Notes on Intuitionistic Fuzzy Sets
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Intuitionistic L-fuzzy essential and closed submodules

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Abstract: Let R be a commutative ring with identity and M be an R-module. An intuitionistic L-fuzzy submodule (ILFSM) C of an intuitionistic L-fuzzy module A of R-module M, is called an intuitionistic L-fuzzy essential submodule in A, if $C \cap B \neq \chi_{\{\theta\}}$ for any non-trivial ILFSM B of A. In this case we say that A is an essential extension of C. Also, if C has no proper essential extension in A, then C is called an intuitionistic L-fuzzy closed submodule in A. Further, for ILFSMs B, C of A, C is called complement of B in A if C is maximal with the property that $B \cap C = \chi_{\{\theta\}}$. We study these mentioned notations which are generalization of the notions of essential submodule, closed submodule and complement of a submodule in the intuitionistic L-fuzzy module theory. We prove many basic properties of both these concepts.

Keywords: Intuitionistic L-fuzzy submodule, Intuitionistic L-fuzzy essential submodule, Intuitionistic L-fuzzy closed submodule.
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1 Introduction

Let M be a unitary module over a commutative ring R with zero element θ. Recall that a submodule K of an R-module M is called an essential submodule of M denoted by $K \trianglelefteq_e M$, if for every submodule N of M, $K \cap N = \{\theta\}$ implies that $N = \{\theta\}$. Equivalently, $K \cap N \neq \{\theta\}$ for all non-zero submodule N of M. In this case, M is called an essential extension of K. A submodule K of M is called closed in M written as $K \trianglelefteq_c M$ if and only if M is the only essential extension of K, that is if N is any proper submodule of M such that $K \trianglelefteq_e N$, then $K = N$. A submodule K of a module M is called complement for a submodule N of M if it is maximal with respect to the property that $K \cap N = \{\theta\}$. For more information about essential submodules, closed submodules and complement submodule, we refer to [1, 8, 15].

Atanassov and Stoeva [2] generalized the notion of L-fuzzy subset given by Goguen [5] to an intuitionistic L-fuzzy subset, where L is any complete lattice with a complete order reversing involution N. Wang and He in [14] and Deschrijver and Kerre in [4] studied the relationship between intuitionistic fuzzy sets and L-fuzzy sets and some extensions of fuzzy set theory. Palaniappan and others in [11] have studied intuitionistic L-fuzzy subgroups. Meena and Thomas in [10] have discussed the notion of intuitionistic L-fuzzy subrings. Sharma et al. [7, 12, 13] have discussed intuitionistic L-fuzzy submodules, intuitionistic L-fuzzy prime and primary submodule of a module. In this paper we introduce and study the concepts of intuitionistic L-fuzzy essential submodule, intuitionistic L-fuzzy closed submodule and the complement of intuitionistic L-fuzzy submodule of a module and establish some results.

2 Preliminaries

Throughout this paper R is a commutative ring with identity, M a unitary R-module and L stands for a complete lattice with least element 0 and greatest element 1. θ denotes the zero element of M. An element $\alpha \in L, 1 \neq \alpha$, is called a prime element in L if for all $a; b \in L$ if $a \wedge b \leq \alpha$ implies $a \leq \alpha$ or $b \leq \alpha$ (see [3]).

Definition 1 ([7]). *Let $(L, \leq)$ be a complete lattice with an evaluative order reversing operation $N : L \to L$. Let X be a non-empty set. An intuitionistic L-fuzzy set A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to L$ and $\nu_A : X \to L$ define the degree of membership and the degree of non-membership for every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$. A complete order reversing involution is a map $N : L \to L$ such that:*

(i) $N(0_L) = 1_L$ *and* $N(1_L) = 0_L$;

(ii) If $\alpha \leq \beta$, *then* $N(\beta) \leq N(\alpha)$;

(iii) $N(N(\alpha)) = \alpha$;

(iv) $N(\vee_{i=1}^{n}\alpha_i) = \wedge_{i=1}^{n}N(\alpha_i)$ *and* $N(\wedge_{i=1}^{n}\alpha_i) = \vee_{i=1}^{n}N(\alpha_i)$.

We also denote an intuitionistic L-fuzzy set by simply $ILFS$ and the set of all $ILFS$'s on X by $ILFS(X)$.

