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Numerical solution of intuitionistic fuzzy differential equations by Runge–Kutta Method of order four

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Abstract: This paper presents solution for first order fuzzy differential equation by Runge–Kutta method of order four. This method is discussed in detail and this is followed by a complete error analysis. The accuracy and efficiency of the proposed method is illustrated by solving an intuitionistic fuzzy initial value problem.
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1 Introduction

The idea of intuitionistic fuzzy set was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. Atanassov [2] explored the concept of fuzzy set theory [10] by intuitionistic fuzzy set (IFS) theory. The existence and uniqueness of the solution of a differential equation with intuitionistic fuzzy data has been studied by several authors [8, 9]. Numerical Solution Of Intuitionistic Fuzzy Differential Equations By Euler and Taylor Methods has been introduced in [4]. In this paper, intuitionistic fuzzy Cauchy problem is solved numerically by Runge–Kutta of order four method based on [4] and establish that this method is better than Euler method.

This paper is organised as follows: In Section 2, some basic results on intuitionistic fuzzy sets and the metric space, which have been discussed in [7, 9], are given. Section 3 contains intuitionistic fuzzy differential equation whose numerical solution is the main interest of this paper. Solving numerically the intuitionistic fuzzy differential equation by Runge–Kutta method of order four is discussed in Section 4. We illustrate an example and conclusion in the last section.

2 Preliminairies

An intuitionistic fuzzy set (IFS) $A \in X$ is defined as an object of the following form

$$A = \{(x, \mu_A(x), \nu_A(x), x \in X)\},$$

where the functions $\mu_A, \nu_A(x) : X \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$, respectively, and for every $x \in X$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Obviously, each ordinary fuzzy set may be written as

$$\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}.$$

Definition 2.1. *[2] The value of*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is called the degree of non-determinacy (or uncertainty) of the element $x \in X$ to the intuitionistic fuzzy set A.

Remark 2.1. *Clearly, in the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in X$.*

we denote by

$$IF_1 = \{\langle u, v\rangle : R \to [0, 1]^2, \;\; \forall x \in \mathbb{R} \;\; 0 \leq u(x) + v(x) \leq 1\}$$

the collection of all intuitionistic fuzzy number by IF_1. An element $\langle u, v\rangle$ of IF_1 is called intuitionistic fuzzy number if it satisfies the following conditions:

(i) is normal, i.e., there exists x_0, $x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous.

(iv) $supp(u) = cl\{x \in \mathbb{R} : v(x) < 1\}$ is bounded.

Remark 2.2. *If $\langle u, v\rangle$ a fuzzy number, so we can see $[\langle u, v\rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v\rangle]^\alpha$ as $[1 - v]^\alpha$.*

A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u, v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v:

$$u(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } \; a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{if } \; a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

$$v(x)=\begin{cases}\frac{a_2-x}{a_2-a_1'} & \text{if } a_1'\leq x\leq a_2\\ \frac{x-a_2}{a_3'-a_2} & \text{if } a_2\leq x\leq a_3'\\ 1 & \text{otherwise}\end{cases}$$

where $a_1'\leq a_1\leq a_2\leq a_3\leq a_3'$ and $u(x),v(x)\leq 0.5$ for $u(x)=v(x),\ \forall x\in\mathbb{R}$.

This TIFN is denoted by $\langle u,v\rangle$=$\langle a_1,a_2,a_3;a_1^{'},a_2,a_3^{'}\rangle$ where,

$$[\langle u,v\rangle]_\alpha=[a_1+\alpha(a_2-a_1),a_3-\alpha(a_3-a_2)],$$

$$[\langle u,v\rangle]^\alpha=[a_1'+\alpha(a_2-a_1'),a_3'-\alpha(a_3'-a_2)].$$

Definition 2.2. *[7] Let $\langle u,v\rangle$ an element of IF_1 and $\alpha\in[0,1]$, we define the following sets:*

$$[\langle u,v\rangle]_l^+(\alpha)=\inf\{x\in\mathbb{R}|u(x)\geq\alpha\},$$

$$[\langle u,v\rangle]_r^+(\alpha)=\sup\{x\in\mathbb{R}|u(x)\geq\alpha\},$$

$$[\langle u,v\rangle]_l^-(\alpha)=\inf\{x\in\mathbb{R}|v(x)\leq 1-\alpha\},$$

$$[\langle u,v\rangle]_r^-(\alpha)=\sup\{x\in\mathbb{R}|v(x)\leq 1-\alpha\}.$$

Remark 2.3.

