tilde{x}^I_{ij}}(x_{ij}) > 0\right\},$$
$$\forall\, r,j, \quad \mathrm{S}(\tilde{y}^I_{rj}) = \left\{y_{rj} \mid \mu_{\tilde{y}^I_{rj}}(y_{rj}) > 0\right\},$$
$$\forall\; i,k, \quad \mathrm{S}(\tilde{c}^I_{ik}) = \left\{c_{ik} \mid \mu_{\tilde{c}^I_{ik}}(c_{ik}) > 0\right\},$$

The α-cuts of $\tilde{x}^I_{ij}$, $\tilde{y}^I_{rj}$ and $\tilde{c}^I_{ik}$ are defined, respectively, as follows.

$$\begin{aligned}
&\forall\, i,j \text{ and } 0 < \alpha \le 1\\
&(\tilde{x}^I_{ij})_\alpha = \left\{x_{ij} \in \mathrm{S}(\tilde{x}^I_{ij}) \mid \mu_{\tilde{x}^I_{ij}}(x_{ij}) \ge \alpha\right\} = \left[(x_{ij})^L_\alpha, (x_{ij})^U_\alpha\right]\\
&\quad = \left[\min_{x_{ij}}\left\{x_{ij} \in \mathrm{S}(\tilde{x}^I_{ij}) \mid \mu_{\tilde{x}^I_{ij}}(x_{ij}) \ge \alpha\right\}, \max_{x_{ij}}\left\{x_{ij} \in \mathrm{S}(\tilde{x}^I_{ij}) \mid \mu_{\tilde{x}^I_{ij}}(x_{ij}) \ge \alpha\right\}\right].
\end{aligned}$$

$$\begin{aligned}
&\forall\, r,j \text{ and } 0 < \alpha \le 1\\
&(\tilde{y}^I_{rj})_\alpha = \left\{y_{rj} \in \mathrm{S}(\tilde{y}^I_{rj}) \mid \mu_{\tilde{y}^I_{rj}}(y_{rj}) \ge \alpha\right\} = \left[(y_{rj})^L_\alpha, (y_{rj})^U_\alpha\right]\\
&\quad = \left[\min_{y_{rj}}\left\{y_{rj} \in \mathrm{S}(\tilde{y}^I_{rj}) \mid \mu_{\tilde{y}^I_{rj}}(y_{rj}) \ge \alpha\right\}, \max_{y_{rj}}\left\{y_{rj} \in \mathrm{S}(\tilde{y}^I_{rj}) \mid \mu_{\tilde{y}^I_{rj}}(y_{rj}) \ge \alpha\right\}\right].
\end{aligned}$$

and

$$\begin{aligned}&\forall i,k \text{ and } 0<\alpha\le 1\\ (\tilde{c}^I_{ik})_\alpha &= \left\{c_{ik}\in \mathrm{S}(\tilde{c}^I_{ik}) \mid \mu_{\tilde{c}^I_{ik}}(c_{ik})\ge\alpha\right\} = \left[(c_{ik})^L_\alpha,(c_{ik})^U_\alpha\right]\\ &= \left[\min_{c_{ik}}\left\{c_{ik}\in \mathrm{S}(\tilde{c}^I_{ik}) \mid \mu_{\tilde{c}^I_{ik}}(c_{ik})\ge\alpha\right\},\ \max_{c_{ik}}\left\{c_{ik}\in \mathrm{S}(\tilde{c}^I_{ik}) \mid \mu_{\tilde{c}^I_{ik}}(c_{ik})\ge\alpha\right\}\right].\end{aligned}$$

Further, using these α-cuts, IFCEM can be transformed into a crisp model. Owing to the input and output data being intuitionistic fuzzy numbers, the efficiency score is also an intuitionistic fuzzy number denoted by $\tilde{\mathrm{CE}}_k{}^I$ with membership function $\mu_{\tilde{\mathrm{CE}}_k{}^I}$. Let $\mathrm{S}(\tilde{\mathrm{CE}}_k{}^I)$ be the support of $\tilde{\mathrm{CE}}_k{}^I$:

$$\forall\, k,\quad \mathrm{S}(\tilde{\mathrm{CE}}_k{}^I) = \left\{\mathrm{CE}_k \mid \mu_{\tilde{\mathrm{CE}}_k{}^I}(\mathrm{CE}_k)>0\right\}$$

and let the α-cut be

$$\begin{aligned}&\forall\, k \text{ and } 0<\alpha\le 1\\ (\tilde{\mathrm{CE}}_k{}^I)_\alpha &= \left\{\mathrm{CE}_k\in \mathrm{S}(\tilde{\mathrm{CE}}_k{}^I) \mid \mu_{\tilde{\mathrm{CE}}_k{}^I}(\mathrm{CE}_k)\ge\alpha\right\} = \left[(\mathrm{CE}_k)^L_\alpha,(\mathrm{CE}_k)^U_\alpha\right]\\ &= \left[\min_{\mathrm{CE}_k}\left\{\mathrm{CE}_k\in \mathrm{S}(\tilde{\mathrm{CE}}_k{}^I) \mid \mu_{\tilde{\mathrm{CE}}_k{}^I}(\mathrm{CE}_k)\ge\alpha\right\},\max_{\mathrm{CE}_k}\left\{\mathrm{CE}_k\in \mathrm{S}(\tilde{\mathrm{CE}}_k{}^I) \mid \mu_{\tilde{\mathrm{CE}}_k{}^I}(\mathrm{CE}_k)\ge\alpha\right\}\right].\end{aligned}$$

5.1 Model based on α-cut

Aghayi [1] developed CE models in fuzzy enviornment. Here, we propose CE models in the IF enviornment. Next, using α-cut in the proposed Model 3, we obtain the lower and upper bound CE-DEA models. Model 3 is reduced to Models 4 and 5.

