Notes on Intuitionistic Fuzzy Sets
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On extended intuitionistic fuzzy matrices

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Abstract: In this paper the concept of intuitionistic fuzzy index matrix is extended. Its basic properties are discussed and statements are formulated.
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1 Introduction

Here, as a continuation of [3, 4, 5], we extend the concepts of Index Matrix (IM) and Intuitionistic Fuzzy IM (IFIM) introducing the concept of an Extended IFIM (EIFIM).

In Section 2, we give the definition of an EIFIM and the standard operations over two EIFIMs, as well as examples. In the next sections, essentially new operations over EIFIMs are introduced and their properties are discussed.

Initially, we give some remarks on Intuitionistic Fuzzy Sets (IFSs, see, e.g., [2, 5]) and especially, of their partial case, Intuitionistic Fuzzy Pairs (IFPs; see [7]). The IFP is an object with the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process and which conponents (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$.

First, in [7] we defined the relations

$$\begin{array}{lll} x < y & \text{iff} & a < c \text{ and } b > d \\ x \leq y & \text{iff} & a \leq c \text{ and } b \geq d \\ x > y & \text{iff} & a > c \text{ and } b < d \\ x \geq y & \text{iff} & a \geq c \text{ and } b \leq d \\ x = y & \text{iff} & a = c \text{ and } b = d \end{array}$$

Second, we defined analogous of operations "conjunction" and "disjunction":

$$x \&_1 y = \langle \min(a, c), \max(b, d) \rangle$$
$$x \vee_1 y = \langle \max(a, c)), \min(b, d) \rangle$$
$$x + y = \langle a + c - a.c, b.d \rangle$$
$$x.y = \langle a.c, b + d - b.d \rangle$$
$$x@y = \langle \frac{a+c}{2}, \frac{b+d}{2} \rangle.$$

Third, we defined analogues of operations "implication" and "negation". In Table 1 the existing currently 45 negations are given. In some of these definitions, we use the functions sg and $\overline{\text{sg}}$ that are defined by:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Table 1

$\neg_1$	$\langle x, b, a \rangle$
$\neg_2$	$\langle x, \overline{\text{sg}}(a), \text{sg}(a) \rangle$
$\neg_3$	$\langle x, b, a.b + a^2 \rangle$
$\neg_4$	$\langle x, b, 1 - b \rangle$
$\neg_5$	$\langle x, \overline{\text{sg}}(1 - b), \text{sg}(1 - b) \rangle$
$\neg_6$	$\langle x, \overline{\text{sg}}(1 - b), \text{sg}(a) \rangle$
$\neg_7$	$\langle x, \overline{\text{sg}}(1 - b), a \rangle$
$\neg_8$	$\langle x, 1 - a, a \rangle$
$\neg_9$	$\langle x, \overline{\text{sg}}(a), a \rangle$
$\neg_{10}$	$\langle x, \overline{\text{sg}}(1 - b), 1 - b \rangle$
$\neg_{11}$	$\langle x, \text{sg}(b), \overline{\text{sg}}(b) \rangle$
$\neg_{12}$	$\langle x, b.(b + a), \min(1, a.(b^2 + a + b.a)) \rangle$
$\neg_{13}$	$\langle x, \text{sg}(1 - a), \overline{\text{sg}}(1 - a) \rangle$
$\neg_{14}$	$\langle x, \text{sg}(b), \overline{\text{sg}}(1 - a) \rangle$
$\neg_{15}$	$\langle x, \overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - a) \rangle$
$\neg_{16}$	$\langle x, \overline{\text{sg}}(a), \overline{\text{sg}}(1 - a) \rangle$
$\neg_{17}$	$\langle x, \overline{\text{sg}}(1 - b), \overline{\text{sg}}(b) \rangle$
$\neg_{18}$	$\langle x, b.\text{sg}(a), a.\text{sg}(b) \rangle$
$\neg_{19}$	$\langle x, b.\text{sg}(a), 0 \rangle$
$\neg_{20}$	$\langle x, b, 0 \rangle$
$\neg_{21}$	$\langle x, \min(1 - a, \text{sg}(a)), \min(a, \text{sg}(1 - a)) \rangle$
$\neg_{22}$	$\langle x, \min(1 - a, \text{sg}(a)), 0 \rangle$
$\neg_{23}$	$\langle x, 1 - a, 0 \rangle$
$\neg_{24}$	$\langle x, \min(b, \text{sg}(1 - b)), \min(1 - b, \text{sg}(b)) \rangle$
$\neg_{25}$	$\langle x, \min(b, \text{sg}(1 - b)), 0 \rangle$

