Monotone Measures of Intuitionistic Fuzzy Sets

Piotr Nowak
Systems Research Institute Polish Academy of Sciences
ul. Newelska 6, 01-447 Warszawa, Poland
pnowak@ibspan.waw.pl

Abstract

Our main goal is to extend the domain of a monotone measure to a class of intuitionistic fuzzy sets. The extension is made by using of the Choquet integral.

Keywords: Monotone Measure, Fuzzy Measure, Intuitionistic Fuzzy Set, Choquet Integral, Probability of an Intuitionistic Fuzzy Event

1 Introduction

In the real world we often meet imprecisely defined notions. Fuzzy sets were introduced to deal with them. The intuitionistic fuzzy sets, defined by Atanassov in [1], can be regarded as generalizations of fuzzy sets, because they are described by two functions: one expressing the degree of membership and the other one expressing the degree of nonmembership, which do not need to sum up to one.

Classical measure theory and classical theory of integration do not describe all real world situations too. Therefore monotone measures, called fuzzy measures are considered. It is not possible to integrate with respect to them in the classical way, since their additivity is not assumed. Choquet integral, proposed by Vitali ([10]) and Choquet ([4]), is one of the most important integrals defined without the assumption of the σ-additivity of the measure.

Real- or complex-valued integrals can be treated as functionals defined on a certain space of functions. One of the fundamental properties of this functionals is that the integral of the characteristic function of a set is equal to its measure. Therefore it is obvious that an extension of a monotone measure to intuitionistic fuzzy sets is the first step to define an integral of imprecisely defined functions or random variables.

The extension of the domain of a monotone measure to a class of intuitionistic fuzzy sets is the main goal of this paper. We also prove some basic properties of such extended measures. The proposed definition is in agreement with the notion of probability of intuitionistic fuzzy events, which was introduced and discussed by Gerstenkorn and Mańko (see [6]) and Szmidt and Kacprzyk ([8], [9]) as well as Grzegorzewski and Mrówka in [7].

Section 2 of the paper includes basic definitions and notations. Section 3 is devoted to the definition and basic properties of the proposed extension of a monotone measure to a class of intuitionistic fuzzy sets. In its third subsection we show that the probability of intuitionistic fuzzy events is an extended monotone measure.

2 Intuitionistic Fuzzy Sets

We first recall the definition of intuitionistic fuzzy sets and introduce some basic notations.

Definition 1 *Let Ω denote a universe of discourse. Then an intuitionistic fuzzy set A in Ω is a set of ordered triples*

$$A = \{\langle \omega, \mu_A(\omega), \nu_A(\omega)\rangle : \omega \in \Omega\},$$

where $\mu_A, \nu_A : \Omega \to [0,1]$ are such that $0 \leq \mu_A(\omega) + \nu_A(\omega) \leq 1$ for each $\omega \in \Omega$. The family of all intuitionistic fuzzy sets on Ω is denoted by $IFS(\Omega)$.

For each ω the numbers $\mu_A(\omega)$ and $\nu_A(\omega)$ represent the degree of membership and degree of nonmembership of an element $\omega \in \Omega$ to A, respectively. The function $\pi_A(\omega) = 1 - \mu_A(\omega) - \nu_A(\omega)$ is interpreted as hesitation margin.

Definition 2 *Let $A, B \in IFS(\Omega)$ with $A = \{\langle \omega, \mu_A(\omega), \nu_A(\omega)\rangle : \omega \in \Omega\}$ and $B = \{\langle \omega, \mu_B(\omega), \nu_B(\omega)\rangle : \omega \in \Omega\}$. $A \subseteq B \Leftrightarrow$ $\forall \omega \in \Omega$ $\mu_A(\omega) \leq \mu_B(\omega)$ and $\nu_A(\omega) \geq \nu_B(\omega)$. Consequently, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. $\varnothing$ denotes the intuitionistic fuzzy set with $\mu_\varnothing(\omega) = 0$ and $\nu_\varnothing(\omega) = 1$ for each $\omega \in \Omega$.*

For arbitrary $f, g : \Omega \to [0,1]$ we denote by $f \wedge g$ and $f \vee g$ the functions defined by formulas

$$\begin{aligned} f \wedge g(\omega) &= \min(f(\omega), g(\omega)); \\ f \vee g(\omega) &= \max(f(\omega), g(\omega)), \ \omega \in \Omega. \end{aligned}$$

We repeat Atanassov's definition of the basic operations on intuitionistic fuzzy sets.

