Notes on Intuitionistic Fuzzy Sets
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Related fixed point theorems in intuitionistic fuzzy metric spaces satisfying an implicit relation

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Abstract: In this paper, we introduce a new class of implicit relation to present an extended version of a fixed point theorem of Popa [23] in the framework of intuitionistic fuzzy metric space.

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1 Introduction

The concept of fuzzy sets was introduced initially by Zadeh [31] in 1965. George and Veeramani [10] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [13]. In 1986 Atanassov [6] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [1]. Recently, many authors have proved fixed point theorems involving intuitionistic fuzzy sets (see [7, 11, 14, 17–19, 28, 30] and references therein). Motivated by some work of V. Popa et al. via implicit relation, we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition. In this paper, we prove a related fixed point theorem for four mappings using an implicit relation. Our theorem generalizes the theorem of Popa [23] and other theorems in literature.

2 Preliminaries

For the terminologies and basic properties of intuitionistic fuzzy metric, we begin with some definitions, as follows.

Definition 1 ([26]). *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-norm if it satisfies the following conditions:*

1) $*$ *is associative and commutative,*

2) $*$ *is continuous,*

3) $a * 1 = a$ *for all* $a \in [0,1]$*,*

4) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$*; for each* $a, b, c, d \in [0,1]$.

Two typical examples of a continuous t-norm are $a * b = ab$ *and* $a * b = \min\{a, b\}$.

Definition 2 ([26]). *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-conorm if it satisfies the following conditions:*

1) $\Diamond$ *is associative and commutative,*

2) $\Diamond$ *is continuous,*

3) $a \Diamond 0 = a$ *for all* $a \in [0,1]$*,*

4) $a \Diamond b \le c \Diamond d$ *whenever* $a \le c$ *and* $b \le d$*; for each* $a, b, c, d \in [0,1]$.

Definition 3 ([10]). *A* 3*-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous t-norm and* M *is a fuzzy metric on* $X^2 \times (0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*,*

1) $M(x, y, t) > 0$*,*

2) $M(x, y, t) = 1$ *if and only if* $x = y$*,*

3) $M(x, y, t) = M(y, x, t)$*,*

4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$*,*

5) $M(x, y, .) : (0, \infty) \to [0,1]$ *is continuous.*

Definition 4 ([1]). *A* 5*-tuple* $(X, M, N, *, \Diamond)$ *is called an intuitionistic fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous t-norm,*$\Diamond$ *is a continuous t-conorm and* M, N *are a fuzzy metric on* $X^2 \times (0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*,*

1) $M(x, y, t) + N(x, y, t) \le 1$.

2) $M(x, y, t) > 0$*,*

3) $M(x, y, t) = 1$ *if and only if* $x = y$*,*

4) $M(x, y, t) = M(y, x, t)$*,*

5) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$*,*

6) $M(x, y, .) : (0, \infty) \to [0,1]$ *is continuous*

7) $M(x, y, t) > 0$*,*

8) $N(x, y, t) = 0$ *if and only if* $x = y$*,*

9) $N(x, y, t) = N(y, x, t)$*,*

10) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$*,*

11) $N(x, y, .) : (0, \infty) \to [0,1]$ *is continuous.*

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ respectively denote the degree of nearness and degree of nonnearness between x and y with respect to t.

Remark 1. *In intuitionistic fuzzy metric space, $M(x, y, t)$ is non-decreasing and $N(x, y, t)$ is non-increasing for all $x, y \in X$.*

Example 1 ([27]). *Let (X, d) be a metric space. Define t-norm $a * b = \min\{a, b\}$ and t-conorm $a \Diamond b = \max\{a, b\}$ and for all $x, y \in X$ and $t > 0$,*

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space induced by the metric d. It is obvious that $N(x, y, t) = 1 - M(x, y, t)$.

Definition 5 ([1]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space(IFM space), then*

1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$, $\lim_{n \to \infty} N(x_n, x, t) = 0$; $\forall\, t > 0$.

2) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0$;$\forall\, t > 0$ and $p > 0$.