Remark 1. *When $\mu_A(x) = N(\nu_A(x))$, for all $x \in X$, then A is called L-fuzzy set. We use the notation $A = (\mu_A, \nu_A)$ to denote the intuitionistic L-fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.*

For $A, B \in ILFS(X)$ we say $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

If $f : X \to Y$ is a mapping $A \in ILFS(X)$ and $B \in ILFS(Y)$, then $f(A) \in ILFS(Y)$ and $f^{-1}(B) \in ILFS(X)$ are defined as follows:

$$f(A)(y) = \begin{cases} (\sup\{\mu_A(x) \mid x \in f^{-1}(y)\}, \inf\{\nu_A(x) | x \in f^{-1}(y)\}), & \text{if } f^{-1}(y) \neq \varnothing \\ (0,1), & \text{otherwise} \end{cases}$$

$\forall y \in Y$. Also, $f^{-1}(B)(x) = (\mu_B(f(x)), \nu_B(f(x))), \forall x \in X$.

For $A \in ILFS(X)$ and $\alpha, \beta \in L$ with $\alpha \leq N(\beta)$, define $A_{(\alpha,\beta)} = \{x \in X \mid \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$. Then $A_{(\alpha,\beta)}$ is called the (α, β)-cut set of A. In particular, we denote $A_{(1,0)}$ by A_*. Of course, $A_* = \{x \in X \mid \mu_A(x) = 1$ and $\nu_A(x) = 0\}$. The support of an $ILFS$ A is denoted by A^* and is defined as $A^* = \{x \in X \mid \mu_A(x) > 0$ and $\nu_A(x) < 1\}$.

Definition 2 ([12]). *Let $A = (\mu_A, \nu_A)$ be an ILFS of X and $Y \subseteq X$. Then the intuitionistic L-fuzzy characteristic function $\chi_Y = (\mu_{\chi_Y}, \nu_{\chi_Y})$ on Y is defined as*

$$\mu_{\chi_Y}(y) = \begin{cases} 1, & \text{if } y \in Y \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{\chi_Y}(y) = \begin{cases} 0, & \text{if } y \in Y \\ 1, & \text{otherwise}. \end{cases}$$

The following are two very basic definitions given in [10] and [12].

Definition 3 ([10]). *Let $A \in ILFS(R)$. Then A is called an intuitionistic L-fuzzy ideal $(ILFI)$ of R if for all $x, y \in R$, the following are satisfied:*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;

(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Definition 4 ([7, 12]). *Let $A \in ILFS(M)$. Then A is called an intuitionistic L-fuzzy module $(ILFM)$ of M if for all $x, y \in M, r \in R$, the following are satisfied:*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(rx) \geq \mu_A(x)$;

(iii) $\mu_A(\theta) = 1$;

(iv) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$;

(v) $\nu_A(rx) \leq \nu_A(x)$;

(vi) $\nu_A(\theta) = 0$.

Let $IF_L(M)$ denote the set of all intuitionistic L-fuzzy R-modules of M and $ILFI(R)$ denote the set of all intuitionistic L-fuzzy ideals of R. We note that when $R = M$, then $A \in IF_L(M)$ if and only if $\mu_A(\theta) = 1, \nu_A(\theta) = 0$ and $A \in ILFI(R)$.

If L is regular and $A, B \in IF_L(M)$, then A^*, B^* are submodules of M. Further we see that $(A+B)^* = A^* + B^*$ and $(A \cap B)^* = A^* \cap B^*$. Also, $A^* = \{\theta\}$ if and only if $A = \chi_{\{\theta\}}$ (see [7]).

3 Intuitionistic L-fuzzy essential submodules

In this section, we extend the concept of an essential submodule of an R-module in the intuitionistic L-fuzzy setting and prove some results.

Definition 5. *Let M be an R-module and $A, C \in IF_L(M)$ be such that $\chi_{\{\theta\}} \neq C \subseteq A$. Then C is called an intuitionistic L-fuzzy essential submodule of A if $C \cap B \neq \chi_{\{\theta\}}$ $\forall B \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq B \subseteq A$. We denote this by $C \trianglelefteq_e A$, and we also say that A is an intuitionistic L-fuzzy essential extension of C.*

In particular, when $A = \chi_M$, then C is called an intuitionistic L-fuzzy essential submodule of M, written as $C \trianglelefteq_e \chi_M$ or $C \trianglelefteq_e M$, if $C \cap B \neq \chi_{\{\theta\}}$ $\forall B \neq \chi_{\{\theta\}} \in IF_L(M)$.