$\neg_{26}$	$\langle x, b, a.b + \overline{\text{sg}}(1-a)\rangle$
$\neg_{27}$	$\langle x, 1-a, a.(1-a) + \overline{\text{sg}}(1-a)\rangle$
$\neg_{28}$	$\langle x, b, (1-b).b + \overline{\text{sg}}(b)\rangle$
$\neg_{29}$	$\langle x, \max(0, b.a + \overline{\text{sg}}(1-b)), \min(1, a.(b.a + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(1-a))\rangle$
$\neg_{30}$	$\langle x, a.b, a.(a.b + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(1-a)\rangle$
$\neg_{31}$	$\langle x, \max(0, (1-a).a + \overline{\text{sg}}(a)), \min(1, a.((1-a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1-a))\rangle$
$\neg_{32}$	$\langle x, (1-a).a, a.((1-a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1-a)\rangle$
$\neg_{33}$	$\langle x, b.(1-b) + \overline{\text{sg}}(1-b), (1-b).(b.(1-b) + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(b))\rangle$
$\neg_{34}$	$\langle x, b.(1-b), (1-b).(b.(1-b) + \overline{\text{sg}}(1-b)) + \overline{\text{sg}}(b)\rangle$
$\neg_{35}$	$\langle \frac{b}{2}, \frac{1+a}{2}\rangle$
$\neg_{36}$	$\langle \frac{b}{3}, \frac{2+a}{3}\rangle$
$\neg_{37}$	$\langle \frac{2b}{3}, \frac{2a+1}{3}\rangle$
$\neg_{38}$	$\langle \frac{1-a}{3}, \frac{2+a}{3}\rangle$
$\neg_{39}$	$\langle \frac{b}{3}, \frac{3-b}{3}\rangle$
$\neg_{40}$	$\langle \frac{2-2a}{3}, \frac{1+2a}{3}\rangle$
$\neg_{41}$	$\langle \frac{2b}{3}, \frac{3-2b}{3}\rangle$
$\neg_{42,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda}$, where $\lambda \geq 1$
$\neg_{43,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1}$, where $\gamma \geq 1$
$\neg_{44,\alpha,\beta}$	$\langle \frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta}$, where $\alpha \geq 1, \beta \in [0, \alpha]$
$\neg_{45,\varepsilon,\eta}$	$\langle \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle$

2 Basic definition

Extending [3, 4], the basic definition of the IFIM-extension is given.

Let I be a fixed set. By IFIM with index sets K and L $(K, L \subset I)$, we denote the object:

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$.

For briefly, we can mention the above object by $[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$), where

$$K = \{k_1, k_2, ..., k_m\},$$

$$L = \{l_1, l_2, ..., l_n\},$$

for $1 \leq i \leq m$, and $1 \leq j \leq n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0,1].$$

Now, the new object is defined by:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0,1],$$

$$\alpha_1^k, \beta_1^k, \alpha_1^k + \beta_1^k \in [0,1],$$

$$\alpha_1^l, \beta_1^l, \alpha_1^l + \beta_1^l \in [0,1]$$

and here and below,

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \leq i \leq m\},$$

$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \leq j \leq n\}.$$

Let

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K : (\alpha_i^k < \alpha_i^p) \ \& \ (\beta_i^k > \beta_i^p)).$$

$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K : (\alpha_i^k \leq \alpha_i^p) \ \& \ (\beta_i^k \geq \beta_i^p)).$$

3 Operations over EIFIMs

For the EIMs $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], B = [P^*, Q^*, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other specific ones.