Definition 6 ([2]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space in which every Cauchy sequence is convergent, then $(X, M, N, *, \Diamond)$ is said to be a complete fuzzy metric space.*

Lemma 1 ([2]). *Let $\{u_n\}$ be a sequence in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. If there exists a constant $k \in (0, 1)$ such that $M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)$ and $N(u_n, u_{n+1}, kt) \leq N(u_{n-1}, u_n, t)$ for all $t > 0$, then $\{u_n\}$ is a Cauchy sequence in X.*

Lemma 2 ([2]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1)$, $M(x, y, kt) \geq M(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$ then $x = y$.*

The following theorem is proved by V. Popa [23].

Theorem 1. *Let (X, d) and (Y, ρ) be complete metric spaces. Let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities:*

$$F(d(SAx, TBx'), d(x, x'), d(x, SAx), d(x', TBx'), \rho(Ax, Bx')) \leq 0,$$

$$G(\rho(BSy, ATy'), \rho(y, y'), \rho(y, BSy), \rho(y', ATy'), d(Sy, Ty')) \leq 0,$$

for all x, x' in X and y, y' in Y, where $F, G \in \mathscr{F}_5$. If one of the mappings A, B, T and S is continuous, then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

3 Implicit relation

Implicit relations play an important role in establishing of the fixed point theorems (see [3–5, 15, 20, 22]). Our implicit relation can be described as follows: Let Φ be the set of the functions $\varphi_1, \varphi_2, \theta_1, \theta_2 : [0,1]^5 \to \mathbb{R}$ such that, for every $u, v, w \in (0,1)$

(H_1) $\varphi_1, \varphi_2, \theta_1, \theta_2$ are continuous in each coordinate variable,

(H_2) φ_1, θ_1 are nonincreasing in second and 3rd variables,

(H_3) φ_2, θ_2 is nonincreasing in second and 4th variables,

(H_4) $\varphi_1(u, v, u, v, w) \geq 0 \Rightarrow u \geq \min\{v, w\}$.

(H_5) $\theta_1(u, v, u, v, w) \leq 0 \Rightarrow u \leq \max\{v, w\}$.

(H_6) $\varphi_2(u, v, v, u, w) \geq 0 \Rightarrow u \geq \min\{v, w\}$.

(H_7) $\theta_2(u, v, v, u, w) \leq 0 \Rightarrow u \leq \max\{v, w\}$.

(H_8) $\varphi_1(u, 1, v, 1, 1) \geq 0$ or $\varphi_1(u, u, 1, 1, v) \geq 0 \Rightarrow u \geq v$.

(H_9) $\theta_1(u, 0, v, 0, 0) \leq 0$ or $\theta_1(u, u, 0, 0, v) \leq 0 \Rightarrow u \leq v$.

(H_{10}) $\varphi_2(u, 1, v, 1, 1) \geq 0$ or $\varphi_2(u, v, v, v, 1) \geq 0 \Rightarrow u \geq v$.

(H_{11}) $\theta_2(u, 0, v, 0, 0) \leq 0$ or $\theta_2(u, v, v, v, 0) \leq 0 \Rightarrow u \leq v$.

The above definitions, results and implicit relation motivated us to prove new related fixed point theorems for four mappings on intuitionistic fuzzy metric spaces by using implicit relation.

4 Main result

The main result of this paper is the following theorem.

Theorem 2. *Let $(X, M_1, N_1, *, \diamondsuit)$ and $(Y, M_2, N_2, *, \diamondsuit)$ be a complete intuitionistic fuzzy metric spaces with $M_1(x, x', t) \to 1$ as $t \to \infty$ for all $x, x' \in X$ and $M_2(y, y', t) \to 1$ as $t \to \infty$ for all $y, y' \in Y$. Let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying:*

$$\varphi_1(M_1(SAx, TBx', kt), M_1(x, x', t), M_1(x, SAx, t), M_1(x', TBx', t), M_2(Ax, Bx', t)) \geq 0 \tag{4.1}$$

$$\theta_1(N_1(SAx, TBx', kt), N_1(x, x', t), N_1(x, SAx, t), N_1(x', TBx', t), N_2(Ax, Bx', t)) \leq 0 \tag{4.2}$$

$$\varphi_2(M_2(BSy, ATy', kt), M_2(y, y', t), M_2(y, BSy, t), M_2(y', ATy', t), M_1(Sy, Ty', t)) \geq 0 \tag{4.3}$$