Definition 3 *Let $A, B \in IFS(\Omega)$ with $A = \{\langle \omega, \mu_A(\omega), \nu_A(\omega)\rangle : \omega \in \Omega\}$ and $B = \{\langle \omega, \mu_B(\omega), \nu_B(\omega)\rangle : \omega \in \Omega\}$. Then*
$A^c = \{\langle \omega, \nu_A(\omega), \mu_A(\omega)\rangle : \omega \in \Omega\};$
$A \cup B =$
$\{\langle \omega, \mu_A \vee \mu_B(\omega), \nu_A \wedge \nu_B(\omega)\rangle : \omega \in \Omega\};$
$A \cap B =$
$\{\langle \omega, \mu_A \wedge \mu_B(\omega), \nu_A \vee \nu_B(\omega)\rangle : \omega \in \Omega\}.$
Two intuitionistic fuzzy sets are disjoint if and only if $A \cap B = \varnothing$.

3 Extension of Monotone Measures

3.1 Monotone Measures

Definition 4 *Let Ω be an arbitrary set and let $\mathcal{F}$ be a σ-algebra of its subsets. Then every function $\mu : \mathcal{F} \to \mathbb{R}^+ = [0, \infty)$ satisfying the properties:*
1. $\mu(\varnothing) = 0$,
2. *for every $A \subset B \in \mathcal{F}$ $\mu(A) \leq \mu(B)$*
is called a monotone measure or a fuzzy measure. We say that μ is continuous from below if for every sequence $\{C_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $C \in \mathcal{F}$

$$C_n \nearrow C \Rightarrow \mu(C_n) \nearrow \mu(C).$$

In general we do not assume that μ is continuous from below. If Ω is finite, it is often assumed that $\mathcal{F} = 2^\Omega$.

Definition 5 *A monotone measure $\mu : \mathcal{F} \to \mathbb{R}^+$ is supermodular, if for every $A, B \in \mathcal{F}$*

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

It is called submodular, if for every $A, B \in \mathcal{F}$

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

If the above inequalities hold for each pair of disjoint events A and B, then μ is called superadditive and subadditive respectively. μ is called additive if it is both sub- and superadditive.

For an arbitrary set I we denote with $\#I$ the cardinality of I.

Definition 6 *A monotone measure $\mu : \mathcal{F} \to \mathbb{R}^+$ is called k-monotone, $k \geq 2$, if for every $A_1, A_2, ..., A_k \in \mathcal{F}$*

$$\mu\left(\cup_{i=1}^k A_i\right) + \sum_{\substack{I \subset \{1,2...,k\}, \\ I \neq \varnothing}} (-1)^{\#I} \mu(\cap_{i \in I} A_i) \geq 0$$

while it is k-alternating, $k \geq 2$, if for every $A_1, A_2, ..., A_k \in \mathcal{F}$

$$\mu\left(\cap_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,2...,k\},\\ I \neq \varnothing}} (-1)^{\#I} \mu\left(\cup_{i \in I} A_i\right) \leq 0.$$

μ is called totally monotone if it is k-monotone for every $k \geq 2$.

With any function $f : \Omega \to \mathbb{R}^+$ we associate the family $C_f = \{C_f(x) : x \in \mathbb{R}^+\}$, with $C_f(x) = \{\omega \in \Omega : f(\omega) > x\}$. Each function $f : \Omega \to \mathbb{R}$ introduces on Ω a semi-order relation:

$$\omega_1 <_f \omega_2 \Leftrightarrow f(\omega_1) < f(\omega_2).$$

Definition 7 *Two functions: $f, g : \Omega \to \mathbb{R}$ are commonotone if and only if there exists no pair ω_1, ω_2 in Ω such that $\omega_1 <_f \omega_2$ and $\omega_2 <_g \omega_1$.*

Definition 8 *Let $\mathcal{F}(\Omega)$ be the set of all $\mathcal{F}$-measurable functions from Ω to $\mathbb{R}^+$, i.e. the functions satisfying the condition: $C_f \subset \mathcal{F}$. Let X belong to $\mathcal{F}(\Omega)$ and let μ be a monotone measure on $\mathcal{F}$. Let $G_\mu^X(x) = \mu(C_X(x))$. Then the Choquet integral of X is defined by the formula $(C)\int_\Omega X d\mu = \int_0^\infty G_\mu^X(x)\, dx$.*

The above integral can be treated as the Riemann integral, since $G_\mu^X(x)$ is a non-decreasing function of x.