Proposition 1. *Let M be an R-module and $A, C \in IF_L(M)$ be such that $C \trianglelefteq_e A$. Then $C^* \trianglelefteq_e A^*$, but the converse is true when L is regular.*

Proof. Firstly, let $A, C \in IF_L(M)$ be such that $C \trianglelefteq_e A$. To show that $C^* \trianglelefteq_e A^*$.

As $C \trianglelefteq_e A$. Then $C \cap D \neq \chi_{\{\theta\}}, \forall D \in IF_L(M), \chi_{\{\theta\}} \neq D \subseteq A$.

Let $\{\theta\} \neq N$ be a submodule of M. Define $D = \chi_N$. Clearly, $\chi_{\{\theta\}} \neq D \in IF_L(M)$ and $D \subseteq A$ and therefore $C \cap D \neq \chi_{\{\theta\}}$. Therefore, there exists $\theta \neq x \in N$ such that $x \in (C \cap D)^*$ and so $(C \cap D)^* \neq \{\theta\}$, i.e., $C^* \cap D^* \neq \{\theta\}$. Hence $C^* \trianglelefteq_e A^*$.

Conversely, suppose that L is regular and $C^* \trianglelefteq_e A^*$. We want to show that $C \trianglelefteq_e A$. For this we consider any $\chi_{\{\theta\}} \neq D \subseteq A$, where $D \in IF_L(M)$. Then $D^* \neq \{\theta\}$ and $D^* \subseteq A^*$. Therefore, $C^* \cap D^* \neq \{\theta\} \Rightarrow (C \cap D)^* \neq \{\theta\}$. This means that there exists $\theta \neq x \in M$ such that $x \in (C \cap D)^*$. Therefore, $C \cap D \neq \chi_{\{\theta\}}$. Hence $C \trianglelefteq_e A$. □

Example 1. *Let N be an essential submodule of R-module M. Then the intuitionistic L-fuzzy submodule A of M defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = \theta \\ \alpha, & \text{if } x \in N - \{\theta\} \\ 0, & \text{if } x \notin N \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = \theta \\ \beta, & \text{if } x \in N - \{\theta\} \\ 1, & \text{if } x \notin N \end{cases},$$

where $\alpha, \beta \in L \setminus \{0,1\}$ with $\alpha \leq N(\beta)$, is an intuitionistic L-fuzzy essential submodule of M.

Example 2. *Let $L = \{0, a, b, 1\}$ be a diamond lattice with $a \vee b = 1$ and $a \wedge b = 0$ so that $N(a) = b$ and $N(b) = a$. Consider $M = \{0, 1, 2, \ldots, 11\}$ under addition and multiplication module 12 as Z-module. Consider $A, B, C \in IF_L(M)$ as follows:*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ a, & \text{if } x \in \{4, 8\} \\ 0, & \text{if } x \notin \{0, 4, 8\} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ b, & \text{if } x \in \{4, 8\} \\ 1, & \text{if } x \notin \{0, 4, 8\} \end{cases} ;$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ a, & \text{if } x \in \{2, 4, 6, 8, 10\} \\ 0, & \text{if } x \notin \{0, 2, 4, 6, 8, 10\} \end{cases} ; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ b, & \text{if } x \in \{2, 4, 6, 8, 10\} \\ 1, & \text{if } x \notin \{0, 2, 4, 6, 8, 10\} \end{cases} .$$

Here $A \subseteq B$ but A is not essential in B. As there is $\chi_{\{\theta\}} \neq C \in IF_L(M)$ such that $C \subseteq B$ and $A \cap C = \chi_{\{\theta\}}$, where

$$\mu_C(x) = \begin{cases} 1, & \text{if } x = 0 \\ a, & \text{if } x = 6 \\ 0, & \text{if } x \notin \{0, 6\} \end{cases} ; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x = 0 \\ b, & \text{if } x = 6 \\ 1, & \text{if } x \notin \{0, 6\} \end{cases} .$$

Also, $B \trianglelefteq_e M$ but A is not essential in M.

Theorem 1. *Let L be regular, $A, C \in IF_L(M)$ be such that $\chi_{\{\theta\}} \neq C \subseteq A$. Then C is an intuitionistic L-fuzzy essential submodule of A if and only if for each $\theta \neq x \in M$, with $x \in A^*$, there exists $r \in R$ such that $rx \neq \theta$ and $rx \in C^*$.*

Proof. Assume that for each $\theta \neq x \in M$ with $x \in A^*$ there exists $0 \neq r \in R$ such that $rx \in C^*$. We want to show that $C \trianglelefteq_e A$. Take any $B \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq B \subseteq A$. We will show that $C \cap B \neq \chi_{\{\theta\}}$.