Model 4.

$$(\mathrm{CE}_k)^L_\alpha = \min_{\substack{(x_{ij})^L_\alpha\le x_{ij}\le(x_{ij})^U_\alpha\\ (y_{rj})^L_\alpha\le y_{rj}\le(y_{rj})^U_\alpha\\ (c_{ik})^L_\alpha\le c_{ik}\le(c_{ik})^U_\alpha}} \begin{cases}\min \mathrm{CE}_k = \dfrac{\sum_{i=1}^m c_{ik}x_i}{\sum_{i=1}^m c_{ik}x_{ik}}\\ \sum_{j=1}^n \lambda_j x_{ij} \le x_i, & i=1,2,\ldots,m,\\ \sum_{j=1}^n \lambda_j y_{rj} \ge y_{rk}, & r=1,2,\ldots,s,\\ \lambda_j, x_i\ge 0.\end{cases}$$

Model 5.

$$(\mathrm{CE}_k)^U_\alpha = \max_{\substack{(x_{ij})^L_\alpha\le x_{ij}\le(x_{ij})^U_\alpha\\ (y_{rj})^L_\alpha\le y_{rj}\le(y_{rj})^U_\alpha\\ (c_{ik})^L_\alpha\le c_{ik}\le(c_{ik})^U_\alpha}} \begin{cases}\min \mathrm{CE}_k = \dfrac{\sum_{i=1}^m c_{ik}x_i}{\sum_{i=1}^m c_{ik}x_{ik}}\\ \sum_{j=1}^n \lambda_j x_{ij} \le x_i, & i=1,2,\ldots,m,\\ \sum_{j=1}^n \lambda_j y_{rj} \ge y_{rk}, & r=1,2,\ldots,s,\\ \lambda_j, x_i\ge 0.\end{cases}$$

Models 4 and 5 are employed for evaluating the lower and upper bounds of cost efficiency, respectively, based on the α-cut. Therefore, the CE-DEA Model 6, presented below, determines the lower bound of CE_k. Specifically, the input price for DMU_k is adjusted at the lower bound level in the objective function. In the constraints, the levels of inputs and outputs are tailored against the DMU being assessed while supporting the other DMUs. Inputs are set at upper bounds, and outputs are set at lower bounds for DMU_k, while inputs are arranged at lower bounds and outputs are arranged at upper bounds for other units. Hence, the lower bound CE-DEA Model 6 for DMU_k is as follows.

Model 6.

$$\min(\text{CE}_k)_\alpha^L = \frac{\sum_{i=1}^m (c_{ik})_\alpha^L x_i}{\sum_{i=1}^m (c_{ik})_\alpha^L (x_{ik})_\alpha^U}$$

subject to

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j (x_{ij})_\alpha^L + \lambda_k (x_{ik})_\alpha^U \quad \leq \quad x_i, \qquad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j (y_{rj})_\alpha^U + \lambda_k (y_{rk})_\alpha^L \quad \geq \quad (y_{rk})_\alpha^L, \qquad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \geq 0.$$

In the below-presented Model 7, the upper bound of CE_k is evaluated using a CE-DEA approach. The objective function adjusts the input price of DMU_k at the upper bound level, while the levels of inputs and outputs in the constraints are customized against the evaluated DMU and in support of the other DMUs. For DMU_k, the inputs are set at lower bounds and outputs at upper bounds, whereas for other units, inputs are arranged at upper bounds and outputs at lower bounds. In this way, the upper bound CE-DEA Model 7 for DMU_k is obtained.

Model 7.

$$\min(\text{CE}_k)_\alpha^U = \frac{\sum_{i=1}^m (c_{ik})_\alpha^U x_i}{\sum_{i=1}^m (c_{ik})_\alpha^U (x_{ik})_\alpha^L}$$

subject to

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j (x_{ij})_\alpha^U + \lambda_k (x_{ik})_\alpha^L \quad \leq \quad x_i, \qquad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j (y_{rj})_\alpha^L + \lambda_k (y_{rk})_\alpha^U \quad \geq \quad (y_{rk})_\alpha^U, \qquad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \geq 0.$$

The α-cuts of $\tilde{x}_{ij}^I, \tilde{y}_{rj}^I$ and $\tilde{c}_{ik}^I$ for $\alpha \in (0, 1]$ are given by

$$(\tilde{x}_{ij}^I)_\alpha = [(x_{ij})_\alpha^L, (x_{ij})_\alpha^U] = \left[\alpha x_{ij}^M + (1-\alpha) x_{ij}^L, \alpha x_{ij}^M + (1-\alpha) x_{ij}^U\right],$$

$$(\tilde{y}_{rj}^I)_\alpha = [(y_{rj})_\alpha^L, (y_{rj})_\alpha^U] = \left[\alpha y_{rj}^M + (1-\alpha) y_{rj}^L, \alpha y_{rj}^M + (1-\alpha) y_{rj}^U\right],$$

$$(\tilde{c}_{ik}^I)_\alpha = [(c_{ik})_\alpha^L, (c_{ik})_\alpha^U] = \left[\alpha c_{ik}^M + (1-\alpha)c_{ik}^L, \alpha c_{ik}^M + (1-\alpha)c_{ik}^U\right].$$

Using α-cuts, Models 6 and 7 are transformed into Models 8 and 9, respectively.

Model 8. For $\alpha \in (0, 1]$,

$$\min(\text{CE}_k)_\alpha^L = \frac{\sum_{i=1}^m \left(\alpha c_{ik}^M + (1-\alpha)c_{ik}^L\right) x_i}{\sum_{i=1}^m \left(\alpha c_{ik}^M + (1-\alpha)c_{ik}^L\right)\left(\alpha x_{ik}^M + (1-\alpha)x_{ik}^U\right)}$$

subject to

$$\sum_{\substack{j=1 \\ j\neq k}}^n \lambda_j \left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^L\right) + \lambda_k \left(\alpha x_{ik}^M + (1-\alpha)x_{ik}^U\right) \le x_i, \ i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1 \\ j\neq k}}^n \lambda_j \left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^U\right) + \lambda_k \left(\alpha y_{rk}^M + (1-\alpha)y_{rk}^L\right) \ge \left(\alpha y_{rk}^M + (1-\alpha)y_{rk}^L\right), \ r = 1, 2, \ldots, s,$$

