Notes on Intuitionistic Fuzzy Sets
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Connectedness concept in intuitionistic fuzzy topological spaces

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Abstract: The purpose of this paper is to establish the connectedness in intuitionistic fuzzy topological space. In this paper we give six notions of separatedness, connectedness and total connectedness and one notion of T_1-space in intuitionistic fuzzy topological space. Also, we find a relation between classical topology and intuitionistic fuzzy topology. Further, we show that connectedness in intuitionistic fuzzy topological spaces are productive and we demonstrate some of its features.

Keywords: Fuzzy set, Intuitionistic fuzzy set, Intuitionistic topological space, Intuitionistic fuzzy topological space, Intuitionistic fuzzy connectedness, Intuitionistic fuzzy T_1-space.

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1 Introduction

The basic concept of a fuzzy set was given by Zadeh [40] in 1965, after then fuzzy topology by Chang [13] in 1968. The generalized concept of intuitionistic fuzzy set was introduced by Atanassov [9] which take into account both the degrees of membership and non-membership subject to the condition that their sum does not exceed 1. Coker [11, 12, 14–17] and his colleagues introduced intuitionistic fuzzy topological spaces and connectedness in intuitionistic fuzzy topological spaces was introduced by Ozcag and Coker [29]. Islam et al. [23, 24], S. Das [18], Lee et al. [25, 26], Minana et al. [28], M. Barile [10], R. Srivastava et al. [33, 34], Tiwari et al. [38], Estiaq Ahmed et al. [1–5], L. Ying-Ming et al. [39], Talukder et al. [35], Fang et al. [19], Hasan et al. [20], Tamilmani [37], R. Islam et al. [22], M. K. Ahmad et al. [6], A. M. Ali et al. [7, 8], Ramadan et al. [30], Immaculate et al. [21] and N. X. Tan et al. [36] subsequently developed the study of intuitionistic fuzzy topological spaces by using intuitionistic fuzzy sets. In this paper, we define six new notions of separatedness, connectedness and total connectedness and one notion of T_1-space in intuitionistic fuzzy topological space and some of their features.

2 Notation and preliminaries

Throughout this paper, X is a non-empty set, T is a topology, t is a fuzzy topology, $\mathcal{T}$ is an intuitionistic topology and τ is an intuitionistic fuzzy topology. λ and μ are fuzzy sets, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set. Particularly, by $\underline{0}$ and $\underline{1}$ we denote constant fuzzy sets taking values 0 and 1, respectively.

Definition 2.1 [13]. *Let X be a non-empty set. A family t of fuzzy sets in X is called a fuzzy topology on X if the following conditions hold.*

(1) $\underline{0}, \underline{1} \in t$,
(2) $\lambda \cap \mu \in t$ for all $\lambda, \mu \in t$,
(3) $\cup \lambda_j \in t$ for any arbitrary family $\{\lambda_j \in t, j \in J\}$.

Definition 2.2 [14]. *Suppose that X is a non-empty set. An intuitionistic set A on X is an object having the form $A = (X, A_1, A_2)$, where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set of members of A, while A_2 is called the set of non-members of A. In this paper, we use the simpler notation $A = (A_1, A_2)$ instead of $A = (X, A_1, A_2)$ for an intuitionistic set.*

Definition 2.3 [9]. *Let X be a non-empty set. An intuitionistic fuzzy set A (IFS, in short) in X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where μ_A and ν_A are fuzzy sets in X denoting the degree of membership and the degree of non-membership, respectively, subject to the condition $\mu_A(x) + \nu_A(x) \leq 1$.*

Throughout this paper, we use the simpler notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ for IFSs.

Definition 2.4 [9]. *Let X be a non-empty set and IFSs A, B in X be given by $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ respectively, then*

(a) *$A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,*
(b) *$A = B$ if $A \subseteq B$ and $B \subseteq A$,*
(c) *$\bar{A} = (\nu_A, \mu_A)$,*
(d) *$A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$,*
(e) *$A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$.*

Definition 2.5 [14]. Let $\{A_j = (\mu_{A_j}, \nu_{A_j}), j \in J\}$ be an arbitrary family of IFSs in X. Then

(a) $\cap A_j = (\cap \mu_{A_j}, \cup \nu_{A_j})$,

(b) $\cup A_j = (\cup \mu_{A_j}, \cap \nu_{A_j})$,

(c) $0_\sim = (\underline{0}, \underline{1})$, $1_\sim = (\underline{1}, \underline{0})$.

Definition 2.6 [14]. *An intuitionistic fuzzy topology (IFT, in short) on a non-empty set X is a family τ of IFSs in X satisfying the following axioms:*

(1) $0_\sim, 1_\sim \in \tau$,

(2) $A \cap B \in \tau$, *for all* $A, B \in \tau$,

(3) $\cup A_j \in \tau$ *for any arbitrary family* $\{A_j \in \tau, j \in J\}$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, in short), members of τ are called intuitionistic fuzzy open sets (IFOS, in short).