$$[\langle u,v\rangle]_\alpha=\left[[\langle u,v\rangle]_l^+(\alpha),[\langle u,v\rangle]_r^+(\alpha)\right],$$

$$[\langle u,v\rangle]^\alpha=\left[[\langle u,v\rangle]_l^-(\alpha),[\langle u,v\rangle]_r^-(\alpha)\right].$$

We define the following operations by:

$$[\langle u,v\rangle\oplus\langle z,w\rangle]^\alpha=[\langle u,v\rangle]^\alpha+[\langle z,w\rangle]^\alpha,$$

$$[\lambda\langle u,v\rangle]^\alpha=\lambda[\langle u,v\rangle]^\alpha,$$

$$[\langle u,v\rangle\oplus\langle z,w\rangle]_\alpha=[\langle u,v\rangle]_\alpha+[\langle z,w\rangle]_\alpha,$$

$$[\lambda\langle u,v\rangle]_\alpha=\lambda[\langle u,v\rangle]_\alpha,$$

where $\langle u,v\rangle,\langle z,w\rangle\in IF_1$ and $\lambda\in\mathbb{R}$

Definition 2.3. *Let $\langle u,v\rangle$ and $\langle u',v'\rangle\in IF_1$, the H-difference is the IFN $\langle z,w\rangle\in IF_1$, if it exists, such that*

$$\langle u,v\rangle\ominus\langle u',v'\rangle=\langle z,w\rangle\iff\langle u,v\rangle=\langle u',v'\rangle\oplus\langle z,w\rangle$$

On the space IF_1 we will consider the following L_p-metric,

Theorem 2.1. *[7] For* $1 \le p \le \infty$,

$$\begin{aligned} d_p(\langle u,v\rangle,\langle z,w\rangle) &= (\frac{1}{4})^{\frac{1}{p}}\Big\{\int_0^1 \Big|[\langle u,v\rangle]_r^+(\alpha) - [\langle z,w\rangle]_r^+(\alpha)\Big|^p d\alpha \\ &+ \int_0^1 \Big|[\langle u,v\rangle]_l^+(\alpha) - [\langle z,w\rangle]_l^+(\alpha)\Big|^p d\alpha \\ &+ \int_0^1 \Big|[\langle u,v\rangle]_r^-(\alpha) - [\langle z,w\rangle]_r^-(\alpha)\Big|^p d\alpha \\ & \int_0^1 \Big|[\langle u,v\rangle]_l^-(\alpha) - [\langle z,w\rangle]_l^-(\alpha)\Big|^p d\alpha\Big\}^{\frac{1}{p}} \end{aligned}$$

and for $p = \infty$

$$\begin{aligned} d_\infty(\langle u,v\rangle,\langle z,w\rangle) &= \frac{1}{4}\Big[\sup_{0<\alpha\le 1}\Big|[\langle u,v\rangle]_r^+(\alpha) - [\langle z,w\rangle]_r^+(\alpha)\Big| \\ &+ \sup_{0<\alpha\le 1}\Big|[\langle u,v\rangle]_l^+(\alpha) - [\langle z,w\rangle]_l^+(\alpha)\Big| \\ &+ \sup_{0<\alpha\le 1}\Big|[\langle u,v\rangle]_r^-(\alpha) - [\langle z,w\rangle]_r^-(\alpha)\Big| \\ &+ \sup_{0<\alpha\le 1}\Big|[\langle u,v\rangle]_l^-(\alpha) - [\langle z,w\rangle]_l^-(\alpha)\Big|\Big] \end{aligned}$$

is a metric on IF_1.

Definition 2.4. *[9] Let* $F : [a,b] \to IF_1$ *be an intuitionistic fuzzy valued mapping and* $t_0 \in [a,b]$. *Then* F *is called intuitionistic fuzzy continuous in* t_0 *iff:*

$$\forall(\varepsilon > 0)(\exists\delta > 0)(\forall t \in [a,b]\ \ tel\ \ que\ \ |t - t_0| < \delta) \Rightarrow d_p(F(t), F(t_0)) < \varepsilon.$$

Definition 2.5. *[9]* F *is called intuitionistic fuzzy continuous iff is intuitionistic fuzzy continuous in every point of* $[a,b]$.