$$\lambda_j, \ x_i \ge 0.$$

Model 9. For $\alpha \in (0, 1]$,

$$\min(\text{CE}_k)_\alpha^U = \frac{\sum_{i=1}^m \left(\alpha c_{ik}^M + (1-\alpha)c_{ik}^U\right) x_{ik}}{\sum_{i=1}^m \left(\alpha c_{ik}^M + (1-\alpha)c_{ik}^U\right)\left(\alpha x_{ik}^M + (1-\alpha)x_{ik}^L\right)}$$

subject to

$$\sum_{\substack{j=1 \\ j\neq k}}^n \lambda_j \left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^U\right) + \lambda_k \left(\alpha x_{ik}^M + (1-\alpha)x_{ik}^L\right) \le x_i, \ i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1 \\ j\neq k}}^n \lambda_j \left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^L\right) + \lambda_k \left(\alpha y_{rk}^M + (1-\alpha)y_{rk}^U\right) \ge \left(\alpha y_{rk}^M + (1-\alpha)y_{rk}^U\right), \ r = 1, 2, \ldots, s,$$

$$\lambda_j, \ x_i \ge 0.$$

Definition 7. *Efficient and inefficient DMU based on α-cut:*

- *If $(CE_k)_\alpha^{L*} = (CE_k)_\alpha^{U*} = 1$ for any $\alpha \in (0, 1]$, then DMU_k is called α-based strongly cost-efficient.*
- *If $(CE_k)_\alpha^{U*} = 1$ and $(CE_k)_\alpha^{L*} < 1$ for any $\alpha \in (0, 1]$, then DMU_k is called α-based weakly cost-efficient.*
- *If $(CE_k)_\alpha^{U*} < 1$ and $(CE_k)_\alpha^{L*} < 1$ for any $\alpha \in (0, 1]$, then DMU_k is called α-based cost-inefficient.*

Axiom. $(CE_k)_\alpha^{L*} \le (CE_k)_\alpha^{U*} \, \forall \, \alpha \in (0, 1]$, *i.e., The minimum achievable cost efficiency (CE) is either lower or equal to the maximum achievable CE for the DMU being evaluated.*

5.2 Model based on β-cut

Now, using β-cut, we obtain the lower and upper bound CEDEA Models. Model 3 is reduced to Models 10 and 11 as follows.

Model 10.

$$(\mathrm{CE}_k)'^L_\beta = \min_{\substack{(x_{ij})'^L_\beta \le x_{ij} \le (x_{ij})'^U_\beta \\ (y_{rj})'^L_\beta \le y_{rj} \le (y_{rj})'^U_\beta \\ (c_{ik})'^L_\beta \le c_{ik} \le (c_{ik})'^U_\beta}} \begin{cases} \min \mathrm{CE}_k = \dfrac{\sum_{i=1}^m c_{ik} x_i}{\sum_{i=1}^m c_{ik} x_{ik}} \\ \sum_{j=1}^n \lambda_j x_{ij} \quad \le \quad x_i, \qquad i = 1, 2, \ldots, m, \\ \sum_{j=1}^n \lambda_j y_{rj} \quad \ge \quad y_{rk}, \qquad r = 1, 2, \ldots, s, \\ \qquad \lambda_j, x_i \ge 0. \end{cases}$$

Model 11.

$$(\mathrm{CE}_k)'^U_\beta = \max_{\substack{(x_{ij})'^L_\beta \le x_{ij} \le (x_{ij})'^U_\beta \\ (y_{rj})'^L_\beta \le y_{rj} \le (y_{rj})'^U_\beta \\ (c_{ik})'^L_\beta \le c_{ik} \le (c_{ik})'^U_\beta}} \begin{cases} \min \mathrm{CE}_k = \dfrac{\sum_{i=1}^m c_{ik} x_i}{\sum_{i=1}^m c_{ik} x_{ik}} \\ \sum_{j=1}^n \lambda_j x_{ij} \quad \le \quad x_i, \qquad i = 1, 2, \ldots, m, \\ \sum_{j=1}^n \lambda_j y_{rj} \quad \ge \quad y_{rk}, \qquad r = 1, 2, \ldots, s, \\ \qquad \lambda_j, x_i \ge 0. \end{cases}$$

Models 10 and 11 are applied for the purpose of assessing the lower and upper bounds of cost efficiency with the assistance of the β-cut method. As a result, Model 12 (shown below) is a variant of the CE-DEA model that quantifies the minimum possible CE (CE_k) for DMU_k by utilizing β-cut. Within the objective function, the input price for DMU_k is adjusted to the lower limit level, while in the constraints, the input and output levels are customized to the highest and lowest levels for DMU_k, as well as the opposite arrangement for the other DMUs. Specifically, inputs are set to the highest limit and outputs to the lowest limit for DMU_k, while inputs are set to the lowest limit and outputs to the highest limit for other units. Hence, the resultant lower bound CE-DEA model for DMU_k (Model 12) using β-cut can be represented as follows.

Model 12.

$$\min (\mathrm{CE}_k)'^L_\beta = \frac{\sum_{i=1}^m (c_{ik})'^L_\beta x_i}{\sum_{i=1}^m (c_{ik})'^L_\beta (x_{ik})'^U_\beta}$$

subject to

$$\sum_{\substack{j=1 \\ j \ne k}}^n \lambda_j (x_{ij})'^L_\beta + \lambda_k (x_{ik})'^U_\beta \quad \le \quad x_i, \qquad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1 \\ j \ne k}}^n \lambda_j (y_{rj})'^U_\beta + \lambda_k (y_{rk})'^L_\beta \quad \ge \quad (y_{rk})'^L_\beta, \qquad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \ge 0.$$

The following is Model 13, which is a variant of the CE-DEA model that assesses the maximum achievable cost efficiency of DMU$_k$ by utilizing the β-cut approach. Within the objective function, the input price for DMU$_k$ is adjusted to the upper limit level, while in the constraints, the input and output levels are customized at their lowest and highest levels for DMU$_k$, as well as the opposite configuration for the other DMUs. Thus, the resulting upper bound CE-DEA model for DMU$_k$ (Model 13) using β-cut can be represented as follows.