$$\theta_2(N_2(BSy, ATy', kt), N_2(y, y', t), N_2(y, BSy, t), N_2(y', ATy', t), N_1(Sy, Ty', t)) \leq 0 \tag{4.4}$$

for all x, x'in X and y, y' in Y and for all $t > 0$, where $\varphi_1, \varphi_2, \theta_1, \theta_2 \in \Phi$ and $0 < k < 1$. Then, if one of the mappings A, B, T and S is continuous then SA and TB have a unique fixed point z in X and BS and AT have a unique fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by:

$$Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1}.$$

Using the inequality (4.1) and (4.2), we have successively

$$\varphi_1(M_1(SAx_{2n}, TBx_{2n-1}, kt), M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, SAx_{2n}, t), M_1(x_{2n-1}, TBx_{2n-1}, t), M_2(Ax_{2n}, Bx_{2n-1}, t)) \geq 0,$$

$$\theta_1(N_1(SAx_{2n}, TBx_{2n-1}, kt), N_1(x_{2n}, x_{2n-1}, t), N_1(x_{2n}, SAx_{2n}, t), N_1(x_{2n-1}, TBx_{2n-1}, t), N_2(Ax_{2n}, Bx_{2n-1}, t)) \leq 0,$$

that is,

$$\varphi_1(M_1(x_{2n+1}, x_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n-1}, x_{2n}, t), M_2(y_{2n}, y_{2n+1}, t)) \geq 0.$$

$$\theta_1(N_1(x_{2n+1}, x_{2n}, kt), N_1(x_{2n}, x_{2n-1}, t), N_1(x_{2n}, x_{2n+1}, t), N_1(x_{2n-1}, x_{2n}, t), N_2(y_{2n}, y_{2n+1}, t)) \leq 0.$$

As φ_1 and θ_1 are nonincreasing in the third variable, we get

$$\varphi_1(M_1(x_{2n+1}, x_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, x_{2n+1}, kt), M_1(x_{2n-1}, x_{2n}, t), M_2(y_{2n}, y_{2n+1}, t)) \geq 0.$$

$$\theta_1(N_1(x_{2n+1}, x_{2n}, kt), N_1(x_{2n}, x_{2n-1}, t), N_1(x_{2n}, x_{2n+1}, kt), N_1(x_{2n-1}, x_{2n}, t), N_2(y_{2n}, y_{2n+1}, t)) \leq 0.$$

which implies by (H_4) and (H_5) respectively

$$\begin{aligned} M_1(x_{2n+1}, x_{2n}, kt) &\geq \min\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n}, y_{2n+1}, t)\}. && (4.5)\\ N_1(x_{2n+1}, x_{2n}, kt) &\leq \max\{N_1(x_{2n}, x_{2n-1}, t), N_2(y_{2n}, y_{2n+1}, t)\}. && (4.6)\end{aligned}$$

Using inequality (4.1) and (4.2) again, it follows that

$$\begin{aligned} M_1(x_{2n}, x_{2n-1}, t) &\geq \min\{M_1(x_{2n-1}, x_{2n-2}, t), M_2(y_{2n}, y_{2n-1}, t)\}, && (4.7)\\ N_1(x_{2n}, x_{2n-1}, t) &\leq \max\{N_1(x_{2n-1}, x_{2n-2}, t), N_2(y_{2n}, y_{2n-1}, t)\} && (4.8)\end{aligned}$$

Similarly, using inequality (4.3) and (4.4), we get

$$\varphi_2\left(M_2(y_{2n+1}, y_{2n}, kt), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, t), M_1(x_{2n}, x_{2n-1}, t)\right) \geq 0.$$

$$\theta_2\left(N_2(y_{2n+1}, y_{2n}, kt), N_2(y_{2n}, y_{2n-1}, t), N_2(y_{2n}, y_{2n-1}, t), N_2(y_{2n+1}, y_{2n}, t), N_1(x_{2n}, x_{2n-1}, t)\right) \leq 0.$$

Since φ_2 and θ_2 are nonincreasing in the fourth variable, we obtain

$$\varphi_2\left(M_2(y_{2n+1}, y_{2n}, kt), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t)\right) \geq 0.$$

$$\theta_2\left(M_2(y_{2n+1}, y_{2n}, kt), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t)\right) \leq 0.$$

