Distances and orderings in a family of intuitionistic fuzzy numbers

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Abstract

Two families of metrics in space of intuitionistic fuzzy numbers are considered. A method of ranking intuitionistic fuzzy numbers based on these metrics is also suggested and invesigated.

Keywords: intuitionistic fuzzy numbers, metrics, ranking methods, expected value.

1 Introduction

A membership function of a classical fuzzy set assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The degree of nonbelongingness is just automatically the complement to 1 of the membership degree. However, a human being who expresses the degree of membership of given element in a fuzzy set very often does not express corresponding degree of nonmembership as the complement to 1. This reflects a well known psychological fact that the linguistic negation not always identifies with logical negation. Thus Atanassov [1] introduced the concept of an intuitionistic fuzzy set which is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness, respectively. This idea, which is a natural generalization of usual fuzzy set, seems to be useful in modeling many real life situations.

Ranking fuzzy numbers is one of the fundamental problems of fuzzy arithmetic and fuzzy decision making. It is due to the fact that fuzzy numbers are not linearly ordered. This problem is also important in the case of intuitionistic fuzzy numbers. In this paper we propose and investigate two families of metrics in space of intuitionistic fuzzy numbers. Then we suggest a method of ranking intuitionistic fuzzy numbers based on these metrics.

2 Basic notions

Let X denote a universe of discourse. Then an intuitionistic fuzzy set A in X (see [1], [2]) is a set of ordered triples

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{1}$$

where $\mu_A, \nu_A : X \to [0,1]$ are functions such that

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \qquad \forall x \in X. \tag{2}$$

For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively. For each element $x \in X$ we can compute the so-called, the intuitionistic fuzzy index of x in A defined as follows

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x). \tag{3}$$

Of course, a fuzzy set is a particular case of the intuitionistic fuzzy set with $\nu_A(x) = 1 - \mu_A(x)$.

Atanassov has also defined two kinds of $\alpha-$cuts for intuitionistic fuzzy sets. Namely

$$\begin{aligned} A_\alpha &= \{x \in X : \mu_A(x) \ge \alpha\}, & (4)\\ A^\alpha &= \{x \in X : \nu_A(x) \le \alpha\}. & (5) \end{aligned}$$

Since from now on we will restrict our consideration to intuitionistic fuzzy numbers, henceforth our universe of discourse would be the real line, i.e. $X = \mathcal{R}$. We define an intuitionistic fuzzy number as follows (see [7]):

Definition 1 *An intuitionistic fuzzy subset* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ *of the real line is called an intuitionistic fuzzy number if*

(a) *A is if-normal (i.e. there exist at least two points* $x_0, x_1 \in X$ *such that* $\mu_A(x_0) = 1$ *and* $\nu_A(x_1) = 1$*),*

(b) *A is if-convex (i.e. its membership function μ is fuzzy convex and its nonmembership function is fuzzy concave),*

(c) μ_A *is upper semicontinuous and* ν_A *is lower semicontinuous,*

(d) $\mathrm{supp}A = cl\left(\{x \in X : \nu_A(x) < 1\}\right)$ *is bounded.*

From the definition given above we get at once that for any intuitionistic fuzzy number A there exist eight numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathcal{R}$ such that $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$ and four functions $f_A, g_A, h_A, k_A : \mathcal{R} \to [0,1]$, called the sides of a fuzzy number, where f_A and k_A are nondecreasing and g_A and h_A are nonincreasing, such that we can describe a membership function μ_A in a form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ f_A(x) & \text{if} \quad a_1 \leq x < a_2 \\ 1 & \text{if} \quad a_2 \leq x \leq a_3 \\ g_A(x) & \text{if} \quad a_3 < x \leq a_4 \\ 0 & \text{if} \quad a_4 < x. \end{cases}, \tag{6}$$

while a nonmembership function ν_A has a following form

$$\nu_A(x) = \begin{cases} 1 & \text{if} \quad x < b_1 \\ h_A(x) & \text{if} \quad b_1 \leq x < b_2 \\ 0 & \text{if} \quad b_2 \leq x \leq b_3 \\ k_A(x) & \text{if} \quad b_3 < x \leq b_4 \\ 1 & \text{if} \quad b_4 < x. \end{cases} \tag{7}$$

It is worth noting that each intuitionistic fuzzy number $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ is a conjunction of two fuzzy numbers: A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$. It is seen that $suppA^+ \subseteq suppA^-$.

A useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. Every $\alpha-$cut of a fuzzy number is a closed interval and a family of such intervals describes completely a fuzzy number under study. In the case of intuitionistic fuzzy numbers it is convenient to distinguish following $\alpha-$cuts: $(A^+)_\alpha$ and $(A^-)_\alpha$. It is easily seen that

$$(A^+)_\alpha = \{x \in X : \mu_A(x) \geq \alpha\} = A_\alpha, \tag{8}$$

$$\begin{aligned} (A^-)_\alpha &= \{x \in X : 1 - \nu_A(x) \geq \alpha\} \\ &= \{x \in X : \nu_A(x) \leq 1 - \alpha\} = A^{1-\alpha}. \end{aligned} \tag{9}$$

According to the definition it is seen at once that every $\alpha-$cut $(A^+)_\alpha$ or $(A^-)_\alpha$ is a closed interval. Hence we have $(A^+)_\alpha = [A_L^+(\alpha), A_U^+(\alpha)]$ and $(A^-)_\alpha = [A_L^-(\alpha), A_U^-(\alpha)]$, respectively, where

$$\begin{aligned} A_L^+(\alpha) &= \inf\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\}, \\ A_U^+(\alpha) &= \sup\{x \in \mathcal{R} : \mu_A(x) \geq \alpha\}, \\ A_L^-(\alpha) &= \inf\{x \in \mathcal{R} : \nu_A(x) \leq 1 - \alpha\}, \\ A_U^-(\alpha) &= \sup\{x \in \mathcal{R} : \nu_A(x) \leq 1 - \alpha\}. \end{aligned} \tag{10}$$

If the sides of the fuzzy number A are strictly monotone then by (6) and (7) one can see easily that $A_L^+(\alpha)$, $A_U^+(\alpha)$, $A_L^-(\alpha)$ and $A_U^-(\alpha)$ are inverse functions of f_A, g_A, h_A and k_A, respectively. In general, we may adopt the convention that $f_A^{-1}(\alpha) = A_L^+(\alpha)$, $g_A^{-1}(\alpha) = A_U^+(\alpha)$, $h_A^{-1}(\alpha) = A_L^-(\alpha)$ and $k_A^{-1}(\alpha) = A_U^-(\alpha)$.

3 Distances between intuitionistic fuzzy numbers

Various methods for measuring distances between intuitionistic fuzzy sets are considered in the literature (see [1], [5], [6], [9]). Below we suggest two families of metrics that seem to be useful for measuring distances between intuitionistic fuzzy numbers. We get

Definition 2 *The* $d_p(A,B)$ *distance, indexed by a parameter* $1 \leq p \leq \infty$*, for any two intuitionistic fuzzy numbers* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in \mathcal{R}\}$ *is given by*

$$\begin{aligned} d_p(A,B) = \Bigg(& \frac{1}{4}\int_0^1 \left|A_L^+(\alpha) - B_L^+(\alpha)\right|^p d\alpha \\ & + \frac{1}{4}\int_0^1 \left|A_U^+(\alpha) - B_U^+(\alpha)\right|^p d\alpha \\ & + \frac{1}{4}\int_0^1 \left|A_L^-(\alpha) - B_L^-(\alpha)\right|^p d\alpha \\ & + \frac{1}{4}\int_0^1 \left|A_U^-(\alpha) - B_U^-(\alpha)\right|^p d\alpha \Bigg)^{1/p} \end{aligned} \tag{11}$$

for $1 \leq p < \infty$ *and*

$$\begin{aligned} d_p(A,B) = & \frac{1}{4}\sup_{0<\alpha\leq 1} \left|A_L^+(\alpha) - B_L^+(\alpha)\right| \\ & + \frac{1}{4}\sup_{0<\alpha\leq 1} \left|A_U^+(\alpha) - B_U^+(\alpha)\right| \\ & + \frac{1}{4}\sup_{0<\alpha\leq 1} \left|A_L^-(\alpha) - B_L^-(\alpha)\right| \\ & + \frac{1}{4}\sup_{0<\alpha\leq 1} \left|A_U^-(\alpha) - B_U^-(\alpha)\right| \end{aligned} \tag{12}$$

for $p = \infty$.

Definition 3 *The* $\rho_p(A,B)$ *distance, indexed by a parameter* $1 \le p \le \infty$, *for any two intuitionistic fuzzy numbers* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in \mathcal{R}\}$ *is given by*

$$\begin{aligned}\rho_p(A,B) &= \max\left\{ \sqrt[p]{\int_0^1 \left|A_L^+(\alpha) - B_L^+(\alpha)\right|^p d\alpha}, \right.\\ &\quad \sqrt[p]{\int_0^1 \left|A_U^+(\alpha) - B_U^+(\alpha)\right|^p d\alpha}, \\ &\quad \sqrt[p]{\int_0^1 \left|A_L^-(\alpha) - B_L^-(\alpha)\right|^p d\alpha}, \\ &\quad \left. \sqrt[p]{\int_0^1 \left|A_U^-(\alpha) - B_U^-(\alpha)\right|^p d\alpha} \right\}\end{aligned} \tag{13}$$