Model 13.

$$\min(\text{CE}_k)'^U_\beta = \frac{\sum_{i=1}^m (c_{ik})'^U_\beta x_i}{\sum_{i=1}^m (c_{ik})'^U_\beta (x_{ik})'^L_\beta}$$

subject to

$$\sum_{\substack{j=1\\j\neq k}}^n \lambda_j (x_{ij})'^U_\beta + \lambda_k (x_{ik})'^L_\beta \quad \leq \quad x_i, \qquad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1\\j\neq k}}^n \lambda_j (y_{rj})'^L_\beta + \lambda_k (y_{rk})'^U_\beta \quad \geq \quad (y_{rk})'^U_\beta, \qquad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \geq 0.$$

The β-cuts of $\tilde{x}^I_{ij}, \tilde{y}^I_{rj}$ and $\tilde{c}^I_{ik}$ are given by

$$(\tilde{x}^I_{ij})_\beta = [(x_{ij})'^L_\beta, (x_{ij})'^U_\beta] = \left[\beta x'^L_{ij} + (1-\beta)x^M_{ij}, \beta x'^U_{ij} + (1-\beta)x^M_{ij}\right],$$
$$(\tilde{y}^I_{rj})_\beta = [(y_{rj})'^L_\beta, (y_{rj})'^U_\beta] = \left[\beta y'^L_{rj} + (1-\beta)y^M_{rj}, \beta y'^U_{rj} + (1-\beta)y^M_{rj}\right],$$
$$(\tilde{c}^I_{ik})_\beta = [(c_{ik})'^L_\beta, (c_{ik})'^U_\beta] = \left[\beta c'^L_{ik} + (1-\beta)c^M_{ik}, \beta c'^U_{ik} + (1-\beta)x^M_{ik}\right].$$

Using β-cuts, Models 12 and 13 are transformed into Models 14 and 15, respectively.

Model 14. For $\beta \in [0, 1)$,

$$\min(\text{CE}_k)'^L_\beta = \frac{\sum_{i=1}^m \left(\beta c'^L_{ik} + (1-\beta)c^M_{ik}\right) x_i}{\sum_{i=1}^m \left(\beta c'^L_{ik} + (1-\alpha)c^M_{ik}\right)\left(\beta x'^U_{ik} + (1-\beta)x^M_{ik}\right)}$$

subject to

$$\sum_{\substack{j=1\\j\neq k}}^n \lambda_j \left(\beta x'^L_{ij} + (1-\beta)x^M_{ij}\right) + \lambda_k \left(\beta x'^U_{ik} + (1-\beta)x^M_{ik}\right) \ \leq\ x_i, \quad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1\\j\neq k}}^n \lambda_j \left(\beta y'^U_{rj} + (1-\beta)y^M_{rj}\right) + \lambda_k \left(\beta y'^L_{rk} + (1-\beta)y^M_{rk}\right) \ \geq\ \left(\beta y'^L_{rk} + (1-\beta)y^M_{rk}\right), \quad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \geq 0.$$

Model 15. For $\beta \in [0, 1)$,

$$\min(\mathrm{CE}_k)'^{U}_{\beta} = \frac{\sum_{i=1}^{m} \left(\beta c'^{U}_{ik} + (1-\beta)c^{M}_{ik}\right) x_i}{\sum_{i=1}^{m} \left(\beta c'^{U}_{ik} + (1-\beta)c^{M}_{ik}\right)\left(\beta x'^{L}_{ik} + (1-\beta)x^{M}_{ik}\right)}$$

subject to

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j \left(\beta x'^{U}_{ij} + (1-\beta)x^{M}_{ij}\right) + \lambda_k \left(\beta x'^{L}_{ik} + (1-\beta)x^{M}_{ik}\right) \leq x_i, \quad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j \left(\beta y'^{L}_{rj} + (1-\beta)y^{M}_{rj}\right) + \lambda_k \left(\beta y'^{U}_{rk} + (1-\beta)y^{M}_{rk}\right) \geq \left(\beta y'^{U}_{rk} + (1-\beta)y^{M}_{rk}\right), \quad r = 1, 2, \ldots, s,$$

$$\lambda_j,\ x_i \geq 0.$$

Definition 8. *Efficient and inefficient DMU based on β-cut:*

- *If $(CE_k)'^{L*}_{\beta} = (CE_k)'^{U*}_{\beta} = 1$ for any $\beta \in [0, 1)$, then DMU$_k$ is called β-based strongly cost-efficient.*
- *If $(CE_k)'^{U*}_{\beta} = 1$ and $(CE_k)'^{L*}_{\beta} < 1$ for any $\beta \in [0, 1)$, then DMU$_k$ is called β-based weakly cost-efficient.*
- *If $(CE_k)'^{U*}_{\beta} < 1$ and $(CE_k)'^{L*}_{\beta} < 1$ for any $\beta \in [0, 1)$, then DMU$_k$ is called β-based cost-inefficient.*

Axiom. $(CE_k)'^{L*}_{\beta} \leq (CE_k)'^{U*}_{\beta}\, \forall\, \beta \in [0, 1)$, *i.e., the lower limit of cost efficiency for the DMU under consideration is less than or equal to its upper limit of cost efficiency.*

The overall methodology is presented in the flowchart on Figure 3.

6 Numerical example

This section presents an illustrative numerical instance to explicate the proposed models. The case examined in this instance involves five DMUs, each with two IF inputs and two IF outputs, along with IF input prices corresponding to these two IF inputs, all presented as Triangular Intuitionistic Fuzzy Numbers (TIFNs). The IF input-output data and IF input prices for all 5 DMUs are presented in Table 1.

The efficiencies of DMUs are calculated using the proposed models. The problem is formulated for each DMU by using the proposed IFCEM (Model 2) and solved by using the Models 8, 9, 14 and 15.