From (H_6) and (H_7) respectevely, we have

$$\begin{aligned} M_2(y_{2n}, y_{2n+1}, kt) &\geq \min\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n}, t)\}, && (4.9)\\ N_2(y_{2n}, y_{2n+1}, kt) &\leq \max\{N_1(x_{2n}, x_{2n-1}, t), N_2(y_{2n-1}, y_{2n}, t)\} && (4.10)\end{aligned}$$

and

$$\begin{aligned} M_2(y_{2n-1}, y_{2n}, kt) &\geq \min\{M_1(x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-2}, y_{2n-1}, t)\}. && (4.11)\\ N_2(y_{2n-1}, y_{2n}, kt) &\leq \max\{N_1(x_{2n-2}, x_{2n-1}, t), N_2(y_{2n-2}, y_{2n-1}, t)\} && (4.12)\end{aligned}$$

Using inequalities (4.5), (4.9), and (4.6), (4.10) we have

$$\begin{aligned} M_1(x_{2n+1}, x_{2n}, kt) &\geq \min\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n}, t)\}, && (4.13)\\ N_1(x_{2n+1}, x_{2n}, kt) &\leq \max\{N_1(x_{2n}, x_{2n-1}, t), N_2(y_{2n-1}, y_{2n}, t)\}. && (4.14)\end{aligned}$$

Similarly, from inequalities (4.7), (4.11) and (4.8), (4.12), we have

$$\begin{aligned} M_1(x_{2n}, x_{2n-1}, t) &\geq \min\{M_1(x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-2}, y_{2n-1}, t)\} && (4.15) \\ N_1(x_{2n}, x_{2n-1}, t) &\leq \max\{N_1(x_{2n-2}, x_{2n-1}, t), N_2(y_{2n-2}, y_{2n-1}, t)\}. && (4.16) \end{aligned}$$

It now follows from inequalities (4.13), (4.14) and (4.15), (4.16) that

$$\begin{aligned} M_1(x_{n+1}, x_n, kt) &\geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\} && (4.17) \\ N_1(x_{n+1}, x_n, kt) &\leq \max\{N_1(x_n, x_{n-1}, t), N_2(y_n, y_{n-1}, t)\} && (4.18) \end{aligned}$$

and

$$\begin{aligned} M_2(y_{n+1}, y_n, kt) &\geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\} && (4.19) \\ N_2(y_{n+1}, y_n, kt) &\leq \max\{N_1(x_n, x_{n-1}, t), N_2(y_n, y_{n-1}, t)\}. && (4.20) \end{aligned}$$

It now follows inequalities (4.17), (4.18) and (4.19), (4.20)

$$\begin{aligned} M_1(x_{n+1}, x_n, kt) &\geq M_2(y_n, y_{n-1}, t) && (4.21) \\ M_2(y_{n+1}, y_n, kt) &\geq M_1(x_n, x_{n-1}, t) && (4.22) \end{aligned}$$

and

$$\begin{aligned} N_1(x_{n+1}, x_n, kt) &\leq N_2(y_n, y_{n-1}, t) && (4.23) \\ N_2(y_{n+1}, y_n, kt) &\leq N_1(x_n, x_{n-1}, t). && (4.24) \end{aligned}$$

Using (4.21), (4.22) and (4.23), (4.24) we have for $n = 1, 2, \ldots$

$$\begin{aligned} M_1(x_{n+1}, x_n, t) &\geq M_2\left(y_n, y_{n-1}, \frac{t}{k^2}\right) \\ M_2(y_{n+1}, x_n, t) &\geq M_1\left(x_n, x_{n-1}, \frac{t}{k^2}\right) \end{aligned}$$

and

$$\begin{aligned} N_1(x_{n+1}, x_n, t) &\leq N_2\left(y_n, y_{n-1}, \frac{t}{k^2}\right) \\ N_2(y_{n+1}, y_n, t) &\leq N_1\left(x_n, x_{n-1}, \frac{t}{k^2}\right). \end{aligned}$$

For $n = 1, 2 \ldots$, since $0 \leq k < 1$, from Lemma 2 it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y, respectively. Hence, $\{x_n\}$ converges to z in X and $\{y_n\}$ converges to w in Y.