(a) addition-(max,min)

$$A \oplus_{(\max,\min)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \max(\alpha_i^k, \alpha_r^p), & \text{if } t_u \in K \cap P \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w \in L - Q \\ \beta_s^q, & \text{if } t_w \in Q - L \\ \min(\beta_j^l, \beta_s^q), & \text{if } t_w \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \ \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

(b) addition-(min,max)

$$A \oplus_{(\min,\max)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where $T^*, V^*, \alpha_u^t, \beta_w^v$, have the forms from (a), but

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \ \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

(c) termwise multiplication-(max,min)

$$A \otimes_{(\max,\min)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cap P\},$$

$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cap Q\},$$
$$\alpha_u^t = \min(\alpha_i^k, \alpha_r^p),\ \text{for } t_u = k_i = p_r \in K \cap P,$$
$$\beta_w^v = \min(\beta_j^l, \beta_s^q),\ \text{for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle.$$

(d) termwise multiplication-(min,max)

$$A \otimes_{(\min,\max)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where $T^*, V^*, \alpha_u^t, \beta_w^v$, have the forms from (c), but

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle.$$

(e) multiplication-(max,min)

$$A \odot_{(\max,\min)} B = [T^*, V^*, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = (K \cup (P - L))^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cup (P - L)\},$$
$$V^* = (Q \cup (L - P))^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in Q \cup (L - P)\},$$
$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u = k_i \in K \\ \\ \alpha_r^p, & \text{if } t_u = p_r \in P - L \end{cases},$$
$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - P \\ \\ \beta_s^q, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle =$$
$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \\ \langle \max\limits_{l_j = p_r \in L \cap P} (\min(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \\ \quad \min\limits_{l_j = p_r \in L \cap P} (\max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}))\rangle, & \\ \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases}$$

(f) multiplication-(min,max)

$$A \odot_{(\min,\max)} B = [T^*, V^*, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where $T^*, V^*, \alpha_u^t, \beta_w^v$, have the forms from (e), but

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \langle \max\limits_{l_j = p_r \in L \cap P} (\min(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \quad \min\limits_{l_j = p_r \in L \cap P} (\max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \rangle, & \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

(g) structural subtraction $A \ominus B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$, where

$$T^* = (K - P)^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K - P\},$$

$$V^* = (L - Q)^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L - Q\},$$

for the set–theoretic subtraction operation and

$$\alpha_u^t = \alpha_i^k, \text{ for } t_u = k_i \in K - P,$$

$$\beta_w^v = \beta_j^l, \text{ for } v_w = l_j \in L - Q$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

(e) negation of an EIFIM $\neg A = [T^*, V^*, \{\neg\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$, where $\neg$ is one of the above (or another) negations.

(f) termwise subtraction

$$A -_{\text{max,min}} B = A \oplus_{\text{max,min}} \neg B,$$

$$A -_{\text{min,max}} B = A \oplus_{\text{min,max}} \neg B.$$

4 Relations over EIFIMs

Let the two IFIMs $A = [K^*, L^*, \{\langle a_{k,l}, b_{k,l} \rangle\}]$ and $B = [P^*, Q^*, \{\langle c_{p,q}, d_{p,q} \rangle\}]$ be given. We shall introduce the following (new) definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion"*.

Definition 1: The strict relation "inclusion about dimension" is

$$A \subset_d B \text{ iff } ((K^* \subset P^*) \& (L^* \subset Q^*)) \vee (K^* \subseteq P^*) \& (L^* \subset Q^*) \vee (K^* \subset P^*) \& (L^* \subseteq Q^*))$$
$$\& (\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l} \rangle = \langle c_{k,l}, d_{k,l} \rangle).$$