Let $x \in M$ be such that $x \neq \theta$ and $x \in B^*$. As $B \subseteq A$, therefore $B^* \subseteq A^*$ implies that $x \in A^*$. From the given, there exists $r \neq 0 \in R$ such that $rx \neq 0$ and $rx \in C^*$, where $\chi_{\{\theta\}} \neq C \subseteq B$. Therefore, $\mu_B(rx) \geq \mu_C(rx) > 0$ and $\nu_B(rx) \leq \nu_C(rx) < 1 \Rightarrow rx \in B^*$. Thus, $rx \in C^* \cap B^* = (C \cap B)^*$ and so $C \cap B \neq \chi_{\{\theta\}}$.

Conversely, suppose that $C \trianglelefteq_e A$. Let $\theta \neq x \in M$ with $x \in A^*$. To show that there exists $r \in R$ such that $rx \in C^*$. Now for every $r \in R$, we have $\mu_A(rx) \geq \mu_A(x) > 0$ and $\nu_A(rx) \leq \nu_A(x) < 1 \Rightarrow rx \in A^*$.

Consider the non-zero submodule $N = Rx$ of M. Define $B = A|_N$, then $B \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq B \subseteq A$. As $C \trianglelefteq_e A$, therefore $C \cap B \neq \chi_{\{\theta\}}$, so $(C \cap B)^* \neq \{\theta\}$, i.e., $C^* \cap B^* \neq \{\theta\}$ and therefore there exists $\theta \neq y \in M$ such that $y \in B^*$ and $y \in C^*$. But $B^* = N = Rx$. Thus, there exists $0 \neq r \in R$ such that $rx = y \in C^*$. This completes the proof. □

Theorem 2. *Let $A, B, C \in IF_L(M)$ be such that $C \subseteq B \subseteq A$. Then $C \trianglelefteq_e A$ if and only if $C \trianglelefteq_e B$ and $B \trianglelefteq_e A$.*

Proof. Assume that $C \trianglelefteq_e A$. Then $C \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq D \subseteq A$.

Since $B \subseteq A$, it follows that $C \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq D \subseteq B$. $\Rightarrow C \trianglelefteq_e B$.

Also since $C \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq D \subseteq A$ and also since $C \subseteq B$ we get $B \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq D \subseteq A$. Hence $B \trianglelefteq_e A$.

Conversely, suppose that $C \trianglelefteq_e B$ and $B \trianglelefteq_e A$. We want to show that $C \trianglelefteq_e A$.

Since $B \trianglelefteq_e A$ we have $B \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq D \subseteq A$. Then $B \cap D \in IF_L(M)$ satisfies $\chi_{\{\theta\}} \neq B \cap D \subseteq B$ and therefore, since $C \trianglelefteq_e B$, we get $C \cap (B \cap D) \neq \chi_{\{\theta\}}$. Since $C \subseteq B$ it follows that $C \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq D \subseteq A$. Therefore $C \trianglelefteq_e A$. □

Theorem 3. *Let $C_1, C_2, A_1, A_2 \in IF_L(M)$. If $C_1 \trianglelefteq_e A_1$ and $C_2 \trianglelefteq_e A_2$, then $C_1 \cap C_2 \trianglelefteq_e A_1 \cap A_2$.*

Proof. Let $D \in IF_L(M)$ be such that $\chi_{\{\theta\}} \neq D \subseteq A_1 \cap A_2 \subseteq A_2$. Then since $C_2 \trianglelefteq_e A_2$ we have $C_2 \cap D \neq \chi_{\{\theta\}}$. Since $D \subseteq A_1$, we get $\chi_{\{\theta\}} \neq C_2 \cap D \subseteq A_1$. Therefore since $C_1 \trianglelefteq_e A_1$, we get $C_1 \cap (C_2 \cap D) \neq \chi_{\{\theta\}}$. Thus we get $(C_1 \cap C_2) \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$, $\chi_{\{\theta\}} \neq D \subseteq A_1 \cap A_2$. Hence $C_1 \cap C_2 \trianglelefteq_e A_1 \cap A_2$. □

Remark 2. *Let $C_1, C_2, A \in IF_L(M)$. If $C_1 \trianglelefteq_e A$ and $C_2 \trianglelefteq_e A$, then $C_1 \cap C_2 \trianglelefteq_e A$.*