Now suppose that A is continuous. Then

$$\lim Ax_{2n} = Az = \lim y_{2n+1} = w$$

and so $Az = w$. Using inequality (4.1) and (4.2) we get

$$\varphi_1\left(M_1(SAz, TBx_{2n-1}, kt), M_1(z, x_{2n-1}, t), M_1(z, SAz, t), M_1(x_{2n-1}, TBx_{2n-1}, t), M_2(Az, Bx_{2n-1}, t)\right) \geq 0,$$

$$\theta_1\left(N_1(SAz, TBx_{2n-1}, kt), N_1(z, x_{2n-1}, t), N_1(z, SAz, t), N_1(x_{2n-1}, TBx_{2n-1}, t), N_2(Az, Bx_{2n-1}, t)\right) \leq 0,$$

that is,

$$\varphi_1\left(M_1(Sw, x_{2n}, kt), M_1(z, x_{2n-1}, t), M_1(z, Sw, t), M_1(x_{2n-1}, x_{2n}, t), M_2(w, y_{2n}, t)\right) \geq 0.$$

$$\theta_1\left(N_1(Sw, x_{2n}, kt), N_1(z, x_{2n-1}, t), N_1(z, Sw, t), N_1(x_{2n-1}, x_{2n}, t), N_2(w, y_{2n}, t)\right) \leq 0.$$

Letting n tend to infinity, we have

$$\begin{aligned} \varphi_1(M_1(Sw, z, kt), 1, M_1(z, Sw, t), 1, 1) &\geq 0, \\ \theta_1(N_1(Sw, z, kt), 0, N_1(z, Sw, t), 0, 0) &\leq 0 \end{aligned}$$

from (H_8) and (H_9), we get

$$\begin{aligned} M_1(Sw, z, kt) &\geq M_1(z, Sw, t), \\ N_1(Sw, z, kt) &\leq N_1(z, Sw, t), \end{aligned}$$

and so $Sw = z = SAz$. On the other hand, using inequality (4.3) and (4.4) we have successively

$$\varphi_2\left(M_2(BSw, ATy_{2n}, kt), M_2(w, y_{2n}, t), M_2(w, BSw, t), M_2(y_{2n}, ATy_{2n}, t), M_1(Sw, Ty_{2n}, t)\right) \geq 0,$$

$$\theta_2\left(N_2(BSw, ATy_{2n}, kt), N_2(w, y_{2n}, t), N_2(w, BSw, t), N_2(y_{2n}, ATy_{2n}, t), N_1(Sw, Ty_{2n}, t)\right) \geq 0,$$

then

$$\begin{aligned} \varphi_2\left(M_2(Bz, y_{2n+1}, kt), M_2(w, y_{2n}, t), M_2(w, Bz, t), M_2(y_{2n}, y_{2n+1}, t), M_1(z, x_{2n}, t)\right) &\geq 0. \\ \theta_2\left(N_2(Bz, y_{2n+1}, kt), N_2(w, y_{2n}, t), N_2(w, Bz, t), N_2(y_{2n}, y_{2n+1}, t), N_1(z, x_{2n}, t)\right) &\leq 0. \end{aligned}$$

Letting n tend to infinity, we have

$$\begin{aligned} \varphi_2(M_2(Bz, w, kt), 1, M_2(w, Bz, t), 1, 1) &\geq 0 \\ \theta_2(N_2(Bz, w, kt), 0, N_2(w, Bz, t), 0, 0) &\leq 0 \end{aligned}$$

Thus, from (H_{10}) and (H_{11}), we get

$$\begin{aligned} M_2(Bz, w, kt) &\geq M_2(w, Bz, t) \\ N_2(Bz, w, kt) &\leq N_2(w, Bz, t), \end{aligned}$$

and so $w = Bz = BSw$. Using inequalities (4.1), (4.2) and (4.3), (4.4) we have respectively:

$$\begin{aligned} z &= Tw \text{ and } z = Tw = TBz \\ w &= ATw. \end{aligned}$$

The same result holds also if one of the mappings B, S, T is continuous . To prove the uniqueness, suppose that TB and SA have a second distinct common fixed point z'. Then, using inequalities (4.1) and (4.2), we get