Definition 2: The non-strict relation "inclusion about dimension" is

$$A \subseteq_d B \text{ iff } (K^* \subseteq P^*)\&(L^* \subseteq Q^*)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 3: The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (K^* = P^*)\&(L^* = Q^*)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 4: The non-strict relation "inclusion about value" is

$$A \subseteq_v B \text{ iff } (K^* = P^*)\&(L^* = Q^*)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 5: The strict relation "inclusion" is

$$A \subset B \text{ iff } ((K^* \subset P^*)\&(L^* \subset Q^*)) \vee (K^* \subseteq P^*)\&(L^* \subset Q^*) \vee (K^* \subset P^*)\&(L^* \subseteq Q^*))$$

$$\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

Definition 6: The non-strict relation "inclusion" is

$$A \subseteq B \text{ iff } (K^* \subseteq P^*)\&(L^* \subseteq Q^*)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

It can be directly seen that for every two IFIMs A and B,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

Operations "reduction" and "projection" coincide with the respective operations defined over IMs in [3], while hierarhical operations over IMs, described in [3, 6], are not applied here.

5 Level operators over EIFIMs

Let the EIFIM $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$ be given.

Let for $i = 1, 2, 3 : \rho_i, \sigma_i, \rho_i + \sigma_i \in [0, 1]$ be fixed.

In [2, 5], some level operators are defined. One of them, for a given IFS

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle | x \in E\}$$

is

$$N_{\alpha,\beta}(X) = \{\langle x, \mu_X(x), \nu_X(x)\rangle | x \in E \;\&\; \mu_X(x) \geq \alpha \;\&\; \nu_X(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ are fixed and $\alpha + \beta \leq 1$.

Here, its analogues are intoduced. They are three: $N^1_{\rho_1,\sigma_1}, N^2_{\rho_2,\sigma_2}, N^3_{\rho_3,\sigma_3}$ and influence over K-, L-indices and $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$-elements, respectively. The three operators can be applied over an EIFIM A sequentially, or simultaneously. In the first case, theis forms are

$$N^1_{\rho_1,\sigma_1}(A) = [N_{\rho_1,\sigma_1}(K^*), L^*, \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$$

only for $\langle k_i, \alpha_i^k, \beta_i^k\rangle \in N_{\rho_1,\sigma_1}(K^*)$ and for each $\langle l_j, \alpha_j^l, \beta_j^l\rangle \in L^*$;

$$N^2_{\rho_2,\sigma_2}(A) = [K^*, N_{\rho_2,\sigma_2}(L^*), \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$$

for each $\langle k_i, \alpha_i^k, \beta_i^k\rangle \in K^*$ and only for $\langle l_j, \alpha_j^l, \beta_j^l\rangle \in N_{\rho_2,\sigma_2}(L^*)$;

$$N^3_{\rho_3,\sigma_3}(A) = [K^*, L^*, \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \mu_{k_i,l_j} \geq \rho_3 \ \&\ \nu_{k_i,l_j} \leq \sigma_3 \\ \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases},$$

In the second case, their form is

$$(N^1_{\rho_1,\sigma_1}, N^2_{\rho_2,\sigma_2}, N^3_{\rho_3,\sigma_3})(A) = [N_{\rho_1,\sigma_1}(K^*), N_{\rho_2,\sigma_2}(L^*), \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } \langle k_i, \alpha_i^k, \beta_i^k\rangle \in N_{\rho_1,\sigma_1}(K^*) \\ & \text{and } \langle l_j, \alpha_j^l, \beta_j^l\rangle \in N_{\rho_2,\sigma_2}(L^*) \\ & \text{and } \mu_{k_i,l_j} \geq \rho_3 \ \&\ \nu_{k_i,l_j} \leq \sigma_3 \\ \\ \langle 0, 1\rangle, & \text{if } \langle k_i, \alpha_i^k, \beta_i^k\rangle \in N_{\rho_1,\sigma_1}(K^*) \\ & \text{and } \langle l_j, \alpha_j^l, \beta_j^l\rangle \in N_{\rho_2,\sigma_2}(L^*) \\ & \text{and } \mu_{k_i,l_j} < \rho_3 \ \vee\ \nu_{k_i,l_j} > \sigma_3 \end{cases},$$

6 Aggregation operations over EIFIMs

Let the EIFIM

$$A = \begin{array}{l|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l\rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l\rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l\rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k\rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k\rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k\rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

be given and let $k_0 \notin K$ and $l_0 \notin L$ be two fixed indeces.