3.2 Extended Monotone Measures

We denote by $IFS(\mathcal{F})$ the family of all intuitionistic fuzzy sets in Ω for which μ_A and ν_A belong to $\mathcal{F}(\Omega)$. We say that two intuitionistic fuzzy sets $A, B \in IFS(\mathcal{F})$ are commonotone if μ_A and μ_B are commonotone and ν_A and ν_B are commonotone.

The extension of a monotone measure to $IFS(\mathcal{F})$ is defined as follows.

Definition 9 *Let μ be a monotone measure on a σ-algebra $\mathcal{F}$ of subsets of Ω. For each $A \in IFS(\mathcal{F})$ the extension $\tilde{\mu}$ of the measure μ is the interval $\tilde{\mu}(A) = [\mu_{\min}(A), \mu_{\max}(A)]$, where $\mu_{\min}(A)$ and $\mu_{\max}(A)$ are the Choquet integrals of the form*

$$\mu_{\min}(A) = (C)\int_\Omega \mu_A d\mu \text{ and}$$
$$\mu_{\max}(A) = (C)\int_\Omega (1 - \nu_A)\, d\mu.$$

To show some basic properties of the extended measure, we introduce the following notations: $\mathbf{0} = [0, 0]$, $\boldsymbol{\mu} = [\mu(\Omega), \mu(\Omega)]$ and $\boldsymbol{\mu}(A) = [\mu(A), \mu(A)]$ for $A \in \mathcal{F}$. We also introduce the following order on the family of intervals:

$$[a, b] \leq [c, d] \Leftrightarrow a \leq c \quad \text{and} \quad b \leq d.$$

Proposition 10 *The following properties hold:*
(a) $\tilde{\mu}(\varnothing) = \mathbf{0}$, $\tilde{\mu}(\Omega) = \boldsymbol{\mu}$ and for any crisp set $A \in \mathcal{F}$ $\tilde{\mu}(A) = \boldsymbol{\mu}(A)$.
Let $A, B \in IFS(\mathcal{F})$.
(b) $A \subseteq B \Rightarrow \tilde{\mu}(A) \leq \tilde{\mu}(B)$, i.e. $\tilde{\mu}$ is monotone.
(c) If μ is supermodular, then $\tilde{\mu}$ is supermodular and if it is submodular, then $\tilde{\mu}$ is submodular. Moreover, $\tilde{\mu}$ is sub- and superadditive if μ is sub- and superadditive respectively.
(d) If A and B are commonotone, then

$$\tilde{\mu}(A \cup B) + \tilde{\mu}(A \cap B) = \tilde{\mu}(A) + \tilde{\mu}(B).$$

Theorem 11 *If a monotone measure μ is n-monotone for $n \geq 2$, then $\tilde{\mu}$ is also n-monotone. Furthermore, if μ is n-alternating,for $n \geq 2$, then $\tilde{\mu}$ is also n-alternating.*

From Theorem 11 it easily follows that if a monotone measure μ on a σ-algebra $\mathcal{F}$ of subsets of Ω is totally monotone, then $\tilde{\mu}$ is totally monotone on $IFS(\mathcal{F})$.

Theorem 12 *Let μ be continuous from below, $\{A_n\}_{n=1}^{\infty} \subset IFS(\mathcal{F})$ and $A_n \subset A_{n+1}$ for each $n \geq 1$. Let $A = \cup_{n=1}^{\infty} A_n$. Then* $\lim_{n\to\infty} \tilde{\mu}(A_n) = \tilde{\mu}(A)$.