Theorem 4. *Let L be regular $C, A \in IF_L(M)$ where $C \subseteq A$. Let $f : N \to M$ be a module homomorphism such that $f(B) \subseteq A$ where $B \in IF_L(N)$. If $C \trianglelefteq_e A$ then $f^{-1}(C) \trianglelefteq_e B$.*

Proof. Given $C \trianglelefteq_e A$. We want to show that $f^{-1}(C) \trianglelefteq_e B$. For this we have to show that $f^{-1}(C) \cap D \neq \chi_{\{\theta\}}$, $\forall D \in IF_L(M)$, $\chi_{\{\theta\}} \neq D \subseteq B$. That is to show that for given $D \in IF_L(M)$, $\chi_{\{\theta\}} \neq D \subseteq B$, there exists $\theta \neq x \in N$ such that $\mu_{f^{-1}(C) \cap D}(x) \neq 0$ and $\nu_{f^{-1}(C) \cap D}(x) \neq 1$; i.e., $\mu_{f^{-1}(C)}(x) \wedge \mu_D(x) \neq 0$ and $\nu_{f^{-1}(C)}(x) \vee \nu_D(x) \neq 1$, i.e., $\mu_C(f(x)) \wedge \mu_D(x) \neq 0$ and $\nu_C(f(x)) \vee \nu_D(x) \neq 1$.

Now, we claim that if $f(D) = \chi_{\{\theta\}}$, then $D \subseteq f^{-1}(C)$.

Let for all $z \in M$ with $f^{-1}(z) \neq \emptyset$, we have $\mu_{f(D)}(z) = \mu_{\chi_{\{\theta\}}}(z)$ and $\nu_{f(D)}(z) = \nu_{\chi_{\{\theta\}}}(z)$. Therefore, we have:

$$\vee\{\mu_D(x) | x \in N, f(x) = z\} = \begin{cases} 1, & \text{if } z = \theta \\ 0, & \text{if } z \neq \theta \end{cases}$$

$\Rightarrow \vee\{\mu_D(x) \mid x \in N, f(x) = z\} = 0$ if $z \neq \theta$

$\Rightarrow \{\mu_D(x) \mid x \in N, f(x) = z\} = \{0\}$ if $z \neq \theta$

$\Rightarrow \mu_D(x) = 0$ if $f(x) \neq \theta$. Similarly, we can show that $\nu_D(x) = 1$ if $f(x) \neq \theta$. Thus we have $\mu_C(f(x)) = \mu_C(\theta) = 1$ and $\nu_C(f(x)) = \nu_C(\theta) = 0$ if $f(x) = \theta$. Therefore, $\mu_D(x) \leq \mu_C(f(x))$ and $\nu_D(x) \geq \nu_C(f(x))$, $\forall x \in N$, i.e., $\mu_D(x) \leq \mu_{f^{-1}(C)}(x)$ and $\nu_D(x) \geq \nu_{f^{-1}(C)}(x)$, $\forall x \in N$. Therefore, in this case we have $D \subseteq f^{-1}(C)$ and so we get $f^{-1}(C) \cap D = D \neq \chi_{\{\theta\}}$.

If $f(D) \neq \chi_{\{\theta\}}$, to prove that $f^{-1}(C) \cap D \neq \chi_{\{\theta\}}$ for $D \in IF_L(M)$, $\chi_{\{\theta\}} \neq D \subseteq B$, we have $D \subseteq B \Rightarrow f(D) \subseteq f(B) \Rightarrow f(D) \subseteq A$ (as $f(B) \subseteq A$).

Therefore, if $f(D) \neq \chi_{\{\theta\}}$, since $C \trianglelefteq_e A$ we get $C \cap f(D) \neq \chi_{\{\theta\}}$. From this we get that there exist some $x \neq \theta \in M$ such that $\mu_{f(D)}(x) \neq 0$ and $\nu_{f(D)}(x) \neq 1$, which further implies that there exists some $y \in N$ such that $f(y) = x$ and $\mu_D(y) \neq 0$; $\nu_D(y) \neq 1$.