Now, we introduce the following 18 operations over it.

(a) (max,max)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \max\limits_{1\le i\le m} \alpha_i^k, \min\limits_{1\le i\le m} \beta_i^k \rangle & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_1}, \min\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \ldots & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_n}, \min\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(b) (max,ave)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \max\limits_{1\le i\le m} \alpha_i^k, \min\limits_{1\le i\le m} \beta_i^k \rangle & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_1}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_1} \rangle & \ldots & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_n}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_n} \rangle \end{array},$$

(c) (max,min)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \max\limits_{1\le i\le m} \alpha_i^k, \min\limits_{1\le i\le m} \beta_i^k \rangle & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_1}, \max\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \ldots & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_n}, \max\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(d) (ave,max)-row-aggregation

$$\rho_{(\min,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \frac{1}{m} \sum\limits_{i=1}^{m} \alpha_i^k, \frac{1}{m} \sum\limits_{i=1}^{m} \beta_i^k \rangle & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_1}, \min\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \ldots & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_n}, \min\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(e) (ave,ave)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \frac{1}{m} \sum\limits_{i=1}^{m} \alpha_i^k, \frac{1}{m} \sum\limits_{i=1}^{m} \beta_i^k \rangle & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_1}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_1} \rangle & \ldots & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_n}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_n} \rangle \end{array},$$

(f) (ave,min)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \frac{1}{m} \sum\limits_{i=1}^{m} \alpha_i^k, \frac{1}{m} \sum\limits_{i=1}^{m} \beta_i^k \rangle & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_1}, \max\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \ldots & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_n}, \max\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(g) (min,max)-row-aggregation

$$\rho_{(\min,\max)}(A, k_0)$$

$$= \begin{array}{l|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \min\limits_{1\le i\le m} \alpha_i^k, \max\limits_{1\le i\le m} \beta_i^k \rangle & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_1}, \min\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \ldots & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_n}, \min\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(h) (min,ave)-row-aggregation

$$\rho_{(\max,\max)}(A,k_0)$$

$$= \begin{array}{c|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \min\limits_{1\le i\le m} \alpha_i^k, \max\limits_{1\le i\le m} \beta_i^k \rangle & \langle \frac{1}{m}\sum\limits_{i=1}^{m} \mu_{k_i,l_1}, \frac{1}{m}\sum\limits_{i=1}^{m} \nu_{k_i,l_1} \rangle & \dots & \langle \frac{1}{m}\sum\limits_{i=1}^{m} \mu_{k_i,l_n}, \frac{1}{m}\sum\limits_{i=1}^{m} \nu_{k_i,l_n} \rangle \end{array},$$

(i) (min,min)-row-aggregation

$$\rho_{(\max,\max)}(A,k_0)$$

$$= \begin{array}{c|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_0, \langle \min\limits_{1\le i\le m} \alpha_i^k, \max\limits_{1\le i\le m} \beta_i^k \rangle & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_1}, \max\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \dots & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_n}, \max\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(j) (max,max)-column-aggregation

$$\sigma_{max}(A,l_0) = \begin{array}{c|c} & l_0, \langle \max\limits_{1\le i\le m} \alpha_j^l, \min\limits_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_1,l_j}, \min\limits_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_i,l_j}, \min\limits_{1\le j\le n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_m,l_j}, \min\limits_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(k) (max,ave)-column-aggregation

$$\sigma_{max}(A,l_0) = \begin{array}{c|c} & l_0, \langle \max\limits_{1\le i\le m} \alpha_j^l, \min\limits_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n} \mu_{k_1,l_j}, \frac{1}{n}\sum\limits_{j=1}^{n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n} \mu_{k_i,l_j}, \frac{1}{n}\sum\limits_{j=1}^{n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n} \mu_{k_m,l_j}, \frac{1}{n}\sum\limits_{j=1}^{n} \nu_{k_m,l_j} \rangle \end{array},$$

