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Operations and Relations over Reduced Generalized Nets

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Abstract: The operations and the relations over Generalized Nets (GNs) are defined in [1, 2] under specific conditions. The main goal of the current paper is to show their connection with the reducing operators over GNs. Two theorems will be formulated and proven for that purpose.
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1 Introduction

Generalized Nets (GNs) are extensions of Petri Nets [4, 3]. They are a means of modelling parallel and concurrent processes. The concept of GN was introduced in year of 1982. Its properties, some of its applications and all aspects of the theory were described in a series of more than 100 papers, published in AMSE Press. In 1991, they were collected in [1]. Later results, published

between 1991 and 2007, were included in [2].
The operations and relations defined over GNs are part of the algebraic aspect of the GNs theory which is the oldest one. These operations are valid on certain conditions stated in [1, 2]. They are slightly changed over the years. The basic definitions of operations and relations over transitions and GNs that are relevant to the current paper are shown in the next section. Section 3 gives short remarks on the reducing operators over GNs. The definitions of the operations over reduced GNs are given in Section 4. The relations over reduced GNs are described in Section 5.

2 Operations and Relations over Transitions and GNs

The relations and operations defined over GN's transitions will be listed here the way they are defined in the GN's theory so far. These definitions are closely related to the statements in the next sections of the current paper.

Let $Z_i = \langle L'_i, L''_i, t^i_1, t^i_2, r^i, M^i, \overset{i}{\square} \rangle$ is a transition in a GN.
The following relations over GN's transitions are defined in [2]:

- $Z_1 = Z_2 \iff (\forall i : 1 \leq i \leq 7)(pr_i Z_1 = pr_i Z_2)$,
  where $pr_i Z$ is the i-th projection of the Z, i.e.
  $pr_i Z \in \{L'_i, L''_i, t^i_1, t^i_2, r^i, M^i, \overset{i}{\square}\}$ for $(1 \leq i \leq 7)$
- $$\begin{aligned} Z_1 \subset Z_2 \iff & (\forall i : 1 \leq i \leq 2)(pr_i Z_1 \subset pr_i Z_2)\& \\ & (pr_3 Z_2 \leq pr_3 Z_1 \leq pr_3 Z_2 + pr_4 Z_2)\& \\ & (pr_3 Z_1 + pr_4 Z_1 \leq pr_3 Z_2 + pr_4 Z_2)\& \\ & (\forall i : 5 \leq i \leq 6)(pr_i Z_1 \subset_1 pr_i Z_2)\&(pr_7 Z_1 \subset_2 pr_7 Z_2), \end{aligned}$$
  where $\subset_1$ is a relation of inclusion over index matrices and $\subset_2$ is a relation of inclusion over Boolean expressions [2, 1].

The following operations over GN's transitions are defined in [2]:

- a *union* of two transitions is the transition
  $$\begin{aligned} Z_1 \cup Z_2 = \langle & L'_1 \cup L'_2, L''_1 \cup L''_2, \min(t^1_1, t^2_1), \max(t^1_1 + t^1_2, t^2_1 + t^2_2) \\ & - \min(t^1_1, t^2_1), r^1 + r^2, M^1 + M^2, \vee(\overset{1}{\square}, \overset{2}{\square}) \rangle; \end{aligned}$$
- an *intersection* of two transitions is the transition
  $$Z_1 \cap Z_2 = \langle L'_1 \cap L'_2, L''_1 \cap L''_2, \max(t^1_1, t^2_1), \min(t^1_1 + t^1_2, t^2_1 + t^2_2)$$

$$-\max(t_1^1, t_1^2), r^1 \times r^2, M^1 \times M^2, \wedge(\square^1, \square^2)\rangle;$$

- a *composition* of two transitions is the transition

$$Z_1 \circ Z_2 = \langle L_1' \cup (L_2' - L_1''), L_2'' \cap (L_1'' - L_2'), t_1^1, \max(t_1^1 + t_2^1, t_1^2 + t_2^2)$$

$$-\min(t_1^1, t_1^2), r^1 \circ r^2, M^1 \circ M^2, \vee(\square^1, \overline{\square}^2)\rangle,$$

  where $\overline{\square}$ is a result of removing all identifier which are elements of the set $L_1'' \cup L_2'$ from $\square$.

If $L_1' \cap L_2' = \emptyset$ and $L_1'' \cap L_2'' = \emptyset$, $Z_1 \cap Z_2 = Z_\emptyset$, where $Z_\emptyset$ is the empty transition.
Let E_1 and E_2 are two given GNs and for $i = 1, 2$:

$$E_i = \langle\langle A_i, \pi_A^i, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle T_i, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle.$$

The operations defined over GNs are listed below.