for $1 \le p < \infty$ *and*

$$\begin{aligned}\rho_p(A,B) &= \max\left\{ \sup_{0<\alpha\le 1} \left|A_L^+(\alpha) - B_L^+(\alpha)\right|, \right.\\ &\quad \sup_{0<\alpha\le 1} \left|A_U^+(\alpha) - B_U^+(\alpha)\right|, \\ &\quad \sup_{0<\alpha\le 1} \left|A_L^-(\alpha) - B_L^-(\alpha)\right|, \\ &\quad \left. + \sup_{0<\alpha\le 1} \left|A_U^-(\alpha) - B_U^-(\alpha)\right| \right\}\end{aligned} \tag{14}$$

for $p = \infty$.

Let IFN' denote a space of all intuitionistic fuzzy numbers. We may partition IFN' into disjoint sets in such a way that two intuitionistic fuzzy numbers A and B belong to the same set if and only if the corresponding functions $A_L^+(\alpha)$, $A_U^+(\alpha)$, $A_L^-(\alpha)$, $A_U^-(\alpha)$ and $B_L^+(\alpha)$, $B_U^+(\alpha)$, $B_L^-(\alpha)$, $B_U^-(\alpha)$ differ only on a set of measure zero. This way we obtain a space IFN of equivalence classes. It is not misleading that we regard elements of the space IFN as intuitionistic fuzzy numbers and by integrals in (11) and (13) we mean the integrals of arbitrary representative of the class containing A.

A following theorem holds

Theorem 1 *Spaces* (IFN, d_p) *and* (IFN, ρ_p) *for* $1 \le p \le \infty$ *are metric spaces.*

4 Ranking intuitionistic fuzzy numbers

It is known that there is no unique linear ordering in a family of fuzzy numbers. Thus ranking fuzzy numbers is one of the fundamental problems of fuzzy arithmetic. The same is true in the case of intuitionistic fuzzy numbers. Below we suggest a method of ranking intuitionistic fuzzy numbers based on metrics introduced in the previous section. This method is a direct generalization of the method for ranking classical fuzzy numbers presented in [4].

Let us start from some definitions. Suppose $\mathcal{A} \subset IFN$ is a subfamily of all intuitionistic fuzzy numbers.

Definition 4 *An intuitionistic fuzzy number* $L(\mathcal{A})$ *is called the lower horizon of a given subfamily* $\mathcal{A}$ *if* $\sup(suppL(\mathcal{A})) \le \inf(suppA)$ *for any* $A \in \mathcal{A}$. *Similarly, an intuitionistic fuzzy number* $U(\mathcal{A})$ *is called the upper horizon of a given subfamily* $\mathcal{A}$ *if* $\inf(suppU(\mathcal{A})) \ge \sup(suppA)$ *for any* $A \in \mathcal{A}$.

It is obvious that $\mathcal{A}$ may have one or more horizons. For a fixed intuitionistic fuzzy number we may consider following subfamilies of IFN:

Definition 5 *Let* $H \in IFN$. *A subfamily* $\mathcal{H}_L(H) \subset IFN$ *of a form*

$$\mathcal{H}_L(H) = \{A \in IFN : \sup(suppH) \le \inf(suppA)\} \tag{15}$$

is said to be lower-dominated by the intuitionistic fuzzy number H. *Similarly, a subfamily* $\mathcal{H}_U(H) \subset IFN$ *of a form*

$$\mathcal{H}_U(H) = \{A \in IFN : \inf(suppH) \ge \sup(suppA)\} \tag{16}$$

is said to be upper-dominated by the intuitionistic fuzzy number H.

A following lemma is straightforward

Theorem 2 *Let* $H \in IFN$. *Then* $H = L(\mathcal{H}_L(H))$ *and* $H = U(\mathcal{H}_U(H))$.

Now we may propose two following orders:

Definition 6 *Let* $A, B \in \mathcal{A} \subset IFN$. *Moreover, let* $H = L(\mathcal{A})$ *and let* d *be a metric in* IFN. *The relation* $\succ_L$ *in* $\mathcal{A} \times \mathcal{A}$ *given by*

$$A \succ_L B \Longleftrightarrow d(A,H) \ge d(B,H) \tag{17}$$

is called the order respect to the lower horizon H.