Thus for this y we have $\mu_{C\cap f(D)}(y) \neq 0$ and $\nu_{C\cap f(D)}(y) \neq 1$, i.e., $\mu_C(f(y)) \wedge \mu_{f(D)}(f(y)) \neq 0$ and $\nu_C(f(y)) \vee \nu_{f(D)}(f(y)) \neq 1$. This implies that both $\mu_C(f(y)) > 0$, $\nu_C(f(y)) < 1$ and $\mu_{f(D)}(f(y)) > 0, \nu_{f(D)}(f(y)) < 1$. Since L is regular, we get $\mu_{f^{-1}(C)}(y) > 0, \nu_{f^{-1}(C)}(y) < 1$ and $\mu_D(y) > 0, \nu_D(y) < 1 \Rightarrow \mu_{f^{-1}(C)}(y) \wedge \mu_D(y) \neq 0$ and $\nu_{f^{-1}(C)}(y) \vee \nu_D(y) \neq 1 \Rightarrow f^{-1}(C) \cap D \neq \chi_{\{\theta\}}$. Thus, we get $f^{-1}(C) \trianglelefteq_e B$. □

Theorem 5. *Let L be a regular and $C_1, C_2, A_1, A_2 \in IF_L(M)$. If $C_i \trianglelefteq_e A_i$, $i = 1, 2$. If $C_1 \cap C_2 = \chi_{\{\theta\}}$, then $A_1 \cap A_2 = \chi_{\{\theta\}}$ and $C_1 \oplus C_2 \trianglelefteq_e A_1 \oplus A_2$.*

Proof. Since $C_i \trianglelefteq_e A_i$, $i = 1, 2$. Then by proposition (3.2) we get $C_i^* \trianglelefteq_e A_i^*$, $i = 1, 2$. Also because $C_1 \cap C_2 = \chi_{\{\theta\}}$, the sum $C_1 + C_2$ is the direct sum $C_1 \oplus C_2$. Since $C_1 \cap C_2 \trianglelefteq_e A_1 \cap A_2$, it follows that $A_1 \cap A_2 \neq \chi_{\{\theta\}}$ and so the sum $A_1 + A_2$ is also the direct sum $A_1 \oplus A_2$. Therefore, since L is regular we have the direct sum of R-modules $C_1^* \oplus C_2^*$ and $A_1^* \oplus A_2^*$. Since $C_i^* \trianglelefteq_e A_i^*$, $i = 1, 2$ we get $C_1^* \oplus_e C_2^* \trianglelefteq A_1^* \oplus A_2^*$. From this it follows that $C_1 \oplus C_2 \trianglelefteq_e A_1 \oplus A_2$. □

Remark 3. *Let L be a regular and $C_1, C_2, A \in IF_L(M)$. If $C_i \trianglelefteq_e A$, $i = 1, 2$. If $C_1 \cap C_2 = \chi_{\{\theta\}}$, then $C_1 \oplus C_2 \trianglelefteq_e A$.*

Proposition 2. *Let $A, B \in IFI(R)$. Let B be an intuitionistic L-fuzzy prime ideal of R such that A is not subset of B. Then $A \trianglelefteq_e R$.*

Proof. Let $C \in IFI(R)$ be such that $A \cap C \subseteq B$. Since $AC \subseteq A \cap C \subseteq B$ implies $AC \subseteq B$. As B is intuitionistic L-fuzzy prime ideal of R. Therefore either $A \subseteq B$ or $C \subseteq B$. But given that A is not a subset of B, so $C \subseteq B$ which implies that $A \trianglelefteq_e R$. This completes the proof. □

4 Complement of an intuitionistic L-fuzzy module

In this section we extend the concept of a complement of a submodule in the intuitionistic L-fuzzy setting and prove some results.

Definition 6. *Let M be an R-module and $A, B, C \in IF_L(M)$ be such that $B \subseteq A$. Then C is called an intuitionistic L-fuzzy complement of B in A if $C \subseteq A$ and C is maximal with the property that $B \cap C = \chi_{\{\theta\}}$. We say that C is complement of B in A.*

Theorem 6. *Let L be regular and M be an R-module. If C is complement of B in A. Then C^* is complement of B^* in A^*.*

Proof. Since C is complement of B in A. Therefore, C is the maximal intuitionistic L-fuzzy submodule of A with the property that $B \cap C = \chi_{\{\theta\}}$. Then $B^* \cap C^* = \{\theta\}$. It remains to show that C^* is the maximal one with this property. Let N be a submodule of M such that $C^* \subseteq N$ and $B^* \cap N = \{\theta\}$. Since $\mu_C(x) > 0, \nu_C(x) < 1$ for all $x \in C^*$. So let $p = \inf\{\mu_C(x) \mid x \in C^*\}$ and $q = \sup\{\nu_C(x) \mid x \in C^*\}$. Then $p, q \in L\backslash\{0, 1\}$ such that $p \leq N(q)$. Choose $\alpha, \beta \in L\backslash\{0, 1\}$ such that $0 < \alpha \leq p$ and $q \leq \beta < 1$. Define $D \in ILFS$ as follows:

$$\mu_D(x) = \begin{cases} \mu_C(x), & \text{if } x \in C^* \\ \alpha, & \text{if } x \in N - C^* \\ 0, & \text{if } x \notin N \end{cases}; \quad \nu_D(x) = \begin{cases} \nu_C(x), & \text{if } x \in C^* \\ \beta, & \text{if } x \in N - C^* \\ 1, & \text{if } x \notin N \end{cases}.$$