- The union of two GNs is defined on certain conditions in [1, 2] as follows:

$$E_1 \cup E_2 = \langle\langle A_1 \overline{\cup} A_2, \pi_A^1 \cup \pi_A^2, \pi_L^1 \cup \pi_L^2, c^1 \cup c^2, f^1 \cup f^2, \theta_1^1 \cup \theta_1^2, \theta_2^1 \cup \theta_2^2\rangle,$$

$$\langle K_1 \cup K_2, \pi_K^1 \cup \pi_K^2, \theta_K^1 \cup \theta_K^2\rangle,$$

$$\langle \min(T_1, T_2), GCD(t_1^o, t_2^o), \max_{1 \le i \le 2}(T_i + \frac{t_i^*.t_i^o}{GCD(t_1^o, t_2^o)} - \min(T_1, T_2))\rangle,$$

$$\langle X_1 \cup X_2, \Phi_1 \cup \Phi_2, b_1 \cup b_2\rangle\rangle,$$

  where

$$A_1 \overline{\cup} A_2 = \bigcup_{i=1}^{2} \{Z | (Z \in A_i)(\forall Z' \in A_{3-i})(Z \cap Z' = Z_\emptyset)\} \cup$$

$$\bigcup_{i=1}^{2} \{Z | (\exists Z' \in A_i)(\exists Z'' \in A_{3-i})(Z' \cap Z'' \neq Z_\emptyset) \& (Z = Z' \cup Z'')\}.$$

- The composition of two GNs is defined on certain conditions in [1, 2] as follows:

$$E_1 \circ E_2 = \begin{cases} E_1, & \text{if } T_2 + t_2^* < T_1 \\ E_3, & \text{if } T_1 \le T_2 + t_2^* \end{cases},$$

  where

$$E_3 = \langle\langle A_1 \overline{\cup} A_2, \pi_A^1 \cup \pi_A^2, \pi_L^1 \cup \pi_L^2, c^1 \cup c^2, f^1 \cup f^2, \theta_1^1 \cup \theta_1^2, \theta_2^1 \cup \theta_2^2\rangle,$$
$$\langle K_1 \cup K_2, \pi_K^1 \cup \pi_K^2, \theta_K^1 \cup \theta_K^2\rangle,$$
$$\langle T_1, GCD(t_1^o, t_2^o), \max_{1 \le i \le 2}(T_i + \frac{t_i^*.t_i^o}{GCD(t_1^o, t_2^o)} - T_1)\rangle,$$
$$\langle X_1 \cup X_2, \Phi_1 \cup \Phi_2, b_1 \cup b_2\rangle\rangle.$$

The relations over two GNs are defined in [1, 2]. They are based on the GNs' structure, GNs' functioning and the results of GNs' work. Here they are:

(1) $E_1 = E_2 \iff (\forall 1 \le i \le 4)(pr_i E_1 = pr_i E_2)$,
where

$$pr_i E_1 = pr_i E_2 \iff (dim(pr_i E_1) = dim(pr_i E_2))\&(\forall j_i : 1 \le j_i \le$$
$$dim(pr_i E_1))(pr_{j_i} pr_i E_1 = pr_{j_i} pr_i E_2)$$

and $dim(Y)$ is the dimension of the set Y.

(2) $E_1 =_o E_2 \iff (\forall 1 \le i \le 3)(pr_i E_1 = pr_i E_2)$,

(3) $E_1 \subset_* E_2 \iff (\forall Z_1 \in A_1)(\exists Z_2 \in A_2)(Z_1 \subset Z_2)\&$
$(\pi_A^1 = \pi_A^2|E_1)\&(\pi_L^1 = \pi_L^2|E_1)\&(c^1 = c^2|E_1)\&(f^1 = f^2|E_1)\&(\Theta_1^1 = \Theta_1^2|E_1)\&(\Theta_2^1 = \Theta_2^2|E_1)\&$
$(K_1 \subset K_2)\&(\pi_K^1 = \pi_K^2|E_1)\&(\Theta_K^1 = \Theta_K^2|E_1)\&$
$(T_2 \le T_1 \le T_1 + t_1^* \le T_2 + t_2^*)\&(t_1^o = t_2^o)\&$
$(X_1 \subset X_2)\&(\Phi_1 = \Phi_2|E_1)\&(b_1 \le b_2|E_1)$,
where $g = h|E_1$ represents that the function g is a restriction of the function h over the first GN E_1 and for the functions g and h with $Dom(g) = Dom(h)$: $g \le h \iff (\forall x \in Dom(g))(g(x) = h(x))$.

(4) $E_1{}' \subset_o E_2 \iff (A_1 \subset A_2)\&(\emptyset \neq Q_1^I \subset Q_2^I)\&(\pi_A^1 = \pi_A^2|E_1)\&(\pi_L^1 = \pi_L^2|E_1)\&$
$(c^1 = c^2|E_1)\&(f^1 = f^2|E_1)\&(\Theta_1^1 = \Theta_1^2|E_1)\&(\Theta_2^1 = \Theta_2^2|E_1)\&$
$(K_1 \subset K_2)\&(\pi_K^1 = \pi_K^2|E_1)\&(\Theta_K^1 = \Theta_K^2|E_1)\&$
$(T_1 = T_2)\&(t_1^o = t_2^o)\&(t_1^* \leq t_2^*)$.

(5) $E_1 \subset_o' E_2 \iff (A_1 \subset A_2)\&(\emptyset \neq Q_1^O \subset Q_2^O)\&(\pi_A^1 = \pi_A^2|E_1)\&(\pi_L^1 = \pi_L^2|E_1)\&$
$(c^1 = c^2|E_1)\&(f^1 = f^2|E_1)\&(\Theta_1^1 = \Theta_1^2|E_1)\&(\Theta_2^1 = \Theta_2^2|E_1)\&$
$(K_1 \subset K_2)\&(\pi_K^1 = \pi_K^2|E_1)\&(T_2 \leq T_1)\&(t_1^o = t_2^o)\&(T_1 + t_1^* \leq T_2 + t_2^*)$.