The step-by-step solution process is presented below:

- Problem Formulation: In this step, the problem for each DMU is formulated using the proposed Model 2. For illustration, we will formulate and solve the problem for DMU$_A$. Similarly, the problem can be formulated and solved for all other DMUs also. The problem for DMU$_A$ can be formulated as follows.

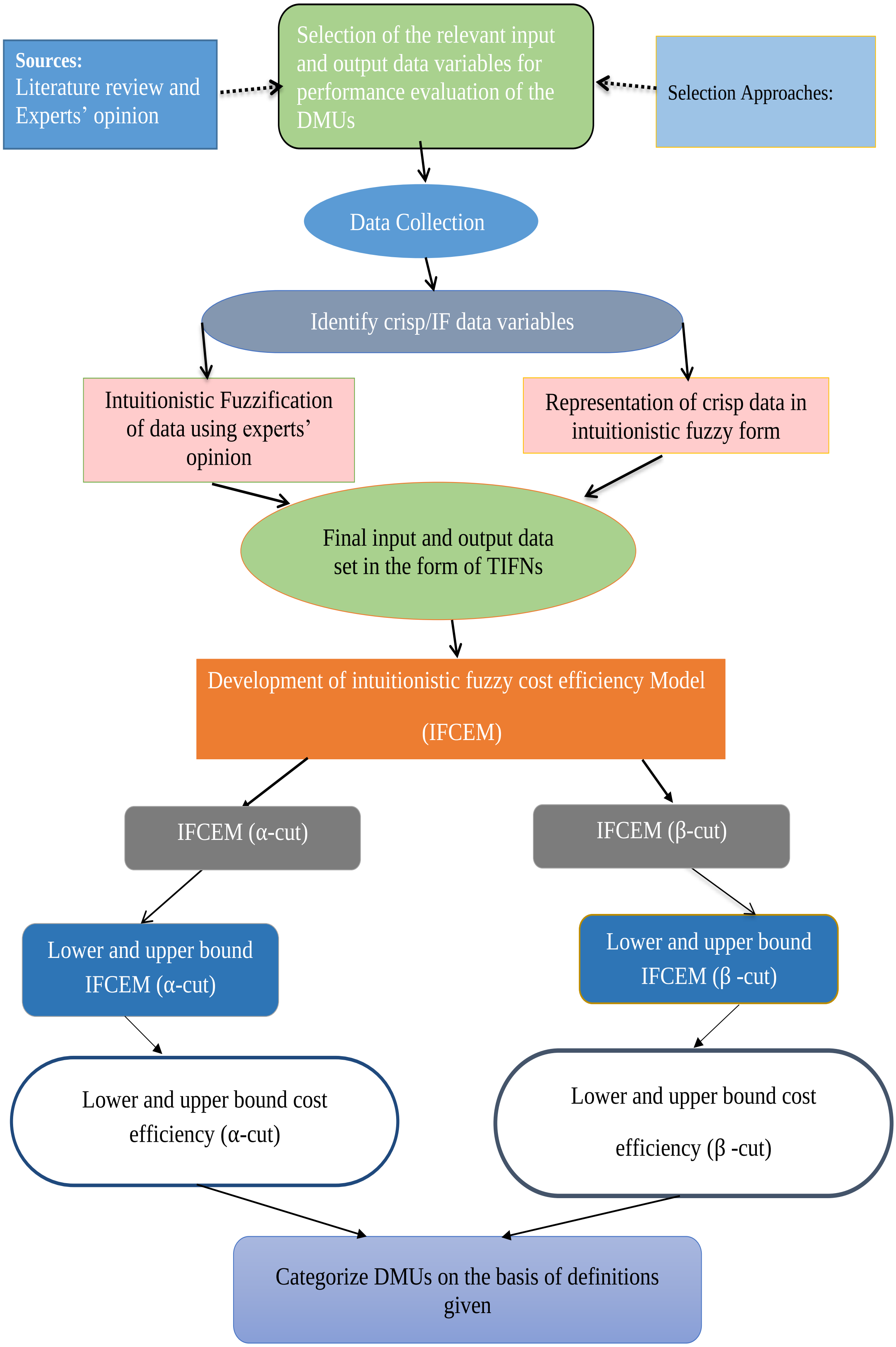

Figure 3. Efficiency of DMUs

Table 1. IF input-output data and IF input prices for DMUs (Source: Self-generated)

	IF Inputs	IF Outputs	Input Prices
DMU	$(x_1^L, x_1^M, x_1^U; x_1'^L, x_1^M, x_1'^U)$	$(y_1^L, y_1^M, y_1^U; y_1'^L, y_1^M, y_1'^U)$	$(c_1^L, c_1^M, c_1^U; c_1'^L, c_1^M, c_1'^U)$
A	(49.4, 53, 56.6; 42.3, 53, 62)	(70, 77.5, 85; 62, 77.5, 80.5)	(4.7, 5, 5.3; 4, 5, 8)
B	(20, 25, 30; 18.5, 25, 31.5)	(41.5, 49.6, 57.7; 36.5, 49.6, 53)	(3.5, 5, 6.5; 3.1, 5, 7.9)
C	(11.5, 18, 24.5; 10.5, 18, 25.2)	(21.6, 26.5, 31.4; 20.6, 26.5, 38.6)	(7.2, 8, 8.8; 7, 8, 11)
D	(12.1, 18, 23.9; 11.6, 18, 25.5)	(34.2, 37.5, 40.8; 28, 37.5, 45.5)	(7.2, 9, 10.8; 6.5, 9, 15.2)
E	(29.1, 32, 34.9; 28.4, 32, 36.6)	(59.2, 64, 68.8; 56, 64, 72)	(6.8, 7, 7.2; 6, 7, 7.3)
DMU	$(x_2^L, x_2^M, x_2^U; x_2'^L, x_2^M, x_2'^U)$	$(y_2^L, y_2^M, y_2^U; y_2'^L, y_2^M, y_2'^U)$	$(c_2^L, c_2^M, c_2^U; c_2'^L, c_2^M, c_2'^U)$
A	(40.5, 45, 49.5; 38.3, 45, 50.2)	(28.8, 35.4, 42; 28.5, 35.4, 42)	(4.5, 5, 5.5; 2.2, 5, 9.2)
B	(41.6, 46.5, 51.4; 40, 46.5, 52.5)	(28.9, 34.7, 40.5; 27, 34.7, 42)	(4.5, 6, 7.5; 4.3, 6, 11.7)
C	(11.9, 15.7, 19.5; 10.5, 15.7, 20.6)	(29.4, 37.6, 45.8; 28.5, 37.6, 47.5)	(6.3, 7, 7.7; 6.2, 7, 12)
D	(20.1, 25.5, 30.9; 18.6, 25.5, 31.5)	(40.9, 47.5, 54.1; 39.5, 47.5, 55)	(5, 5.5, 6; 3, 5.5, 10.5)
E	(20.3, 25, 29.7; 19.5, 25, 28)	(72.9, 76.4, 79.9; 68, 76.4, 82)	(1.75, 2, 2.25; 1.25, 2, 3.7)