Clearly, $D \in IF_L(M)$ such that $C \subseteq D$ and so $D^* = N$. Now $B^* \cap N = \{\theta\} \Rightarrow B^* \cap D^* = \{\theta\}$. This implies $(B \cap D)^* = \{\theta\} \Rightarrow B \cap D = \chi_{\{\theta\}}$. But C is maximal with this property that $B \cap C = \chi_{\{\theta\}}$ so $C = D$ and consequently $C^* = D^* = N$. Hence, C^* is complement of B^* in A^*. □

Remark 4. *The converse of the above theorem is not true. If for any $A, B, C \in IF_L(M)$. The submodule C^* is complement of B^* in A^*. Then C need not be complement of B in A.*

Example 3. *Let $L = [0, 1]$ and let the module $M = Z_6 = \{0, 1, 2, 3, 4, 5\}$ be a module over the ring Z of integers. Define ILFSs A, B, C of M as follows:*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = 3 \\ 0, & \text{if } x \in \{1,2,4,5\} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x = 3 \\ 1, & \text{if } x \in \{1,2,4,5\} \end{cases} ;$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} ; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases} ;$$

$$\mu_C(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.6, & \text{if } x \in \{2,4\} \\ 0, & \text{if } x \in \{1,3,5\} \end{cases} ; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x \in \{2,4\} \\ 1, & \text{if } x \in \{1,3,5\} \end{cases} .$$

It is easy to check that $A, B, C \in IF_L(M)$ such that $A^* = \{0, 3\}$, $B^* = \{0\}$ and $C^* = \{0, 2, 4\}$. Clearly, $A^* \cap C^* = \{0\} = B^*$ and C^* is maximal with this property so C^* is complement of B^* in A^*. But C is not complement of B in A, for if we define the ILFS D on M as follows:

$$\mu_D(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.7, & \text{if } x \in \{2,4\} \\ 0, & \text{if } x \in \{1,3,5\} \end{cases} ; \quad \nu_D(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.1, & \text{if } x \in \{2,4\} \\ 1, & \text{if } x \in \{1,3,5\} \end{cases} .$$

Then $D \in IF_L(M)$ with $C \subseteq D$ and $D \cap B = \chi_{\{0\}}$. This shows that C is not maximal with the property that $C \cap B = \chi_{\{0\}}$.

5 Intuitionistic L-fuzzy closed submodules

In this section we extend the concept of closed submodule in the intuitionistic L-fuzzy setting.

Definition 7. *Let M be an R-module and $A, B, C \in IF_L(M)$. Then C is said to be an intuitionistic L-fuzzy closed submodule of A if $C \subseteq A$ and C has no non-constant (proper) intuitionistic L-fuzzy essential extension in A, i.e., if $\chi_{\{\theta\}} \neq B \subset A$ such that $C \trianglelefteq_e B \Rightarrow B = C$. We write $C \trianglelefteq_c A$ when C is an intuitionistic L-fuzzy closed submodule of A.*

Remark 5. *Note that $\chi_{\{\theta\}}$ and A are always intuitionistic L-fuzzy closed submodules of A.*

Theorem 7. *Let L be regular and M be an R-module. If $A, C \in IF_L(M)$ are such that $C \trianglelefteq_c A$, then $C^* \trianglelefteq_c A^*$.*

Proof. Firstly, let $C \trianglelefteq_c A$. To show that $C^* \trianglelefteq_c A^*$. If possible, let N be a proper submodule of A^* such that $C^* \trianglelefteq_e N$. Then we will show that $N = C^*$.