Theorem 13 *Let μ be continuous from below and let $\{A_n\}_{n=1}^{\infty} \subset IFS(\mathcal{F})$ be pairwise disjoint. If μ is superadditive, then $\tilde{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \tilde{\mu}(A_n)$. If μ is subadditive, then $\tilde{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \tilde{\mu}(A_n)$.*

3.3 Extension of Probability Measures

The following theorem shows relations between monotone measures and probability of intuitionistic fuzzy events.

Theorem 14 *Let Ω be an arbitrary set and let $\mathcal{F}$ be a σ-algebra of its subsets. Let $P : \mathcal{F} \to [0,1]$ be a probability measure (i.e. σ-additive monotone measure with $P(\Omega) = 1$). Then for each $A \in IFS(\mathcal{F})$ $\tilde{P}(A) = P(A)$, where $\tilde{P}$ is the extension of the monotone measure and P is the probability of intuitionistic fuzzy event A(see [6], [7], [8], [9]), i.e. a number from the interval $\left[\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP\right]$.*

Proof. It is enough to prove that

$$\int_\Omega \mu_A dP = \int_0^1 P(\mu_A > x)\, dx \qquad (1)$$

and that

$$1 - \int_\Omega \nu_A dP = \int_0^1 P(1 - \nu_A > x)\, dx. \qquad (2)$$

Let $F_{\mu_A}(x) = P(\mu_A \leq x)$ and $F_{\nu_A}(x) = P(\nu_A \leq x)$.

$$\int_0^1 P(\mu_A > x)\, dx = 1 - \int_0^1 F_{\mu_A}(x)\, dx. \qquad (3)$$

The integral $\int_0^1 F_{\mu_A}(x)\, dx$ can be understood as the Lebesgua'e-Stjeltjes integral. Therefore

$$\begin{aligned} \int_0^1 F_{\mu_A}(x)\, dx &= F_{\mu_A}(x)\, x|_0^1 - \\ \int_0^1 x dF_{\mu_A}(x) &= 1 - \int_0^1 x dF_{\mu_A}(x) = \\ & \quad 1 - \int_\Omega \mu_A dP, \end{aligned} \qquad (4)$$

where the last equality is a known fact from the measure theory. Finally, formulas (3) and (4) give (1). Replacing μ_A by $1 - \nu_A$ we obtain (2). ■

From the above theorem it follows that the extension of a monotone measure, which is a probability measure coincides with the notion of the probability of intuitionistic fuzzy events. In this case the extended measure has all the properties proved in [7].

References

[1] K. Atanassov (1986). Intuitionistic Fuzzy Sets. In *Fuzzy Sets and Systems*, volume 20, pages 87-96.

[2] K. Atanassov (1989). More on Intuitionistic Fuzzy Sets. In *Fuzzy Sets and Systems*, volume 33, pages 37-46.

[3] K. Atanassov (1999). Intuitionistic Fuzzy Sets: Theory and Applications. *Physica-Verlag.*

[4] G. Choquet (1953-1954). Theory of capacities. In *Ann. Inst. Fourier*, volume 5, pages 131-295.

[5] D. Denneberg (2000). Non-additive Measure and Integral, Basic Concepts and Their Role for Applications. In *Fuzzy Measures and Integrals. Theory and Applications.* Physica-Verlag Heilderberg, pages 42-69.

[6] T. Gerstenkorn, J. Mańko (1991). Probability of Fuzzy Intuitionistic Sets. In *BUSEFAL* 45, pages 128-136.

[7] P. Grzegorzewski , E. Mrówka (2002). Probability of Intuitionistic Fuzzy Events. In *Soft Methods in Probability. Statistics and data Analysis.* Springer - Physica Verlag, Heidelberg, 2002, pages 105-115.

[8] E. Szmidt E., J. Kacprzyk (1999a). A Concept of a Probability of an Intuitionistic Fuzzy Event. In *Proceedings of the 1999 IEEE International Fuzzy Systems Conference*, pages 1346-1349, Seoul,1999.

[9] E. Szmidt, J. Kacprzyk (1999b). Probability of Intuitionistic Fuzzy Events and Their Applications in Decision Making. In *Proceedings of the 9th Congreso EUSFLAT'99*, pages 457-460.

[10] G. Vitali (1925). Sulla definizione di integrale delle funzioni di una variabile. In *Ann. Mat. Pura ed Appl.* IV, 2, pages 111-121.