(6) $E_1 \overline{\subset}_o E_2 \iff (A_1 \subset A_2)\&(\pi_A^1 = \pi_A^2|E_1)\&(\pi_L^1 = \pi_L^2|E_1)\&$
$(c^1 = c^2|E_1)\&(f^1 = f^2|E_1)\&(\Theta_1^1 = \Theta_1^2|E_1)\&(\Theta_2^1 = \Theta_2^2|E_1)\&$
$(K_1 \subset K_2)\&(\pi_K^1 = \pi_K^2|E_1)\&(T_2 \leq T_1)\&(t_1^o = t_2^o)\&(T_1 + t_1^* \leq T_2 + t_2^*)$.

(7) $E_1 \subset_o E_2 \iff (E_1{}' \subset_o E_2) \vee (E_1 \subset_o' E_2) \vee (E_1 \overline{\subset}_o E_2)$.

(8) $E_1 \subset_\cup E_2 \iff (A_1 \subset A_2)\&(\pi_A^1 = \pi_A^2|E_1)\&(\pi_L^1 = \pi_L^2|E_1)\&$
$(c^1 = c^2|E_1)\&(f^1 = f^2|E_1)\&(\Theta_1^1 = \Theta_1^2|E_1)\&(\Theta_2^1 = \Theta_2^2|E_1)\&$
$(K_1 \subset K_2)\&(\pi_K^1 = \pi_K^2|E_1)\&(\Theta_K^1 = \Theta_K^2|E_1)\&$
$(T_1 = T_2)\&(t_1^o = t_2^o)\&(T_1 + t_1^* = T_2 + t_2^*)\&$
$(X_1 \subset X_2)\&(\Phi_1 = \Phi_2|E_1)\&(b_1 = b_2|E_1)$.

(9) $E_1 <> E_2 \iff (T_1 + t_1^* < T_2) \vee (T_2 + t_2^* < T_1)$.

(10) $E_1[]E_2 \iff (T_1 < T_2 < T_1 + t_1^*) \vee (T_2 < T_1 < T_2 + t_2^*)$.

The definitions of relations based on the results of the work done by the GNs are shown next.
K_i is the set of a GN's tokens and X_i is a function that assigns initial characteristics to every token when it enters a GN's input place. $X_i(\alpha)$ is the set of all initial characteristics of the token α.

For a given token $\alpha \in K_i$ and a given initial characteristic $x \in X_i$:

$$E_i(\alpha, x) = \begin{cases} \langle \alpha, x_{fin}^{\alpha} \rangle, & \text{if } x \in X_i(\alpha) \\ \langle \alpha, x \rangle, & \text{otherwise} \end{cases},$$

$$E_i\{\alpha, x\} = \begin{cases} \langle \alpha, x, x_1^\alpha, x_2^\alpha, \dots, x_{fin}^\alpha \rangle, & \text{if } x \in X_i(\alpha) \\ \langle \alpha, x \rangle, & \text{otherwise} \end{cases},$$

and x_{fin}^α is the final characteristic of the token α and $x_1^\alpha, x_2^\alpha, \dots, x_{fin-1}^\alpha$ are the rest of its characteristics gain while staying in the GN [1, 2].

(11) $E_1 \sqsubset E_2 \iff (K_1 \subset K_2) \& (X_1 \subset X_2) \& (\forall \alpha \in K_1)(\forall x \in X_1) (E_1(\alpha, x) = E_2(\alpha, x))$,

(12) $E_1 \sqsubset_* E_2 \iff (K_1 \subset K_2) \& (X_1 \subset X_2) \&$
$(\forall \alpha \in K_1)(\forall x \in X_1)(\exists i_1, i_2, \dots, i_s : 1 \leq i < i_1 < \dots < i_s \leq fin)$
$(E_1\{\alpha, x\} = E_2\{\alpha, x\})$,

(13) $E_1 \approx E_2 \iff (E_1 \sqsubset E_2) \& (E_2 \sqsubset E_1)$.

(14) $E_1 \approx_* E_2 \iff (E_1 \sqsubset_* E_2) \& (E_2 \sqsubset_* E_1)$.

3 Reducing Operators over GNs

Let Σ be the class of all GNs,

$$\Omega = \{A, \pi_A, \pi_L, c, f, \theta_1, \theta_2, K, \pi_K, \theta_K, T, t^o, t^*, X, \Phi, b\} \cup \{A_i | 1 \leq i \leq 7\},$$

where $A_i = pr_i A (1 \leq i \leq 7)$, i.e. $A_i \in \{L', L'', t_1, t_2, r, M, \square\}$ be the set of all GN's components.

Let $Y \in \Omega$ then Σ^Y be the class of those GNs which lack the Y component [1, 2].

There is no GN without graphical structure (the component A), without input and output places in its transitions (the components A_1 and A_2 respectively) or one without tokens (the component K). Therefore

$$\Sigma^A = \Sigma^{A_1} = \Sigma^{A_2} = \Sigma^K = \emptyset.$$

If $Y_1, Y_2, \dots, Y_s \in \Omega$ for $s \geq 1$ then $\Sigma^{Y_1, Y_2, \dots, Y_s}$ is called $(Y_1, Y_2, \dots, Y_s)$ - class of reduced GNs.