$$\begin{aligned}\varphi_1\left(M_1(SAz, TBz', kt), M_1(z, z', t), M_1(z, SAz, t), M_1(z', TBz', t), M_2(Az, Bz', t))\right) &\geq 0\\ \theta_1\left(N_1(SAz, TBz', kt), N_1(z, z', t), N_1(z, SAz, t), N_1(z', TBz', t), N_2(Az, Bz', t)\right) &\leq 0\end{aligned}$$

that is

$$\begin{aligned}\varphi_1\left(M_1(z, z', kt), M_1(z, z', t), M_1(z, z, t), M_1(z', z', t), M_2(Az, Bz', t))\right) &\geq 0\\ \theta_1\left(N_1(z, z', kt), N_1(z, z', t), N_1(z, z, t), N_1(z', z', t), N_2(Az, Bz', t)\right) &\leq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_1\left(M_1(z, z', kt), M_1(z, z', t), 1, 1, M_2(Az, Bz', t))\right) &\geq 0\\ \theta_1\left(N_1(z, z', kt), N_1(z, z', t), 0, 0, N_2(Az, Bz', t)\right) &\leq 0\end{aligned}$$

As φ_1 and θ_1 are nonincreasing in second variable, we get

$$\begin{aligned}\varphi_1\left(M_1(z, z', kt), M_1(z, z', kt), 1, 1, M_2(Az, Bz', t))\right) &\geq 0\\ \theta_1\left(N_1(z, z', kt), N_1(z, z', kt), 0, 0, N_2(Az, Bz', t)\right) &\leq 0,\end{aligned}$$

which implies by (H_8) and (H_9)

$$M_1(z, z', kt) \geq M_2(Az, Bz', t) \tag{4.25}$$
$$N_1(z, z', kt) \leq N_2(Az, Bz', t). \tag{4.26}$$

Further, applying inequalities (4.3) and (4.4), we obtain

$$\begin{aligned}\varphi_2\left(M_2(Bz', Az, kt), M_2(Az, Bz', t), M_2(Az, Bz', t), M_2(Bz', Az, t), M_1(z', z, t)\right) &\geq 0\\ \theta_2\left(N_2(Bz', Az, kt), N_2(Az, y', t), N_2(Az, Bz', t), N_2(Bz', Az, t), N_1(z', z, t)\right) &\leq 0.\end{aligned}$$

Using (H_{10}) and (H_{11}), we get

$$M_2(Bz', Az, kt) \geq M_1(z', z, t) \tag{4.27}$$
$$N_2(Bz', Az, kt) \leq N_1(z', z, t). \tag{4.28}$$

By (4.25), (4.26) and (4.27), (4.28)

$$\begin{aligned}M_1(z, z', kt) &\geq M_1(z', z, t)\\ N_1(z, z', kt) &\leq N_1(z', z, t).\end{aligned}$$

Therefore, contradiction with Lemma 2, then, the pair TB and SA have a unique common fixed point. The uniqueness of w follows in a similar manner. The proof is complete. $\square$

References

[1] Alaca, C., Turkoglu, D., & Yildiz, C. (2006). Fixed points in intuitionistic fuzzy metric spaces. *Chaos, Solitons & Fractals*, 29. 1073–1078.

[2] Alaca, C., Altun, I., & Turkoglu, D. (2008). On Compatible Mappings of Type (I) and (II) in Intuitionistic Fuzzy Metric Spaces, *Communications of the Korean Mathematical Society*, 23 (3), 427–446.

[3] Aliouche, A. (2007). Common fixed point theorems via an implicit relation and new properties, *Soochow Journal of Mathematics*, 33 (4), 593–601.

[4] Aliouche, A., & Popa, V. (2008). Common fixed point theorems for occasionally weakly compatible mappings via implicit relations, *Filomat*, 22 (2),99–107.

[5] Altun, I., & Turkoglu D. (2008). Some fixed point theorems on fuzzy metric spaces with implicit relations, *Commun. Korean Math. Soc.*, 23, 111–124.

[6] Atanassov, K. (1986). Intuitionistic Fuzzy sets, *Fuzzy Sets and System*, 20 (1), 87–96.

[7] Beg, I., Gupta, V., & Kanwar, A. (2015). Fixed points on inyuitionistic fuzzy matric spaces using the $E - A$ property, *J. Nonlinear Funct. Anal.*, 2015, Article ID 20.