- **Cost efficiency evaluation for DMU$_A$:**

$$\min \tilde{\text{CE}}_A{}^I = \frac{\sum_{i=1}^{2} \tilde{c}_{iA}^I x_i}{\sum_{i=1}^{2} \tilde{c}_{iA}^I \tilde{x}_{iA}^I}$$

subject to

$$\begin{aligned} \sum_{j=1}^{5} \lambda_j \tilde{x}_{ij}^I &\leq x_i, & i = 1, 2, \\ \sum_{j=1}^{5} \lambda_j \tilde{y}_{rj}^I &\geq \tilde{y}_{rA}^I, & r = 1, 2, \\ x_1, x_2, \lambda_1, ..., \lambda_5 \geq 0. \end{aligned}$$

Now, we will solve the above IFCEM for DMU$_A$ using the proposed models. To determine the lower and upper bound CE efficiency scores, we will utilize Models 8 and 9 for the α-cut and Models 14 and 15 for the β-cut.

Models based on $\boldsymbol{\alpha}$-cut: $\forall\, \alpha \in (0, 1]$

- Lower bound cost efficiency for DMU$_A$

$$\min\,(\text{CE}_A)_\alpha^L = \frac{(5\alpha + 4.7(1-\alpha))\,x_1 + (5\alpha + 4.5(1-\alpha))\,x_2}{(5\alpha + 4.7(1-\alpha))\,(53\alpha + 56.6(1-\alpha)) + (5\alpha + 4.5(1-\alpha))\,(45\alpha + 49.5(1-\alpha))}$$

subject to

$$\begin{aligned} x_1 \geq\ & \lambda_1\,(53\alpha + 56.6(1-\alpha)) + \lambda_2\,(25\alpha + 20(1-\alpha)) + \lambda_3\,(18\alpha + 11.5(1-\alpha)) \\ & + \lambda_4\,(18\alpha + 12.1(1-\alpha)) + \lambda_5\,(32\alpha + 29.1(1-\alpha)), \end{aligned}$$

$$\begin{aligned} x_2 \geq\ & \lambda_1\,(45\alpha + 49.5(1-\alpha)) + \lambda_2\,(46.5\alpha + 41.6(1-\alpha)) + \lambda_3\,(15.7\alpha + 11.9(1-\alpha)) \\ & + \lambda_4\,(25.5\alpha + 20.1(1-\alpha)) + \lambda_5\,(25\alpha + 20.3(1-\alpha)), \end{aligned}$$

$$
\begin{aligned}
(77.5\alpha + 70(1-\alpha)) \leq \lambda_1 (77.5\alpha + 70(1-\alpha)) + \lambda_2 (49.6\alpha + 57.7(1-\alpha)) \\
+ \lambda_3 (26.5\alpha + 31.4(1-\alpha)) + \lambda_4 (37.5\alpha + 40.8(1-\alpha)) \\
+ \lambda_5 (64\alpha + 68.8(1-\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
(35.4\alpha + 28.8(1-\alpha)) \leq \lambda_1 (35.4\alpha + 28.8(1-\alpha)) + \lambda_2 (34.7\alpha + 40.5(1-\alpha)) \\
+ \lambda_3 (37.6\alpha + 45.8(1-\alpha)) + \lambda_4 (47.5\alpha + 54.1(1-\alpha)) \\
+ \lambda_5 (76.4\alpha + 79.9(1-\alpha)),
\end{aligned}
$$

$$x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.$$

- Upper bound cost efficiency for DMU_A

$$\min\,(\text{CE}_A)_\alpha^U = \frac{(5\alpha + 5.3(1-\alpha))\,x_1 + (5\alpha + 5.5(1-\alpha))\,x_2}{(5\alpha + 5.3(1-\alpha))\,(53\alpha + 49.4(1-\alpha)) + (5\alpha + 5.5(1-\alpha))\,(45\alpha + 40.5(1-\alpha))}$$

subject to

$$
\begin{aligned}
x_1 \geq \lambda_1 (53\alpha + 49.4(1-\alpha)) + \lambda_2 (25\alpha + 30(1-\alpha)) + \lambda_3 (18\alpha + 24.5(1-\alpha)) \\
+ \lambda_4 (18\alpha + 23.9(1-\alpha)) + \lambda_5 (32\alpha + 34.9(1-\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
x_2 \geq \lambda_1 (45\alpha + 40.5(1-\alpha)) + \lambda_2 (46.5\alpha + 51.4(1-\alpha)) + \lambda_3 (15.7\alpha + 19.5(1-\alpha)) \\
+ \lambda_4 (25.5\alpha + 30.9(1-\alpha)) + \lambda_5 (25\alpha + 29.7(1-\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
(77.5\alpha + 85(1-\alpha)) \leq \lambda_1 (77.5\alpha + 85(1-\alpha) + \lambda_2 (49.6\alpha + 41.5(1-\alpha)) \\
+ \lambda_3 (26.5\alpha + 21.6(1-\alpha)) + \lambda_4 (37.5\alpha + 34.2(1-\alpha)) \\
+ \lambda_5 (64\alpha + 59.2(1-\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
(35.4\alpha + 42(1-\alpha)) \leq \lambda_1 (35.4\alpha + 42(1-\alpha)) + \lambda_2 (34.7\alpha + 28.9(1-\alpha)) \\
+ \lambda_3 (37.6\alpha + 29.4(1-\alpha)) + \lambda_4 (47.5\alpha + 40.9(1-\alpha)) \\
+ \lambda_5 (76.4\alpha + 72.9(1-\alpha)),
\end{aligned}
$$