The reducing operators defined over GNs bring together an ordinary GN and its reduced ones. They are closely connected to the corresponding classes of reduced GNs [1, 2].

If Y is a component of a given GN E then the operator R_Y reduces E to a GN without the component Y, $R_Y(E) \in \Sigma^Y$.

The reducing operators R_Y for $Y \in \Omega$ can be represented through one universal operator R:

$$(\forall E \in \Sigma)(\forall Y \in \Omega)(R(E, Y) = R_Y(E)).$$

For $Y_1, Y_2 \in \Omega : R(R(E, Y_1), Y_2) = R(R(E, Y_2), Y_1), \forall E \in \Sigma.$

4 Operations over Reduced GNs

The operations over GNs listed above can be transferred over reduced ones in the following way. The result of applying a reducing operator over a union of two GNs is equal to the union of the two corresponding reduced GNs.

Theorem 4.1. *For every two GNs $E_1, E_2 \in \Sigma$ and $\forall Y \in \Omega$:*

$$R_Y(E_1 \cup E_2) = R_Y(E_1) \cup R_Y(E_2).$$

Proof. A complete proof will be given for only a part of the GN components that can be reduced. The proof for the rest of them will be analogous to the presented ones.

Let $Y = \pi_A$.

$$R_{\pi_A}(E_i) = \langle\langle A_i, *, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle T_i, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle,$$

$i = 1, 2$

$$R_{\pi_A}(E_1) \cup R_{\pi_A}(E_2) = \langle\langle A_1 \overline{\cup} A_2, *, \pi_L^1 \cup \pi_L^2, c^1 \cup c^2, f^1 \cup f^2, \theta_1^1 \cup \theta_1^2, \theta_2^1 \cup \theta_2^2\rangle,$$

$$\langle K_1 \cup K_2, \pi_K^1 \cup \pi_K^2, \theta_K^1 \cup \theta_K^2\rangle,$$

$$\langle \min(T_1, T_2), GCD(t_1^o, t_2^o), \max_{1 \le i \le 2}(T_i + \frac{t_i^* . t_i^o}{GCD(t_1^o, t_2^o)} - \min(T_1, T_2))\rangle,$$

$$\langle X_1 \cup X_2, \Phi_1 \cup \Phi_2, b_1 \cup b_2\rangle\rangle = R_{\pi_A}(E_1 \cup E_2).$$

Since $\pi_L, c, f, \theta_1, \theta_2, K, \pi_K, \theta_K, X, \Phi$ and b are treated as sets, just like π_A, the proof of the theorem statement is analogous when one of them is the reduced component.

Let $Y = T$. Therefore the values of T components are assumed to be 0.

$$R_T(E_i) = \langle\langle A_i, \pi_A^i, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle *, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle,$$

$i = 1, 2$ $R_T(E_1) \cup R_T(E_2) = \langle\langle A_1 \overline{\cup} A_2, \pi_A^1 \cup \pi_A^2, \pi_L^1 \cup \pi_L^2, c^1 \cup c^2, f^1 \cup f^2, \theta_1^1 \cup \theta_1^2, \theta_2^1 \cup \theta_2^2\rangle$,
$\langle K_1 \cup K_2, \pi_K^1 \cup \pi_K^2, \theta_K^1 \cup \theta_K^2\rangle, \langle *, GCD(t_1^o, t_2^o), \max(\frac{t_1^*.t_1^o}{GCD(t_1^o,t_2^o)}, \frac{t_2^*.t_2^o}{GCD(t_1^o,t_2^o)})\rangle$,

$$\langle X_1 \cup X_2, \Phi_1 \cup \Phi_2, b_1 \cup b_2\rangle\rangle = R_T(E_1 \cup E_2).$$

Let $Y = t^o$. Therefore the values of t^o components are assumed to be 1.

$R_{t^o}(E_1) \cup R_{t^o}(E_2) = \langle\langle A_1 \overline{\cup} A_2, \pi_A^1 \cup \pi_A^2, \pi_L^1 \cup \pi_L^2, c^1 \cup c^2, f^1 \cup f^2, \theta_1^1 \cup \theta_1^2, \theta_2^1 \cup \theta_2^2\rangle$,

$$\langle K_1 \cup K_2, \pi_K^1 \cup \pi_K^2, \theta_K^1 \cup \theta_K^2\rangle,$$

$$\langle \min(T_1, T_2), *, \max(T_1 + t_1^*, T_2 + t_2^*) - \min(T_1, T_2)\rangle,$$

$$\langle X_1 \cup X_2, \Phi_1 \cup \Phi_2, b_1 \cup b_2\rangle\rangle = R_{t^o}(E_1 \cup E_2).$$

Let $Y = t^*$.

If t^* component (or the duration of the GN functioning) is reduced from E_1 and E_2, it cannot be evaluated in the union $R_{t^*}(E_1) \cup R_{t^*}(E_2)$ either. This component is reduced in $R_{t^*}(E_1) \cup R_{t^*}(E_2)$ as it is in $R_{t^*}(E_1 \cup E_2)$. All other components in the two GNs are obviously equal. Hence, the statement of the theorem is valid.

A proof in the case of reduced transitions' components will be provided next. L', L'' and r components cannot be reduced otherwise the GN integrity will be lost.