(l) (max,min)-column-aggregation

$$\sigma_{max}(A,l_0) = \begin{array}{c|c} & l_0, \langle \max\limits_{1\le i\le m} \alpha_j^l, \min\limits_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \min\limits_{1\le j\le n} \mu_{k_1,l_j}, \max\limits_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \min\limits_{1\le j\le n} \mu_{k_i,l_j}, \max\limits_{1\le j\le n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \min\limits_{1\le j\le n} \mu_{k_m,l_j}, \max\limits_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(m) (ave,max)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \frac{1}{n}\sum\limits_{j=1}^{n} \alpha_j^l, \frac{1}{n}\sum\limits_{j=1}^{n} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_1,l_j}, \min\limits_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_i,l_j}, \min\limits_{1\le j\le n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_m,l_j}, \min\limits_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(n) (ave,ave)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \frac{1}{n}\sum\limits_{j=1}^{n} \alpha_j^l, \frac{1}{n}\sum\limits_{j=1}^{n} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n} \mu_{k_1,l_j}, \frac{1}{n}\sum\limits_{j=1}^{n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n} \mu_{k_i,l_j}, \frac{1}{n}\sum\limits_{j=1}^{n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n} \mu_{k_m,l_j}, \frac{1}{n}\sum\limits_{j=1}^{n} \nu_{k_m,l_j} \rangle \end{array}.$$

(o) (ave,min)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \frac{1}{n}\sum\limits_{j=1}^{n} \alpha_j^l, \frac{1}{n}\sum\limits_{j=1}^{n} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \min\limits_{1\le j\le n} \mu_{k_1,l_j}, \max\limits_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \min\limits_{1\le j\le n} \mu_{k_i,l_j}, \max\limits_{1\le j\le n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \min\limits_{1\le j\le n} \mu_{k_m,l_j}, \max\limits_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(p) (min,max)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \min\limits_{1\le i\le m} \alpha_j^l, \max\limits_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_1,l_j}, \min\limits_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_i,l_j}, \min\limits_{1\le j\le n} \nu_{k_i,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_m,l_j}, \min\limits_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(q) (min,ave)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \min\limits_{1 \le i \le m} \alpha_j^l, \max\limits_{1 \le i \le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \frac{1}{n} \sum\limits_{j=1}^{n} \mu_{k_1, l_j}, \frac{1}{n} \sum\limits_{j=1}^{n} \nu_{k_1, l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \frac{1}{n} \sum\limits_{j=1}^{n} \mu_{k_i, l_j}, \frac{1}{n} \sum\limits_{j=1}^{n} \nu_{k_i, l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \frac{1}{n} \sum\limits_{j=1}^{n} \mu_{k_m, l_j}, \frac{1}{n} \sum\limits_{j=1}^{n} \nu_{k_m, l_j} \rangle \end{array},$$

(r) (min,min)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \min\limits_{1 \le i \le m} \alpha_j^l, \max\limits_{1 \le i \le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \min\limits_{1 \le j \le n} \mu_{k_1, l_j}, \max\limits_{1 \le j \le n} \nu_{k_1, l_j} \rangle \\ \vdots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \min\limits_{1 \le j \le n} \mu_{k_i, l_j}, \max\limits_{1 \le j \le n} \nu_{k_i, l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \min\limits_{1 \le j \le n} \mu_{k_m, l_j}, \max\limits_{1 \le j \le n} \nu_{k_m, l_j} \rangle \end{array}.$$