$$x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.$$

Models based on $\boldsymbol{\beta}$-cut: $\forall\, \beta \in [0, 1)$

- Lower bound cost efficiency for DMU_A

$$\min\,(\text{CE}_A)_\beta^{\prime L} = \frac{(4\beta + 5(1-\beta))\,x_1 + (2.2\beta + 5(1-\beta))\,x_2}{(4\beta + 5(1-\beta))\,(62\beta + 53(1-\beta)) + (2.2\beta + 5(1-\beta))\,(50.2\beta + 45(1-\beta))}$$

subject to

$$
\begin{aligned}
x_1 \geq \lambda_1 (62\beta + 53(1-\beta)) + \lambda_2 (18.5\beta + 25(1-\beta)) + \lambda_3 (10.5\beta + 18(1-\beta)) \\
+ \lambda_4 (11.6\beta + 18(1-\beta)) + \lambda_5 (28.4\beta + 32(1-\beta)),
\end{aligned}
$$

$$
\begin{aligned}
x_2 \geq \lambda_1 (50.2\beta + 45(1-\beta)) + \lambda_2 (40\beta + 46.5(1-\beta)) + \lambda_3 (10.5\beta + 15.7(1-\beta)) \\
+ \lambda_4 (18.6\beta + 25.5(1-\beta)) + \lambda_5 (19.5\beta + 25(1-\beta)),
\end{aligned}
$$

$$
\begin{aligned}
(62\beta + 77.5(1-\beta)) \leq\ & \lambda_1 (62\beta + 77.5(1-\beta)) + \lambda_2 (53\beta + 49.6(1-\beta)) \\
& + \lambda_3 (38.6\beta + 26.5(1-\beta)) \\
& + \lambda_4 (45.5\beta + 37.5(1-\beta)) + \lambda_5 (72\beta + 64(1-\beta)),
\end{aligned}
$$

$$
\begin{aligned}
(28.5\beta + 35.4(1-\beta)) \leq\ & \lambda_1 (28.5\beta + 35.4(1-\beta)) + \lambda_2 (42\beta + 34.7(1-\beta)) \\
& + \lambda_3 (47.5\beta + 37.6(1-\beta)) \\
& + \lambda_4 (55\beta + 47.5(1-\beta)) + \lambda_5 (82\beta + 76.4(1-\beta)),
\end{aligned}
$$

$$x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.$$

- Upper bound cost efficiency for DMU_A

$$\min (\text{CE}_A)'^U_\beta = \frac{(8\beta + 5(1-\beta))\, x_1 + (9.2\beta + 5(1-\beta))\, x_2}{(8\beta + 5(1-\beta))\,(42.3\beta + 53(1-\beta)) + (9.2\beta + 5(1-\beta))\,(38.3\beta + 45(1-\beta))}$$

subject to

$$
\begin{aligned}
x_1 \geq\ & \lambda_1 (42.3\beta + 53(1-\beta)) + \lambda_2 (31.5\beta + 25(1-\beta)) + \lambda_3 (25.2\beta + 18(1-\beta)) \\
& + \lambda_4 (25.5\beta + 18(1-\beta)) + \lambda_5 (36.6\beta + 32(1-\beta)),
\end{aligned}
$$

$$
\begin{aligned}
x_2 \geq\ & \lambda_1 (38.3\beta + 45(1-\beta)) + \lambda_2 (52.5\beta + 46.5(1-\beta)) + \lambda_3 (20.6\beta + 15.7(1-\beta)) \\
& + \lambda_4 (31.5\beta + 25.5(1-\beta)) + \lambda_5 (28\beta + 25(1-\beta)),
\end{aligned}
$$

$$
\begin{aligned}
(80.5\beta + 77.5(1-\beta)) \leq\ & \lambda_1 (80.5\beta + 77.5(1-\beta) + \lambda_2 (36.5\beta + 49.6(1-\beta)) \\
& + \lambda_3 (20.6\beta + 26.5(1-\beta)) \\
& + \lambda_4 (28\beta + 37.5(1-\beta)) + \lambda_5 (56\beta + 64(1-\beta)),
\end{aligned}
$$

$$
\begin{aligned}
(42\beta + 35.4(1-\beta)) \leq\ & \lambda_1 (42\beta + 35.4(1-\beta)) + \lambda_2 (27\beta + 34.7(1-\beta)) \\
& + \lambda_3 (28.5\beta + 37.6(1-\beta)) \\
& + \lambda_4 (39.5\beta + 47.5(1-\beta)) + \lambda_5 (68\beta + 76.4(1-\beta)),
\end{aligned}
$$

$$x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.$$

The linear models are solved by Lingo software for different values of α and β, ranging from 0 to 1 with a step of 0.1. The lower and upper bound cost-efficiencies are presented in Table 2 (using Models 8 and 9) and Table 3 (using Models 14 and 15) for α-cut and β-cut respectively for all DMUs.