Let $Y = t_1$.

When t_1 component is reduced from a GN, Θ_1 is not defined. It can be assumed that each of the transitions can be activated right after it stops functioning. In that case the value of t_1 component is assumed to be $t_1 + t_2$, i.e. the old value for the starting moment plus the duration of the transition functioning.

$$R_{t_1}(E_i) = \langle\langle A_i^*, \pi_A^i, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle T_i, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle,$$

where $A_i^* = \{Z_i^* | Z_i^* = \langle L_i', L_i'', *, t_2^i, r^i, M^i, \overset{i}{\square}\rangle \& Z_i = \langle L_i', L_i'', t_1^i, t_2^i, r^i, M^i, \overset{i}{\square}\rangle \in A_i\}, i = 1, 2$.

The duration of a transition's functioning which is a union of two transitions with a reduced t_1 component can be evaluated as $\max(t_2^1, t_2^2)$.

$$R_{t_1}(E_1) \cup R_{t_1}(E_2) = \langle\langle A_1^* \overline{\cup} A_2^*, \pi_A^1 \cup \pi_A^2, \pi_L^1 \cup \pi_L^2, c^1 \cup c^2, f^1 \cup f^2, \theta_1^1 \cup \theta_1^2, \theta_2^1 \cup \theta_2^2\rangle,$$

$$\langle K_1 \cup K_2, \pi_K^1 \cup \pi_K^2, \theta_K^1 \cup \theta_K^2 \rangle,$$

$$\langle \min(T_1, T_2), GCD(t_1^o, t_2^o), \max_{1 \le i \le 2}(T_i + \frac{t_i^*.t_i^o}{GCD(t_1^o, t_2^o)} - \min(T_1, T_2)) \rangle,$$

$$\langle X_1 \cup X_2, \Phi_1 \cup \Phi_2, b_1 \cup b_2 \rangle \rangle,$$

where

$$A_1^* \overline{\cup} A_2^* = \bigcup_{i=1}^{2} \{Z | (Z \in A_i^*)(\forall Z' \in A_{3-i}^*)(Z \cap Z' = Z_\emptyset)\} \cup$$

$$\bigcup_{i=1}^{2} \{Z | (\exists Z' \in A_i^*)(\exists Z'' \in A_{3-i}^*)(Z' \cap Z'' \neq Z_\emptyset) \& (Z = Z' \cup Z'')\},$$

$$Z' \cup Z'' = \langle L_1' \cup L_2', L_1'' \cup L_2'', *, \max(t_2^1, t_2^2), r^1 + r^2, M^1 + M^2, \vee(\square^1, \square^2) \rangle.$$

Applying the reducing operator R_{t_1} over the GN $E_1 \cup E_2$ will result to the same set of transition with reduced t_1 component as $A_1^* \overline{\cup} A_2^*$.

$$R_{t_1}(E_1) \cup R_{t_1}(E_2) = R_{t_1}(E_1 \cup E_2).$$

Let $Y = t_2$.

When t_2 component is reduced from a GN, Θ_2 is not defined. The values of t_2 components are assumed to be 1. The duration of a transition's functioning which is a union of two transitions with a reduced t_2 component can be evaluated as $\max(t_1^1, t_1^2) - min(t_1^1, t_1^2) + 1$, the same as the value of this duration in the reduced GN $R_{t_1}(E_1 \cup E_2)$.

Let $Y = M$.

If M component (or the arcs' capacities) is reduced from E_1 and E_2, it cannot be evaluated in the union $R_M(E_1) \cup R_M(E_2)$ either. This component is reduced in $R_M(E_1) \cup R_M(E_2)$ as it is in $R_M(E_1 \cup E_2)$. All other components in the two GNs are obviously equal. Hence, the statement of the theorem is valid.

Let $Y = \square$.

If $\square$ component (or the transition's type) is reduced from E_1 and E_2, it cannot be evaluated in the union $E_1 \cup E_2$ either. This component is also reduced in the resulting union $R_\square(E_1) \cup R_\square(E_2)$. The assertions for the last two cases are analogous to the previous ones. $\square$

Corollary 4.1.1. *For every two GNs $E_1, E_2 \in \Sigma$ and $\forall Y \in \Omega$:*

$$R_Y(E_1 \circ E_2) = R_Y(E_1) \circ R_Y(E_2).$$

5 Relations over Reduced GNs

The relations over GNs shown above can be transferred to reduced GNs in the following way. If two GNs are in a certain relation, then the reduction of the same components from both of the GNs does not violate the relation. This statement can be formulated formally as follows:

Theorem 5.1. *For every two GNs $E_1, E_2 \in \Sigma$, for each relation Rel defined over GNs and $\forall Y \in \Omega$:*

$$\textit{if } Rel(E_1, E_2)\textit{, then } Rel(R_Y(E_1), R_Y(E_2)).$$

Proof. In order to prove the theorem, the statement

$$if Rel(E_1, E_2), then Rel(R_Y(E_1), R_Y(E_2))$$

should be proven valid for each relation Rel defined over GNs. A detailed proofs will be given only for the relations $=, \subset_*, ' \subset_o, <>,$ [] and $\sqsubset$. This set of relations is selected based on the fact that they are the basis for the definitions of other relations or because of the close proximity of their definitions to other ones. The proofs for the rest of the relations will be skipped since they are very close to other that will be presented.