Definition 2.6. *A mapping* $F : [a,b] \to IF_1$ *is said to be Hukuhara derivable at* t_0 *if there exists* $F'(t_0) \in IF_1$ *such that both limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t}$$

and

$$\lim_{\Delta t \to 0^+} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t}$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, *which is called the Hukuhara derivative of* F *at* t_0.

3 The intuitionistic fuzzy differential equation

In this section, we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in I \\ x(t_0) &= \langle u_{t_0}, v_{t_0}\rangle \in IF_1 \end{cases}, \tag{1}$$

where $x \in IF_1$ is unknown $I = [t_0, T]$ and $f : I \times IF_1 \to IF_1$. $x(t_0)$ is intuitionistic fuzzy number. Denote the $\alpha-$ level set

$$[x(t)]_\alpha = \Big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big]$$

$$[x(t)]^\alpha = \Big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big]$$

and

$$[x(t_0)]_\alpha = \Big[[x(t_0)]_l^+(\alpha), [x(t_0)]_r^+(\alpha)\Big]$$

$$[x(t_0)]^\alpha = \Big[x(t_0)]_l^-(\alpha), [x(t_0)]_r^-(\alpha)\Big]$$

$$[f(t, x(t))]_\alpha = \Big[f_1^+(t, x(t); \alpha), f_2^+(t, x(t); \alpha)\Big]$$

$$[f(t, x(t))]^\alpha = \Big[f_3^-(t, x(t); \alpha), f_4^-(t, x(t); \alpha)\Big],$$

where

$$\begin{aligned} f_1^+(t, x(t); \alpha) &= \min\Big\{f(t, u) | u \in \big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big]\Big\} \\ f_2^+(t, x(t); \alpha) &= \max\Big\{f(t, u) | u \in \big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big]\Big\} \\ f_3^-(t, x(t); \alpha) &= \min\Big\{f(t, u) | u \in \big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big]\Big\} \\ f_4^-(t, x(t); \alpha) &= \max\Big\{f(t, u) | u \in \big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big]\Big\}. \end{aligned} \tag{2}$$

Denote

$$\begin{aligned} f_1^+(t, x(t); \alpha) &= G\big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big) \\ f_2^+(t, x(t); \alpha) &= H\big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big) \\ f_3^-(t, x(t); \alpha) &= L\big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big) \\ f_4^-(t, x(t); \alpha) &= K\big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big) \end{aligned} \tag{3}$$

Denote by $C(I, IF_1)$ the set of all continuous mappings from I to IF_1.

Defining the metric

$$D(f, g) = \sup_{t \in I} d_\infty((f_{1,t}, f_{2,t}), (g_{1,t}, g_{2,t}))$$

with $f(t) = (f_{1,t}, f_{2,t})$ et $g(t) = (g_{1,t}, g_{2,t})$.

Definition 3.1. *[9] $x : I \to IF_1$ is a solution of the initial value problem* (1)*, if and only if it is continuous and satisfies the integral equation*

$$x(t) = x(t_0) \oplus \int_{t_0}^{t} f(s, x(s))ds.$$

Sufficient conditions for the existence of an unique solution to Eq. (1) are:

1. Continuity of f.
2. Lipschitz condition: for any pair $\big(t,\langle u,v\rangle\big),\big(t,\langle z,w\rangle\big)\in I\times IF_1$, we have

$$d_\infty\Big(f\big(t,\langle u,v\rangle\big),f\big(t,\langle z,w\rangle\big)\Big)\le Kd_\infty\Big(\langle u,v\rangle,\langle z,w\rangle\Big),$$

where $K>0$ is a given constant.

Denote by $C(I\times IF_1,IF_1)$ the set of all continuous mappings from $I\times IF_1$ to IF_1.

Theorem 3.1. *[9] Assume that $f\in C(I\times IF_1,IF_1)$ and satisfies*

$$\begin{aligned}
\big|[f(s,x(s)]_r^+(\alpha)-[f(s,y(s)]_r^+(\alpha)]\big| &\le k\big|[x(s)]_r^+(\alpha)-[y(s)]_r^+(\alpha)]\big|\\
\big|[f(s,x(s)]_l^+(\alpha)-[f(s,y(s)]_l^+(\alpha)]\big| &\le k\big|[x(s)]_l^+(\alpha)-[y(s)]_l^+(\alpha)]\big|\\
\big|[f(s,x(s)]_r^-(\alpha)-[f(s,y(s)]_r^-(\alpha)]\big| &\le k\big|[x(s)]_r^-(\alpha)-[y(s)]_r^-(\alpha)]\big|\\
\big|[f(s,x(s)]_l^-(\alpha)-[f(s,y(s)]_l^-(\alpha)]\big| &\le k\big|[x(s)]_l^-(\alpha)-[y(s)]_l^-(\alpha)]\big|
\end{aligned}$$

with $k|T-t_0|\le 1$, then the initial value problem (1) has an unique solution.