7 Extended modal and level operators defined over EIFIMs

Let, as above, $x = \langle a, b \rangle$ be an IFP and let $\alpha, \beta \in [0, 1]$. Some of the extended modal operators defined over x have the following forms (see [5, 7]):

$$F_{\alpha,1-\alpha}(x) = \langle a + \alpha.(1 - a - b), b + \beta.(1 - a - b) \rangle, \text{ where } \alpha + \beta \le 1$$
$$G_{\alpha,\beta}(x) = \langle \alpha.a, \beta.b \rangle$$
$$H_{\alpha,\beta}(x) = \langle \alpha.a, b + \beta.(1 - a - b) \rangle$$
$$H^*_{\alpha,\beta}(x) = \langle \alpha.a, b + \beta.(1 - \alpha.a - b) \rangle$$
$$J_{\alpha,\beta}(x) = \langle a + \alpha.(1 - a - b), \beta.b \rangle$$
$$J^*_{\alpha,\beta}(x) = \langle a + \alpha.(1 - a - \beta.b), \beta.b \rangle$$

and let the level operators have the forms:

$$P_{\alpha,\beta}x = \langle \max(\alpha, a), \min(\beta, b) \rangle$$
$$Q_{\alpha,\beta}x = \langle \min(\alpha, a), \max(\beta, b) \rangle,$$

for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \le 1$.

Now we define operators over EIFIMs in analogous way with the operators from Section 5. Let $O^1_{\alpha_1,\beta_1}, O^2_{\alpha_2,\beta_2}, O^3_{\alpha_3,\beta_3}$ be three operators and their arguments $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ satisfy the respective conditions, given above. The three operators influence over K-, L-indices and $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$-elements, respectively. They, as in Section 5, can be applied over an EIFIM A sequentially, or simultaneously. In the first case, theis forms are

$$O^1_{\alpha_1,\beta_1}(A)$$

$$= \begin{array}{c|ccccc}
 & l_1, \langle \alpha^l_1, \beta^l_1 \rangle & \dots & l_j, \langle \alpha^l_j, \beta^l_j \rangle & \dots & l_n, \langle \alpha^l_n, \beta^l_n \rangle \\ \hline
k_1, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_1, \beta^k_1 \rangle) & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_i, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_i, \beta^k_i \rangle) & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_m, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_m, \beta^k_m \rangle) & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle
\end{array},$$

$$O^2_{\alpha_2,\beta_2}(A)$$

$$= \begin{array}{c|ccccc}
 & l_1, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_1, \beta^l_1 \rangle) & \dots & l_j, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_j, \beta^l_j \rangle) & \dots & l_n, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_n, \beta^l_n \rangle) \\ \hline
k_1, \langle \alpha^k_1, \beta^k_1 \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_i, \langle \alpha^k_i, \beta^k_i \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_m, \langle \alpha^k_m, \beta^k_m \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle
\end{array},$$

$$O^3_{\alpha_3,\beta_3}(A)$$

$$= \begin{array}{c|ccccc}
 & l_1, \langle \alpha^l_1, \beta^l_1 \rangle & \dots & l_j, \langle \alpha^l_j, \beta^l_j \rangle & \dots & l_n, \langle \alpha^l_n, \beta^l_n \rangle \\ \hline
k_1, \langle \alpha^k_1, \beta^k_1 \rangle & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle) \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_i, \langle \alpha^k_i, \beta^k_i \rangle & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle) \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_m, \langle \alpha^k_m, \beta^k_m \rangle & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle)
\end{array}.$$

In the second case, the form of the three-tuple of operators is

$$(O^1_{\alpha_1,\beta_1}, O^2_{\alpha_2,\beta_2}, O^3_{\alpha_3,\beta_3})(A)$$

$$= \begin{array}{c|ccc}
 & l_1, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_1, \beta^l_1 \rangle) & \dots & l_n, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_n, \beta^l_n \rangle) \\ \hline
k_1, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_1, \beta^k_1 \rangle) & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle) \\
\vdots & \dots & \vdots & \\
k_i, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_i, \beta^k_i \rangle) & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle) \\
\vdots & \dots & \vdots & \\
k_m, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_m, \beta^k_m \rangle) & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle)
\end{array}.$$

8 Conclusion

As we noted in [3], in future, we will discuss the possibility to apply the apparatus of IMs, and especially, of the IFIMs, for representation of configurations in the game-method for modelling, in "transportational (optimization) problem", in OLAP-cubes in data warehouses and in other areas of applications. These new IMs can be used for description of new types of intuitionitic fuzzy relations and intuitionitic fuzzy graphs (see [5]). That will be an object of our next research.

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