The proof for each of the chosen relations will include cases of reduction for part of the GN components. One general statement for all of the relations can be made. If the reduced component Y is not a part of the relation's definition, it does not affect the validity of the statement.

(1) Now, the theorem statement will be proven for the relation $E_1 = E_2$.

If the reduced component Y is a part of $pr_i E_k, i \in [1; 4]$ and $k = 1, 2$, then $\forall j \neq i : pr_j E_1 = pr_j E_2$ follows directly from $E_1 = E_2$.

An example of reducing a component from the first projection of the GN will be shown here. This first projection has the form:

$$pr_1 E_k = \langle A_k, \pi_A^k, \pi_L^k, c^k, f^k, \theta_1^k, \theta_2^k \rangle, k = 1, 2$$

Let $Y = c$.

In the case of reduced c component (capacities of the places) the GNs have the following form:

$R_c(E_k) = \langle\langle A_k, \pi_A^k, \pi_L^k, *, f^k, \theta_1^k, \theta_2^k\rangle, \langle K_k, \pi_K^k, \theta_K^k\rangle, \langle T_k, t_k^o, t_k^*\rangle, \langle X_k, \Phi_k, b_k\rangle\rangle, k = 1, 2$

$pr_i E_1 = pr_i E_2, \forall i \in [2; 4]$ follows directly from the definition of the relation $E_1 = E_2$. None of these components is reduced from E_1 and E_2 therefore $pr_i R_c(E_1) = pr_i R_c(E_2), \forall i \in [2; 4]$.

The proof comes down to evaluating the validity of the statement $pr_1 R_c(E_1) = pr_1 R_c(E_2)$.

The fourth component in $pr_1 E_k, k = 1, 2$ is reduced. The other ones have kept their original values. From $E_1 = E_2$ follows $pr_j pr_1 R_c(E_1) = pr_j pr_1 R_c(E_2), \forall j \neq 4$.

$pr_4 pr_1 R_c(E_1) = \emptyset$ and $pr_4 pr_1 R_c(E_2) = \emptyset$, therefore $pr_4 pr_1 R_c(E_1) = pr_4 pr_1 R_c(E_2)$.

$pr_j pr_1 R_c(E_1) = pr_j pr_1 R_c(E_2), \forall j \in [1; 7] \implies pr_1 R_c(E_1) = pr_1 R_c(E_2)$, hence $R_c(E_1) = R_c(E_2)$

The proof for the components $\pi_A, \pi_L, f, \Theta_1, \Theta_2$ is analogous since they are sets.

Let $Y = t_1$.

In the case of reduced t_1 component (the moment of transition firing) the GNs have the following form:

$R_{t_1}(E_k) = \langle\langle A_k^*, \pi_A^k, \pi_L^k, c^k, f^k, \theta_1^k, \theta_2^k\rangle, \langle K_k, \pi_K^k, \theta_K^k\rangle, \langle T_k, t_k^o, t_k^*\rangle, \langle X_k, \Phi_k, b_k\rangle\rangle, k = 1, 2$, where $A_k^* = \{Z_k^* | Z_k^* = \langle L_k', L_k'', *, t_2^k, r^k, M^k, \overset{k}{\square}\rangle \& Z_k = \langle L_k', L_k'', t_1^k, t_2^k, r^k, M^k, \overset{k}{\square}\rangle \in A_k\}$.

$E_1 = E_2 \implies pr_i E_1 = pr_i E_2, \forall i \in [2; 4]$. None of these components is reduced from E_1 and E_2 therefore $pr_i R_{t_1}(E_1) = pr_i R_{t_1}(E_2), \forall i \in [2; 4]$

$pr_1 E_1 = pr_1 E_2 \implies (dim(A_1) = dim(A_2)) \& (\forall j : 1 \leq j \leq dim(A_1))(pr_j A_1 = pr_j A_2)$.

Each transition in the reduced GN corresponds to exactly one transition in the original GN. All of the transitions' components are transferred to the corresponding reduced GNs without any change, except for the t_1 component, which is reduced. So $(\forall j : 1 \leq j \leq dim(A_1^*))(pr_j A_1^* = pr_j A_2^*)$.

$$(dim(A_1^*) = dim(A_2^*))\&(\forall j : 1 \leq j \leq dim(A_1^*))(pr_j A_1^* = pr_j A_2^*) \implies$$

$$pr_1 R_{t_1}(E_1) = pr_1 R_{t_1}(E_2).$$

$$pr_i R_{t_1}(E_1) = pr_i R_{t_1}(E_2), \forall i \in [1; 4] \implies R_{t_1}(E_1) = R_{t_1}(E_2).$$

For the rest of the components from the first projection or any other projections of the GN the proof is analogous. Therefore the statement is valid for the relation $=$.

(2) For the relation $E_1 =_o E_2$ the proof of the statement is analogous to the one for $E_1 = E_2$ since the definitions of the two relations are very similar (see Section 2).

(3) Now, the theorem statement will be proven for the relation $E_1 \subset_* E_2$.

Based on the definition of the relation (see Section 2) the following conclusion can be made.

When a certain component Y is reduced from the GNs, that does not affect the conditions for the rest of the components. They will remain valid. Therefore in that case the proof of the theorem statement comes down to evaluating the changed condition for this very component only in the corresponding reduced GNs.