Proof. See [9]. □

4 The fourth order Runge–Kutta method

Let

$$[X(t_n)]_\alpha=\Big[[X(t_n)]_l^+(\alpha),[X(t_n)]_r^+(\alpha)\Big]$$

$$[X(t_n)]^\alpha=\Big[[X(t_n)]_l^-(\alpha),[X(t_n)]_r^-(\alpha)\Big]$$

be the exact solutions of (1) and

$$[x(t_n)]_\alpha=\Big[[x(t_n)]_l^+(\alpha),[x(t_n)]_r^+(\alpha)\Big]$$

$$[x(t_n)]^\alpha=\Big[[x(t_n)]_l^-(\alpha),[x(t_n)]_r^-(\alpha)\Big]$$

be approximated solutions at t_n, $0\le n\le N$. The solutions are calculated by grid points at

$$t_0<t_1<t_2<\ldots<t_N=T,\quad h=\frac{T-t_0}{N},\quad t_n=t_0+nh,\quad n=0,1,...N. \tag{4}$$

We recall that

$$\begin{aligned}
f_1^+(t,x(t);\alpha)=&\ G\big(t,[x(t)]_l^+(\alpha),[x(t)]_r^+(\alpha)\big)\\
f_2^+(t,x(t);\alpha)=&\ H\big(t,[x(t)]_l^+(\alpha),[x(t)]_r^+(\alpha)\big)\\
f_3^-(t,x(t);\alpha)=&\ L\big(t,[x(t)]_l^-(\alpha),[x(t)]_r^-(\alpha)\big)\\
f_4^-(t,x(t);\alpha)=&\ K\big(t,[x(t)]_l^-(\alpha),[x(t)]_r^-(\alpha)\big).
\end{aligned} \tag{5}$$

The Runge–Kutta method of order 4 that calculates the value of the function in four intermediate points as follows:

$$[x(t_{n+1})]_l^+(\alpha) = [x(t_n)]_l^+(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \tag{6}$$

where

$$\begin{cases} K_1 &= hG\big(t_n, [x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\big) \\ K_2 &= hG\big(t_n + \frac{h}{2}, [x(t_n)]_l^+(\alpha) + \frac{K_1}{2}, [x(t_n)]_r^+(\alpha) + \frac{K_1}{2}\big) \\ K_3 &= hG\big(t_n + \frac{h}{2}, [x(t_n)]_l^+(\alpha) + \frac{K_2}{2}, [x(t_n)]_r^+(\alpha) + \frac{K_2}{2}\big) \\ K_4 &= hG\big(t_n + h, [x(t_n)]_l^+(\alpha) + K_3, [x(t_n)]_r^+(\alpha) + K_3\big) \end{cases}$$

and

$$[x(t_{n+1})]_r^+(\alpha) = [x(t_n)]_r^+(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \tag{7}$$

where

$$\begin{cases} K_1 &= hH\big(t_n, [x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\big) \\ K_2 &= hH\big(t_n + \frac{h}{2}, [x(t_n)]_l^+(\alpha) + \frac{K_1}{2}, [x(t_n)]_r^+(\alpha) + \frac{K_1}{2}\big) \\ K_3 &= hH\big(t_n + \frac{h}{2}, [x(t_n)]_l^+(\alpha) + \frac{K_2}{2}, [x(t_n)]_r^+(\alpha) + \frac{K_2}{2}\big) \\ K_4 &= hH\big(t_n + h, [x(t_n)]_l^+(\alpha) + K_3, [x(t_n)]_r^+(\alpha) + K_3\big) \end{cases}$$

and

$$[x(t_{n+1})]_l^-(\alpha) = [x(t_n)]_l^-(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \tag{8}$$