Define $B \in ILFS(M)$ as

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in N \\ 0, & \text{if } x \notin N \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in N \\ 1, & \text{if } x \notin N. \end{cases}.$$

It is clear that $B \in IF_L(M)$ such that $B \subseteq A$, i.e., B is a proper intuitionistic L-fuzzy submodule of A. Also, $B^* = N$. But $C^* \trianglelefteq_e N$ implies that $C^* \trianglelefteq_e B^*$ and therefore $C \trianglelefteq_e B$. Also, $C \trianglelefteq_c A$. Therefore, we have $C = B$ which implies that $C^* = B^* = N$. □

Remark 6. *The converse of the above theorem need not be true. See the following example.*

Example 4. *Consider L, M, A as in Example 3. Here, we notice that $B^* = \{0\} \trianglelefteq_c A^*$, but B is not intuitionistic L-fuzzy closed in A.*

Theorem 8. *Let $A, B, C, D \in IF_l(M)$ such that $B \subseteq C \subseteq D \subseteq A$ and $B \trianglelefteq_c C$ and $C \trianglelefteq_c D$, then $B \trianglelefteq_c D$.*

Proof. Since $B \trianglelefteq_c C \Rightarrow B \trianglelefteq_e C$ and if $\chi_{\{\theta\}} \neq F \subset C \in IF_L(M)$ such that $B \trianglelefteq_e F$, then $B = F$ (1).

Also, since $C \trianglelefteq_c D \Rightarrow C \trianglelefteq_e D$ and if $\chi_{\{\theta\}} \neq G \subset D \in IF_L(M)$ such that $C \trianglelefteq_e G$, then $C = G$ (2).

Now, as $B \trianglelefteq_e C$ and $C \trianglelefteq_e D$, then by prop. (3.6), we get $B \trianglelefteq_e D$.
Further, if $\chi_{\{\theta\}} \neq H \subset D \in IF_L(M)$ such that $B \trianglelefteq_e H$, then from (1), we get that $B = H$. Hence $B \trianglelefteq_c D$. □

Proposition 3. *Let L be regular and M be an R-module. If $A, C \in IF_L(M)$, then C is intuitionistic L-fuzzy closed submodule of A if and only if C is intuitionistic L-fuzzy complement of some $B \in IF_L(M)$ such that $B \subseteq A$.*

Proof. Firstly, let $C \trianglelefteq_c A$. Then by Theorem (5.3), we have $C^* \trianglelefteq_c A^*$. Hence C^* is complement of some submodule N, where N is a proper submodule of A^*. Let $B = \chi_N \in ILFS(M)$. Clearly, $B \in IF_L(M)$ is such that $B \subseteq A$ and $B^* = N$. So C^* is complement of B^*. Hence, $B^* \cap C^* = \{\theta\}$ and so $B \cap C = \chi_{\{\theta\}}$. Next we claim that C is maximal with this property.

Suppose that $D \in ILFS(M)$ is such that $D \subseteq A$ and C is intuitionistic L-fuzzy submodule of D, i.e., $C \subseteq D$ such that $B \cap D = \chi_{\{\theta\}}$. So $B^* \cap D^* = \{\theta\}$. But C^* is a submodule of D^* and C^* is a complement of B^*. So $C^* = D^*$. Thus, $C = D$ and hence C is an intuitionistic L-fuzzy complement of B.

Conversely, let C be intuitionistic L-fuzzy complement of B in A. We want to show that C is intuitionistic L-fuzzy closed submodule of A. Suppose that $C \trianglelefteq_c D$, where $D \in IF_L(M)$ is such that $D \subseteq A$. Then $C \cap B \trianglelefteq_c D \cap B$ (by Corollary (3.8)). Hence $\chi_{\{\theta\}} \trianglelefteq_c D \cap B$ and so $D \cap B = \chi_{\{\theta\}}$. But C is an intuitionistic L-fuzzy submodule of D and C is intuitionistic L-fuzzy complement of B, hence $D = C$. Thus C is intuitionistic L-fuzzy closed submodule of A. □

Remark 7. *If L is regular and M is an R-module such that C is intuitionistic L-fuzzy closed submodule of A. Then C being an intuitionistic L-fuzzy complement of some B in A does not imply that B is intuitionistic L-fuzzy complement of C in A. See the following example.*

Example 5. *Consider L, M, A as in Example 3. Interchange B and C. We note that $C = \chi_{\{\theta\}}$ is an intuitionistic L-fuzzy closed submodule of A. Also, C is intuitionistic L-fuzzy complement of B in A. But B is not intuitionistic L-fuzzy complement of C in A.*

6 Conclusion

In this paper, we have introduced the notion of essential submodule, closed submodule and complement of submodule of a module in the intuitionistic L-fuzzy environment to develop the theory of intuitionistic L-fuzzy modules. It has been shown that the converse of many results which hold in general complete lattice L hold only in the case when the lattice L is regular. We have shown that for the existence of complement of an intuitionistic L-fuzzy submodule C, the lattice L should be regular and that C must be closed.

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