The K component is never reduced.

Let $Y = \pi_A$. Therefore π_A components are assumed to be empty sets.

In the case of reduced π_A component (transitions' priorities) the GNs have the following form:

$R_{\pi_A}(E_i) = \langle\langle A_i, *, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle T_i, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle, i = 1, 2$.

$\pi_A^1 = \emptyset$ and $\pi_A^2 = \emptyset$ in the reduced GNs $R_{\pi_A}(E_1)$ and $R_{\pi_A}(E_2)$ respectfully. Therefore the statement $(\pi_A^1 = \pi_A^2 | E_1)$ is valid. The rest of the components in $R_{\pi_A}(E_1)$ and $R_{\pi_A}(E_2)$ fully correspond to the components in E_1 and E_2, so the relations between them remain valid. Therefore $R_{\pi_A}(E_1) \subset_* R_{\pi_A}(E_2)$.

The proof is analogous when the reduced component is one of the following: $\pi_L, c, f, \Theta_1, \Theta_2, \pi_K, \Theta_K$ and Φ, since they are sets and the conditions they are part from are similar to the one for the π_A component.

<u>Let $Y = T$.</u>

In the case of reduced T component (the moment when the GN starts functioning) the GNs have the following form:

$R_T(E_i) = \langle\langle A_i, \pi_A^i, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle *, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle, i = 1, 2.$

The condition $(T_2 \le T_1 \le T_1 + t_1^* \le T_2 + t_2^*)$ in the general case comes down to $(t_1^* \le t_2^*)$ in the case of the corresponding reduced GNs because of the missing T components. It is evaluated as true based on the relation $E_1 \subset_* E_2$.

$E_1 \subset_* E_2 \implies T_2 \le T_1 \implies T_2 + t_1^* \le T_1 + t_1^*, (t_1^* \ge 0)$, but $T_1 + t_1^* \le T_2 + t_2^* \implies T_2 + t_1^* \le T_2 + t_2^* \implies t_1^* \le t_2^*$.

There is no change for the rest of the components in the reduced GNs, therefore the relation is valid for $R_T(E_1)$ and $R_T(E_2)$, i.e. $R_T(E_1) \subset_* R_T(E_2)$.

Similar conclusions can be made when t^* is reduced. The proof in that case is analogous.

<u>Let $Y = t^o$.</u>

When t^o component (the elementary time-step) is reduced its values can be assumed to be 1. Therefore the statement $t_1^o = t_2^o$ from the relation's definition is valid in the case of the reduced GNs $R_{t^o}(E_1)$ and $R_{t^o}(E_2)$, i.e. $R_{t^o}(E_1) \subset_* R_{t^o}(E_2)$.

<u>Let $Y = t_1$.</u>

In the case of reduced t_1 component (the moment when the GN starts functioning) the GNs have the following form:

$R_{t_1}(E_i) = \langle\langle A_i^*, \pi_A^i, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle T_i, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle, i = 1, 2$, where $A_i^* = \{Z_i^* | Z_i^* = \langle L_i', L_i'', *, t_2^i, r^i, M^i, \overset{i}{\square}\rangle \& Z_i = \langle L_i', L_i'', t_1^i, t_2^i, r^i, M^i, \underset{i}{\square}\rangle \in A_i\}$.

The proof comes down to evaluating the validity of the statement $(\forall Z_1^* \in A_1^*)(\forall Z_2^* \in A_2^*)(Z_1^* \subset Z_2^*)$.

The relations between the components that are not reduced in Z_1^* and Z_2^* remain valid. These components are transferred without any change form the original GNs to the corresponding reduced ones.

The statement

$$(\forall Z_1 \in A_1)(\forall Z_2 \in A_2)((pr_3 Z_2 \leq pr_3 Z_1 \leq pr_3 Z_2 + pr_4 Z_2)\&(pr_3 Z_1 + pr_4 Z_1 \leq pr_3 Z_2 + pr_4 Z_2))$$

for the original GNs E_1 and E_2 comes down to

$$(\forall Z_1^* \in A_1^*)(\forall Z_2^* \in A_2^*)(pr_4 Z_1 \leq pr_4 Z_2)$$

for the corresponding reduced ones $R_{t_1} E_1$ and $R_{t_1} E_1$. It is obviously valid since the relation $E_1 \subset_* E_2$ is hold.

$$(\forall Z_1^* \in A_1^*)(\forall Z_2^* \in A_2^*)$$
$$(\forall i : 1 \leq i \leq 2)(pr_i Z_1^* \subset pr_i Z_2^*)\&(pr_4 Z_1^* \leq pr_4 Z_2^*)\&$$
$$(\forall i : 5 \leq i \leq 6)(pr_i Z_1^* \subset_1 pr_i Z_2^*)\&(pr_7 Z_1^* \subset_2 pr_7 Z_2^*))$$

therefore $Z_1^* \subset Z_2^*$ and $E_1 \subset_* E_2$.

The proof for the rest of transitions' component is analogous.

(4) Now, the theorem statement will be proven for the relation $E_1{}' \subset_o E_2$.

The proof will be focused on the case of reduced transition's components only. The rest of them will be skipped because they are part of conditions same as in the previous relation. The proof for them will be analogous to the stated above.