where

$$\begin{cases} K_1 &= hL\big(t_n, [x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\big) \\ K_2 &= hL\big(t_n + \frac{h}{2}, [x(t_n)]_l^-(\alpha) + \frac{K_1}{2}, [x(t_n)]_r^-(\alpha) + \frac{K_1}{2}\big) \\ K_3 &= hL\big(t_n + \frac{h}{2}, [x(t_n)]_l^-(\alpha) + \frac{K_2}{2}, [x(t_n)]_r^-(\alpha) + \frac{K_2}{2}\big) \\ K_4 &= hL\big(t_n + h, [x(t_n)]_l^-(\alpha) + K_3, [x(t_n)]_r^-(\alpha) + K_3\big) \end{cases}$$

and

$$[x(t_{n+1})]_r^-(\alpha) = [x(t_n)]_r^-(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \tag{9}$$

where

$$\begin{cases} K_1 &= hK\big(t_n, [x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\big) \\ K_2 &= hK\big(t_n + \frac{h}{2}, [x(t_n)]_l^-(\alpha) + \frac{K_1}{2}, [x(t_n)]_r^-(\alpha) + \frac{K_1}{2}\big) \\ K_3 &= hK\big(t_n + \frac{h}{2}, [x(t_n)]_l^-(\alpha) + \frac{K_2}{2}, [x(t_n)]_r^-(\alpha) + \frac{K_2}{2}\big) \\ K_4 &= hK\big(t_n + h, [x(t_n)]_l^-(\alpha) + K_3, [x(t_n)]_r^-(\alpha) + K_3\big) \end{cases}$$

Let $G(t,u^+,v^+)$, $H(t,u^+,v)^+$, $L(t,u^-,v)^-$ and $K(t,u^-,v^-)$ be the functions of (3), where u^+, v^+, u^- and v^- are the constants and $u^+ \leq v^+$ and $u^- \leq v^-$. The domain of G and H is

$$M_1 = \{(t,u^+,v^+)\backslash\ t_0 \leq t \leq T,\ \ \infty < u^+ \leq v^+,\ \ -\infty < v^+ < +\infty\}$$

and the domain of L and K is

$$M_2 = \{(t,u^-,v^-)\backslash\ t_0 \leq t \leq T,\ \ \infty < u^- \leq v^-,\ \ -\infty < v^- < +\infty\},$$

where $M_1 \subseteq M_2$.

Theorem 4.1. *Let $G(t,u^+,v^+)$, $H(t,u^+,v^+)$ belong to $C^4(M_1)$ and $L(t,u^-,v^-)$, $K(t,u^-,v^-)$ belong to $C^4(M_2)$ and the partial derivatives of G, H and L, K be bounded over M_1 and M_2 respectively. Then, for arbitrarily fixed $0 \leq \alpha \leq 1$, the numerical solutions of (6), (7), (8) and (9) converge to the exact solutions $[X(t)]_l^+(\alpha)$, $[X(t)]_r^+(\alpha)$, $[X(t)]_l^-(\alpha)$ and $[X(t)]_r^-(\alpha)$ uniformly in t.*

Proof. See [4]. □

5 Example

Consider the intuitionistic fuzzy initial value problem

$$\begin{cases} x'(t) + x(t) &= \sigma(t), \ \ for \ \ all \ \ t \geq 0 \\ x_0 &= (-1, 1, 0, -\frac{3}{2}, \frac{3}{2}) \end{cases} \tag{10}$$

and $\sigma(t) = 2\exp(-t)x_0$.

Applying the method of solution proposed in [9] we get

$$\begin{cases} [x(t)]_l^+(\alpha) = & (\alpha - 1)\exp(-t)(1+2t) \\ [x(t)]_r^+(\alpha) = & (1-\alpha)\exp(-t)(1+2t) \\ [x(t)]_l^-(\alpha) = & (3t + \frac{3}{2})(\alpha - 1))\exp(-t) \\ [x(t)]_r^-(\alpha) = & (3t + \frac{3}{2})(1-\alpha))\exp(-t) \end{cases}$$

Therefore the exact solutions is given by

$$[X(t)]_\alpha = \Big[(\alpha-1)\exp(-t)(1+2t), (1-\alpha)\exp(-t)(1+2t)\Big]$$

$$[X(t)]^\alpha = \Big[(3t+\tfrac{3}{2})(\alpha-1)\exp(-t), (3t+\tfrac{3}{2})(1-\alpha)\exp(-t)\Big],$$

which at $t = 0.3$ are

$$[X(0.3)]_\alpha = \Big[(\alpha-1)\exp(-0.3)(1.6), (1-\alpha)\exp(-0.3)(1.6)\Big]$$

$$[X(0.3)]^\alpha = \Big[(0.9+\tfrac{3}{2})(\alpha-1)\exp(-0.3), (0.9+\tfrac{3}{2})(1-\alpha)\exp(-0.3)\Big].$$