[8] Cho, Y. J. (1997). Fixed points in fuzzy metric spaces, *J. Fuzzy. Math.*, 5 (4), 949–962.

[9] Fisher, B. (1981). Fixed point on two metric spaces, *Glasnik Mat.*, 16 (36), 333–337.

[10] George, A. & Veeramani, P. (1994) On some result in fuzzy metric space, *Fuzzy Sets and Systems*, 64, 395–399.

[11] Hamaizia, T., & Murthy, P. P. (2017). Common Fixed Point Theorems in Relatively Intuitionistic Fuzzy Metric Spaces, *Gazi Unuversity Journal of Science*, 30 (1), 355–362.

[12] Kaleva, O., & Seikkala, S. (1984). On fuzzy metric spaces, *Fuzzy Sets and Systems*, 12, 215–229.

[13] Kramosil, O., & Michalek, J. (1975) Fuzzy metric and statistical metric spaces, *Kybernetica*, 11, 326–334.

[14] Kumar, S., Bhatia, S. S., & Manro, S. (2012). Common Fixed Point Theorems for Weakly Maps Satisfying E.A. Property, in Intuitionistic Fuzzy Metric Spaces Using Implicit Relation. *Global Journal of Science Frontier Research Mathematics and Decision Sciences*, 12.

[15] Kumar, S., & Fisher, B. (2010). A common fixed point theorem in fuzzy metric space using property (E.A) and implicit relation, *Thai J. Math.*,(3), 439–446.

[16] Menger, K. (1942). Statistical metrics, *Proc. Nat. Acad. Sci.*, 28, 535–537.

[17] Manro, S. (2015). A common fixed point theorem for weakly compatible maps satisfying common property (E. A.) and implicit relation in intuitionistic fuzzy metric spaces, *Int. J. Nonlinear Anal. Appl.*, 6 (1), 1–8.

[18] Manro, S., Bouharjera, H., & Singh, S. (2010). A Common fixed point theorem in Intuitionistic fuzzy metric Space by using Sub-Compatible maps, *Int. J. Contemp. Math. Sciences*, 5 (55), 2699– 2707.

[19] Manro, S., Bhatia, S. S., Kumar, S., & Mishra, R. (2011). Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Spaces, *International Mathematical Forum*, 6 (64), 3189–3198.

[20] Manro, S., Kumar, S. & Bhatia, S. S. (2012). Common fixed point theorem in intuitionistic fuzzy metric spaces using common (E. A.) property and implicit relation, *Journal of Advanced Studies in Topology*, 3 (3), 60–68.

[21] Park, J. H. (2004). Intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals*, 22, 1039–1046.

[22] Pathak, H. K., Tiwari, R., & Khan, M. S. (2007). A common fixed point theorem satisfying integral type implicit relation, *Applied Mathematics E-notes*, 7, 222–228.

[23] Popa, V. (2005). A general fixed point theorem for two pairs of mappings on two metric spaces, *Novi Sad J. Math.*, 35 (2), 79–83.

[24] Popa, V. (1999). Some fixed point theorems for compatible mappings satisfying an implicit relation, *Demonstratio Math.*, 32, 157–163.

[25] Rodriguez, L. J., & Ramaguera, S. (2004). The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems*, 147, 273–283.

[26] Schweizer, B., & Sklar, A. (1960). Statistical metric spaces, *Pacific J. Math.*, 10, 313–334.

[27] Sedghi, S., Altun, I., & Shobe, N. (2010). Coupled fixed point theorems for contractions in fuzzy metric spaces, *Non linear Anal.*, 1298–1304.

[28] Sharma, S., Kutukcu, S., & Pathak, A. (2009). Common fixed point theorems for weakly compatible mappings in intuitionistic fuzzy metric spaces, *J. Fuzzy Math.*, 17, 225–240.

[29] Telci, M. (2001). Fixed points on two complete and compact metric spaces, *Applied Mathematics and Mechanics*, 22 (5), 564–568.

[30] Turkoglu, D., Alaca, C., & Yildiz, C. (2006). Compatible maps and Compatible maps of type (α) and (β) in intuitionistic fuzzy metric spaces, *Demonstratio Math.*, 39 (3), 671–684.

[31] Zadeh, L. A. (1965). Fuzzy sets, *Inform and Control*, 8, 338–353.