<u>Let $Y = t_1$.</u>

In the case of reduced t_1 component (the moment when the GN starts functioning) the GNs have the following form as it is mentioned above:

$R_{t_1}(E_i) = \langle\langle A_i^*, \pi_A^i, \pi_L^i, c^i, f^i, \theta_1^i, \theta_2^i\rangle, \langle K_i, \pi_K^i, \theta_K^i\rangle, \langle T_i, t_i^o, t_i^*\rangle, \langle X_i, \Phi_i, b_i\rangle\rangle, i = 1, 2$, where $A_i^* = \{Z_i^* | Z_i^* = \langle L_i', L_i'', *, t_2^i, r^i, M^i, \overset{i}{\square}\rangle \& Z_i = \langle L_i', L_i'', t_1^i, t_2^i, r^i, M^i, \overset{i}{\square}\rangle \in A_i\}$.

The relation $E_1{}' \subset_o E_2$ is hold valid, so $A_1 \subset A_2$ follows directly from the definition.

Each transition in A_i corresponds to exactly one transition in A_i^*. Therefore $A_1^* \subset A_2^*$.

$Q^I_{R_{t_1}(E_1)} = pr_1 A_1^* - pr_2 A_1^* \subset Q^I_{R_{t_1}(E_2)} = pr_1 A_2^* - pr_2 A_2^*$.

The rest of the components in the reduced GNs $R_{t_1}(E_1)$ and $R_{t_1}(E_2)$ fully correspond to the components of E_1 and E_2. Their relations remain as in $E_1{}' \subset_o E_2$.

Therefore $R_{t_1}(E_1)' \subset_o R_{t_1}(E_2)$.

(5) Proving the statement of the theorem for the relations $E_1 \subset_o' E_2, E_1 \overline{\subset}_o E_2$, $E_1 \subset_o E_2$ and $E_1 \subset_\cup E_2$ is analogous to the previous one. Their definitions are very similar to the definition of the previously shown relation.

(6) The relations $E_1 <> E_2$ and $E_1 [] E_2$ are not defined for reduced GNs of class Σ^T or Σ^{t^*}, i.e. when T or t^* components are reduced. The statements are valid in every other case of reduced component because the conditions in their definitions are based on the time components only.

(7) Now, the theorem statement will be proven for the relation $E_1 \sqsubset E_2$. This relation is based on the work that has been done during the GNs functioning. Since the definition of the relation (see Section 2) uses only the components K, X and Φ , they will be the ones that the proof will focus on. The reduction of any other component will not affect the state of the relation.

The K component cannot be reduced by definition (see Section 3).

A common conclusion can be made for all the cases of reduced components. Removing any GN component does not affect the sets of tokens K_i.

Let $Y = X$.

If X component is reduced, it can be assumed that $X_i = \emptyset$. The GN's tokens don't have any initial characteristics. Therefore $R_X(E_i)(\alpha, x) = \langle \alpha, x \rangle$ by definition.

Since $E_1 \sqsubset E_2$, then $K_1 \subseteq K_2$. The sets of tokens in the reduced GNs $R_X E_i$ are the same as in the original ones. Therefore the relation between K_1 and K_2 remain valid for the reduced GNs.

All conditions of the relation are met:

$$(K_1 \subseteq K_2)(\forall \alpha \in K_i) R_X(E_i)(\alpha, x) = \langle \alpha, x \rangle.$$

Therefore $R_X(E_1) \sqsubset R_X(E_2)$.

Let $Y = \Phi$.

If Φ component is reduced, there is no characteristic function to assign new characteristics to every token. Therefore for the reduced GNs $R_\Phi(E_i)$:

$$(\forall \alpha \in K_i)(\forall x \in X_i) R_\Phi(E_i)(\alpha, x) = \langle \alpha, x \rangle.$$

Since $E_1 \sqsubset E_2$ is hold valid, then $K_1 \subseteq K_2 \& X_1 \subseteq X_2$.

But the sets of tokens in the reduced GNs $R_\Phi E_i$ are the same as in the original ones. The functions that assigns initial characteristics to every token that enters the reduced GNs are also the same as the original ones.

$(\forall \alpha \in K_1)(\forall x \in X_1)(R_\Phi(E_i)(\alpha, x) = \langle \alpha, x \rangle)$, hence $R_\Phi(E_1) \sqsubset R_\Phi(E_2)$.

Removing any other component than X and Φ does not affect the sets of tokens K_i and their initial characteristics X_i, neither

$$R_Y(E_1)(\alpha, x) : R_Y(E_i)(\alpha, x) = E_i(\alpha, x)$$

Therefore the corresponding reduced GNs $R_Y(E_i)$ are in the same relation as the original ones E_i.

(8) Proving the statement of the theorem for the relations $E_1 \sqsubset_* E_2, E_1 \approx E_2$ and $E_1 \approx_* E_2$ is analogous to the previous one. The definition of $E_1 \sqsubset_* E_2$ is very similar to the definition of the previously shown relation, while the other two relations are based on $\sqsubset$ and $\sqsubset_*$ respectively.

$\square$

6 Conclusion

Two theorem were formulated and proven regarding the way reducing operators affect the operations and relations defined over GNs.

Applying the reducing operators over a union or a composition of two GNs results in a union or a composition of the corresponding reduced ones. The use of these operators preserve the relations between the nets.

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