Applying the Runge–Kutta method proposed we get:

$$[x(t_{n+1})]_l^+(\alpha) = [x(t_n)]_l^+(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4],$$

where

$$\begin{cases} K_1 &= h\Big[[\sigma(t_n)]_l^+(\alpha) - [x(t_n)]_l^+(\alpha)\Big] \\ K_2 &= h\Big[[\sigma(t_n + \frac{h}{2})]_l^+(\alpha) - [x(t_n)]_l^+(\alpha) - \frac{K_1}{2}\Big] \\ K_3 &= h\Big[[\sigma(t_n + \frac{h}{2})]_l^+(\alpha) - [x(t_n)]_l^+(\alpha) - \frac{K_2}{2}\Big] \\ K_4 &= h\Big[[\sigma(t_n + h)]_l^+(\alpha) - [x(t_n)]_l^+(\alpha) - K_3\Big] \end{cases}$$

and

$$[x(t_{n+1})]_r^+(\alpha) = [x(t_n)]_r^+(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4],$$

where

$$\begin{cases} K_1 &= h\Big[[\sigma(t_n)]_r^+(\alpha) - [x(t_n)]_r^+(\alpha)\Big] \\ K_2 &= h\Big[[\sigma(t_n + \frac{h}{2})]_r^+(\alpha) - [x(t_n)]_r^+(\alpha) - \frac{K_1}{2}\Big] \\ K_3 &= h\Big[[\sigma(t_n + \frac{h}{2})]_r^+(\alpha) - [x(t_n)]_r^+(\alpha) - \frac{K_2}{2}\Big] \\ K_4 &= h\Big[[\sigma(t_n + h)]_r^+(\alpha) - [x(t_n)]_r^+(\alpha) - K_3\Big] \end{cases}$$

and

$$[x(t_{n+1})]_l^-(\alpha) = [x(t_n)]_l^-(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4],$$

where

$$\begin{cases} K_1 &= h\Big[[\sigma(t_n)]_l^-(\alpha) - [x(t_n)]_l^-(\alpha)\Big] \\ K_2 &= h\Big[[\sigma(t_n + \frac{h}{2})]_l^-(\alpha) - [x(t_n)]_l^-(\alpha) - \frac{K_1}{2}\Big] \\ K_3 &= h\Big[[\sigma(t_n + \frac{h}{2})]_l^-(\alpha) - [x(t_n)]_l^-(\alpha) - \frac{K_2}{2}\Big] \\ K_4 &= h\Big[[\sigma(t_n + h)]_l^-(\alpha) - [x(t_n)]_l^-(\alpha) - K_3\Big] \end{cases}$$

and

$$[x(t_{n+1})]_r^-(\alpha) = [x(t_n)]_r^-(\alpha) + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4],$$

where

$$\begin{cases} K_1 &= h\Big[[\sigma(t_n)]_r^-(\alpha) - [x(t_n)]_r^-(\alpha)\Big] \\ K_2 &= h\Big[[\sigma(t_n+\frac{h}{2})]_r^-(\alpha) - [x(t_n)]_r^-(\alpha) - \frac{K_1}{2}\Big] \\ K_3 &= h\Big[[\sigma(t_n+\frac{h}{2})]_r^-(\alpha) - [x(t_n)]_r^-(\alpha) - \frac{K_2}{2}\Big] \\ K_4 &= h\Big[[\sigma(t_n+h)]_r^-(\alpha) - [x(t_n)]_r^-(\alpha) - K_3\Big], \end{cases}$$

where $n = 0, 1, ...N$ and $h = \frac{1}{N}$.

The exact and approximate solutions by Runge–Kutta method are plotted at $t = 0.3$ and $h = 0.25$ in Fig. 1.