Table 2. The lower and upper cost-efficiencies based on α-cuts

DMU	A	B	C	D	E
$\alpha\downarrow$	$[(\text{CE}_A)^L_\alpha, (\text{CE}_A)^U_\alpha]$	$[(\text{CE}_B)^L_\alpha, (\text{CE}_B)^U_\alpha]$	$[(\text{CE}_C)^L_\alpha, (\text{CE}_C)^U_\alpha]$	$[(\text{CE}_D)^L_\alpha, (\text{CE}_D)^U_\alpha]$	$[(\text{CE}_E)^L_\alpha, (\text{CE}_E)^U_\alpha]$
0	[0.474, 1]	[0.346, 0.991]	[0.388, 1]	[0.473, 1]	[0.589, 1]
0.1	[0.494, 0.993]	[0.366, 0.942]	[0.428, 1]	[0.502, 1]	[0.631, 1]
0.2	[0.514, 0.956]	[0.387, 0.896]	[0.472, 1]	[0.534, 1]	[0.675, 1]
0.3	[0.535, 0.921]	[0.409, 0.851]	[0.512, 1]	[0.569, 1]	[0.720, 1]
0.4	[0.557, 0.886]	[0.432, 0.809]	[0.549, 1]	[0.607, 1]	[0.768, 1]
0.5	[0.579, 0.853]	[0.456, 0.769]	[0.589, 1]	[0.649, 1]	[0.818, 1]
0.6	[0.602, 0.821]	[0.481, 0.731]	[0.631, 1]	[0.691, 1]	[0.870, 1]
0.7	[0.626, 0.79]	[0.507, 0.694]	[0.677, 1]	[0.733, 1]	[0.924, 1]
0.8	[0.651, 0.76]	[0.535, 0.659]	[0.726, 0.962]	[0.777, 0.986]	[0.981, 1]
0.9	[0.677, 0.732]	[0.564, 0.626]	[0.778, 0.896]	[0.824, 0.929]	[1, 1]
1	[0.704, 0.704]	[0.594, 0.594]	[0.835, 0.835]	[0.875, 0.875]	[1, 1]

Table 3. The lower and upper cost-efficiencies based on β-cuts

DMU	A	B	C	D	E
$\beta\downarrow$	$[(\text{CE}_A)'^L_\beta, (\text{CE}_A)'^U_\beta]$	$[(\text{CE}_B)'^L_\beta, (\text{CE}_B)'^U_\beta]$	$[(\text{CE}_C)'^L_\beta, (\text{CE}_C)'^U_\beta]$	$[(\text{CE}_D)'^L_\beta, (\text{CE}_D)'^U_\beta]$	$[(\text{CE}_E)'^L_\beta, (\text{CE}_E)'^U_\beta]$
0	[0.704, 0.704]	[0.594, 0.594]	[0.835, 0.835]	[0.875, 0.875]	[1, 1]
0.1	[0.662, 0.738]	[0.554, 0.616]	[0.802, 0.858]	[0.846, 0.909]	[1, 1]
0.2	[0.621, 0.774]	[0.517, 0.638]	[0.77, 0.882]	[0.819, 0.945]	[0.964, 1]
0.3	[0.584, 0.811]	[0.481, 0.662]	[0.739, 0.906]	[0.794, 0.987]	[0.905, 1]
0.4	[0.548, 0.851]	[0.448, 0.688]	[0.71, 0.929]	[0.769, 1]	[0.864, 1]
0.5	[0.515, 0.894]	[0.416, 0.715]	[0.683, 0.954]	[0.746, 1]	[0.824, 1]
0.6	[0.483, 0.938]	[0.386, 0.743]	[0.657, 0.982]	[0.724, 1]	[0.785, 1]
0.7	[0.432, 0.985]	[0.358, 0.773]	[0.632, 1]	[0.703, 1]	[0.748, 1]
0.8	[0.38, 1]	[0.331, 0.804]	[0.608, 1]	[0.684, 1]	[0.712, 1]
0.9	[0.334, 1]	[0.306, 0.837]	[0.586, 1]	[0.665, 1]	[0.678, 1]
1	[0.291, 1]	[0.282, 0.872]	[0.564, 1]	[0.648, 1]	[0.644, 1]

After examining Table 2 and 3; the DMUs can be classified into three groups, according to Definitions 7 and 8, based on α-cut and β-cut techniques, respectively as shown in Table 4.

Table 4. Efficiency categorization

Efficiency	strongly cost-efficient	weakly cost-efficient	cost-inefficient
Based on α-cut	E	A, C, D	B
Based on β-cut	E	A, C, D	B

So, based on the results of Table 2, 3 and 4 we can say that DMU E is strongly cost-efficient, DMUs A, C and D are weakly cost-efficient and DMU B is a cost-inefficient DMU, based on α- and β-cuts.

7 Conclusion

The present study entails the expansion of the traditional DEA cost efficiency model to incorporate the intuitionistic fuzzy nature of the data, in the context of DEA. The conventional CE-DEA model is used for crisp data, but for all intents and purposes of real-life problems, we developed IFCEMs. The proposed IFCEM is a fully intuitionistic fuzzy model that can be applied to all types of data, i.e., we do not need any longer conventional CE-DEA models developed for crisp and fuzzy data because every crisp and fuzzy number can be represented as IF numbers. This study develops lower and upper IFCEMs based on α and β-cuts, respectively, by using arithmetic operations on the conventional CE DEA model. The current work attempts to generalize the conventional CE DEA models into IFCE DEA models and give new insight into solving IFCE DEA models. However, the proposed models are developed by taking TIFNs.

The proposed IFCEMs are applied to a numerical example in which we have 5 DMUs, each having two inputs, two outputs and inputs prices; all are taken as TIFNs. Using α and β-cuts of IFCEMs, we obtained lower and upper-cost efficiencies of these 5 DMUs presented in Tables 2 and 3. By Definitions 7 and 8, we conclude that DMUs A, C and D are weakly cost-efficient, DMU B is cost-inefficient and DMU E is strongly cost-efficient.

DMs can use trapezoidal fuzzy numbers (TrFNs), L-R type fuzzy numbers, or any other form of IF numbers to represent the imprecision and hesitation in input-output and cost data. Second, in the present work, the proposed IF CE-DEA approach is studied under constant returns to scale (CRS), but this can be extended to the variable returns to scale (VRS) approach.

In the future, research can explore results from other DEA models in unfavorable settings, such as stochastic and robust, and develop new models and approaches for the performance evaluation of DMUs based on cost and revenue efficiencies.

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