Definition 7 *Let* $A, B \in \mathcal{A} \subset IFN$. *Moreover, let* $H = U(\mathcal{A})$ *and let* d *be a metric in* IFN. *The relation* $\succ_U$ *in* $\mathcal{A} \times \mathcal{A}$ *given by*

$$A \succ_U B \Longleftrightarrow d(A,H) \le d(B,H) \tag{18}$$

is called the order respect to the upper horizon H.

Of course, using different metrics, e.g. d_p or ρ_p given above, we may obtain different orders. Anyway, the following theorem holds

Theorem 3 *Let $H \in IFN$. Then $\mathcal{H}_L(H)$ is quasiordered by the relation $\succ_L$, while $\mathcal{H}_U(H)$ is quasiordered by the relation $\succ_U$. Moreover, both relations $\succ_L$and $\succ_U$ are connected.*

It should be noticed that both relations $\succ_L$ and $\succ_U$ are not antisymmetric and hence they are only quasi-ordering relations, not ordering relations. However, every quasi-ordering determines an equivalence relation and an ordering relation (on equivalence classes) in a natural way. Therefore the last theorem is of great importance since it makes possible to rank any subfamily of intuitionistic fuzzy numbers which is lower-dominated or upper-dominated that is very common in practical applications.

Now we show some properties of the proposed quasi-orderings based on the metrics introduced on Sec. 3. Namely

Theorem 4 *The quasi-order $\succ_L$ with respect to the lower horizon, based on the metric d_1 (i.e. d_p for $p = 1$), does not depend on the choice of the lower horizon. Similarly, the quasi-order $\succ_U$ with respect to the upper horizon, based on the metric d_1, does not depend on the choice of the upper horizon.*

Theorem 5 *The quasi-orders $\succ_L$ with respect to the lower horizon, based on the metric d_∞, ρ_1 and ρ_∞, do not depend on the choice of the lower horizon, provided that the horizon is a crisp number. Similarly, the quasi-orders $\succ_U$ with respect to the upper horizon, based on the metric d_∞, ρ_1 and ρ_∞, do not depend on the choice of the upper horizon, provided that the horizon is a crisp number.*

5 Expected value and orderings

The expected interval of an intuitionistic fuzzy number $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ is a crisp interval $\widetilde{EI}(A)$ given by (see [7])

$$\widetilde{EI}(A) = \left[\widetilde{E_*}(A), \widetilde{E^*}(A)\right] \tag{19}$$

where

$$\begin{aligned} \widetilde{E_*}(A) &= \tfrac{b_1+a_2}{2} + \tfrac{1}{2}\int_{b_1}^{b_2} h_A(x)dx - \tfrac{1}{2}\int_{a_1}^{a_2} f_A(x)dx, \\ \widetilde{E^*}(A) &= \tfrac{a_3+b_4}{2} + \tfrac{1}{2}\int_{a_3}^{a_4} g_A(x)dx - \tfrac{1}{2}\int_{b_3}^{b_4} k_A(x)dx. \end{aligned} \tag{20}$$

As in the case of the classical fuzzy sets we will define the expected value of intuitionistic fuzzy number as follows (compare [3], [8]):

Definition 8 *The expected value $\widetilde{EV}(A)$ of an intuitionistic fuzzy number $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{R}\}$ is the center of the expected interval of that intuitionistic fuzzy number, i.e.*

$$\widetilde{EV}(A) = \frac{\widetilde{E_*}(A) + \widetilde{E^*}(A)}{2}. \tag{21}$$

A following theorem holds

Theorem 6 *Let $\succ_L$ and $\succ_U$ denote the quasi-order with respect to the lower and upper horizon, respectively, based on the metric d_1 (i.e. d_p for $p = 1$). Then for any $A, B \in IFN$ we get*

$$A \succ_L B \Longleftrightarrow \widetilde{EV}(A) \geq \widetilde{EV}(B) \tag{22}$$

and

$$A \succ_U B \Longleftrightarrow \widetilde{EV}(A) \geq \widetilde{EV}(B). \tag{23}$$

As a natural consequence of Th. 4 and Th. 6 we get

Corollary *Quasi-order $\succ_L$ and $\succ_U$ with respect to the lower and upper horizon, respectively, based on the metric d_1, do not depend on the choice of any horizon. Moreover, these two quasi-orders are equivalent, i.e.*

$$A \succ_L B \Longleftrightarrow A \succ_U B. \tag{24}$$

6 Conclusions

In this paper we have introduced two families of metrics in a space of intuitionistic fuzzy numbers. We have also proposed a method of ranking intuitionistic fuzzy numbers based on the suggested metrics and considered some properties of these methods. Orderings based on the metric d_1 are of a special interest because of the close relationship with the concept of expected value of an intuitionistic fuzzy number. It is worth noting that these results are direct generalizations of the results obtained for the classical fuzzy numbers.

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