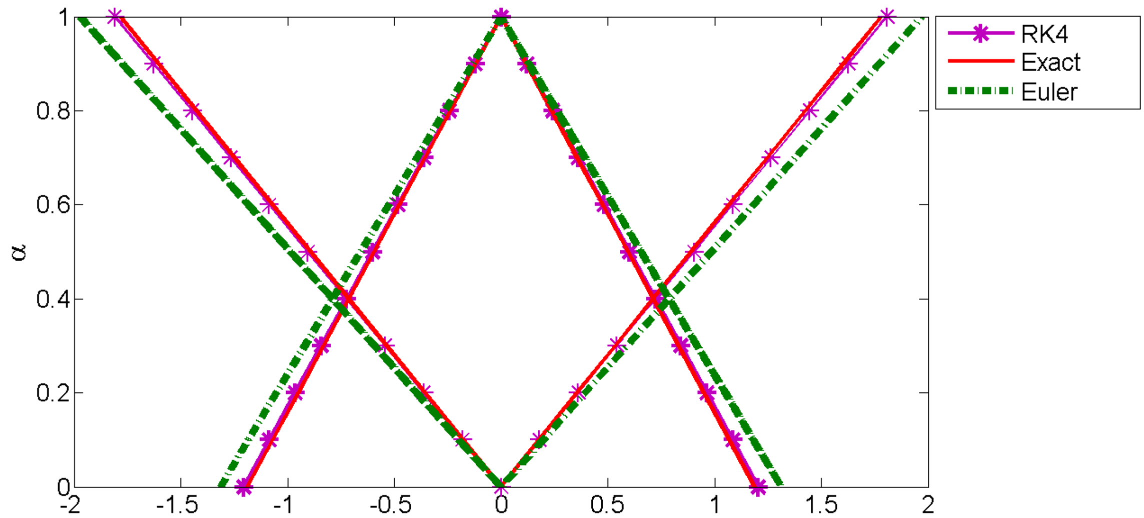

Figure 1: $h = 0.25$

The error between the Euler and the 2nd-order and 4th-order Taylor method and 4th-order Runge–Kutta method is plotted in Fig. 2.

6 Conclusion

In this work, we have used the proposed fourth-order Runge–Kutta method to find a numerical solution of intuitionistic fuzzy differential equations. Taking into account the convergence order of the Euler method is $O(h)$, higher order of convergence $O(h^4)$ is obtained by the Runge–Kutta method of order 4. Comparison of the solutions of this example shows that the Runge–Kutta method of order 4 and the 4th-order Taylor method give a better solution than the Euler method.

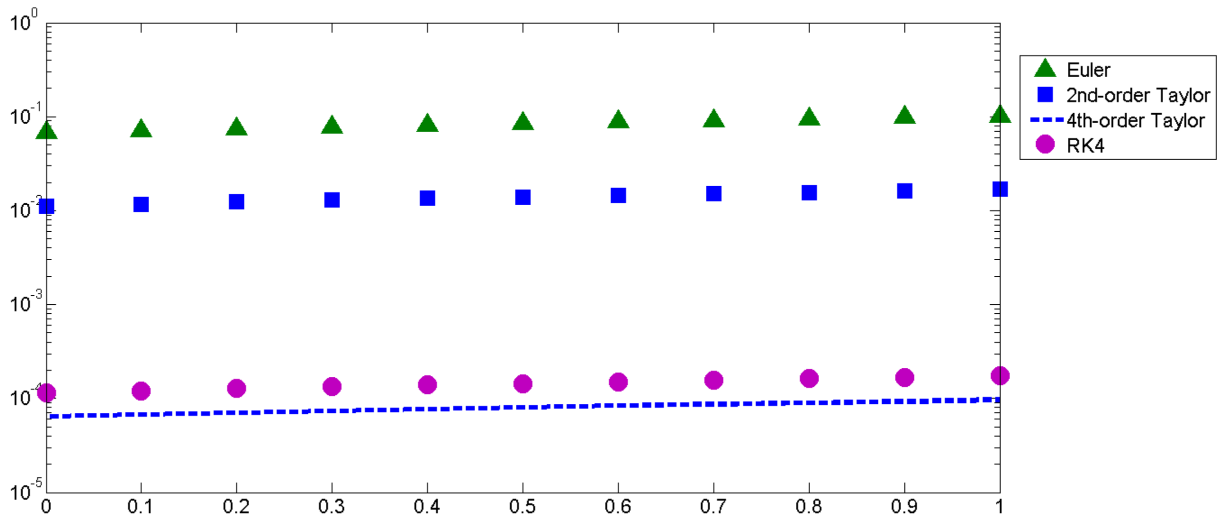

Figure 2: $h = 0.25$

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