Definition 2.7 [32]. *Let $A = (\mu_A, \nu_A)$ be an IFS in X and U be a non-empty subset of X. The restriction of A to U is an IFS in U, denoted by $A|U$ and defined by $A|U = (\mu_A|U, \nu_A|U)$.*

Definition 2.8 [23]. *Let (X, τ) be an intuitionistic fuzzy topological space and U be a non-empty subset of X, then $\tau_U = \{A|U : A \in \tau\}$ is an intuitionistic fuzzy topology on U and (U, τ_U) is called subspace of (X, τ).*

Definition 2.9 [32]. *Let $\alpha, \beta \in (0,1)$ and $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP for short) $p^x_{(\alpha,\beta)}$ of X defined by $p^x_{(\alpha,\beta)} = \langle x, \mu_p, \nu_p \rangle$, for $y \in X$*

$$\mu_p(y) = \begin{cases} \alpha & if \quad y = x \\ 0 & if \quad y \neq x \end{cases} \text{ and } \nu_p(y) = \begin{cases} \beta & if \quad y = x \\ 1 & if \quad y \neq x \end{cases}.$$

In this case, x is called the support of $p^x_{(\alpha,\beta)}$. An IFP $p^x_{(\alpha,\beta)}$ is said to belong to an IFS $A = \langle x, \mu_A, \nu_A \rangle$ of X, denoted by $p^x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \nu_A(x)$.

Proposition 2.1 [32]. *An IFS A in X is the union of all IFP belonging to A.*

Definition 2.10 [11]. *Let $A = (x, \mu_A, \nu_A)$ and $B = (y, \mu_B, \nu_B)$ be IFSs in X and Y respectively. Then the product of IFSs A and B denoted by $A \times B$ is defined by $A \times B = \left\{(x, y), \mu_A \overset{\times}{\underset{\cdot}{}} \mu_B, \nu_A \underset{\times}{\overset{\cdot}{}} \nu_B)\right\}$ where $\left(\mu_A \overset{\times}{\underset{\cdot}{}} \mu_B\right)(x, y) = \min\left(\mu_A(x), \mu_B(y)\right)$ and $\left(\nu_A \underset{\times}{\overset{\cdot}{}} \nu_B\right)(x, y) = \max\left(\nu_A(x), \nu_B(y)\right)$ for all $(x, y) \in X \times Y$. Obviously, $0 \leq \left(\mu_A \overset{\times}{\underset{\cdot}{}} \mu_B\right) + \left(\nu_A \underset{\times}{\overset{\cdot}{}} \nu_B\right) \leq 1$. This definition can be extended to an arbitrary family of IFSs.*

Definition 2.11 [34]. *Two disjoint non-empty intuitionistic fuzzy subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of an IFTS (X, τ) are said to be separated if there exist $U_i \in \tau$ $(i = 1,2)$ such that $U_1 \supseteq A, U_2 \supseteq B$ and $U_1 \cap A = U_2 \cap B = 0_\sim$.*

Definition 2.12 [34]. Let (X, τ) be an IFTS and A be an IFS in X which is strictly positive, i.e., $A(x) \gg 0_\sim$ (i.e., $\mu_A(x) > 0, \nu_A(x) < 1, \forall x \in X$). A pair $U_1, U_2 \in \tau$ is called (C_1)-separation of A if $U_1 \neq A$, $U_2 \neq A$, $U_1 \cup U_2 = A$ and $U_1 \cap U_2 = 0_\sim$.

Definition 2.13 [31]. *A fuzzy topological space X is said to be disconnected if $X = A \cup B$, where A and B are non-empty open fuzzy sets in X such that $A \cap B = \emptyset$. Hence a fuzzy topological space X cannot be represented as the union of two non-empty, disjoint open fuzzy sets on X.*

Definition 2.14 [27]. *Let (X, τ) be an intuitionistic fuzzy topological space. A family $\{(\mu_{G_i}, \nu_{G_i}) : i \in J\}$ of IFOS in X is called open cover of X if $\cup \mu_{G_i} = 1$ and $\cap \nu_{G_i} = 0$. If every* *open cover of X has a finite subcover, then X is said to be intuitionistic fuzzy compact (IF-compact, in short).*

Definition 2.15 [27]. *A family* $\{(\mu_{G_i}, \nu_{G_i}): i \in J\}$ *of IFOS in X is called* (α, β)*-level open cover of X if* $\cup\, \mu_{G_i} \geq \alpha$ *and* $\cap\, \nu_{G_i} \leq \beta$ *with* $\alpha + \beta \leq 1$. *If every* (α, β)*-level open cover of X has a finite subcover, then X is said to be* (α, β)*-level IF-compact.*

3 Connectedness in intuitionistic fuzzy topological space

In this section we define six new definitions of separatedness, connectedness and total connectedness and one notions of T_1-space in intuitionistic fuzzy topological space (IFTS, in short) and established several properties of these notions.

Definition 3.1. *Two disjoint non-empty intuitionistic fuzzy subsets* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *of an IFTS X are said to be separated if A and B neither contain a limit point of the other, i.e., A and B are separated iff* $A \cap \bar{B} = (0,1)$ *and* $\bar{A} \cap B = (0.1)$.

Definition 3.2. *Two IFS's* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *in X are called Q-separated for an IFTS* (X, τ) *if and only if there exist closed (open) IFS's* $G = (\mu_G, \nu_G)$ *and* $H = (\mu_H, \nu_H)$ *in X such that* $A \subseteq G, B \subseteq H$ *and* $A \cap B = (0,1) = G \cap H$.

Definition 3.3. *An intuitionistic fuzzy subsets* $A = (\mu_A, \nu_A)$ *of an IFTS X is disconnected if there exist open intuitionistic fuzzy subsets* $G = (\mu_G, \nu_G)$ *and* $H = (\mu_H, \nu_H)$ *of X such that* $(A \cap G) \cup (A \cap H) = (1,0)$ *and* $(A \cap G) \cap (A \cap H) = (0,1)$. *In this case,* $G \cup H$ *is called a disconnection.*

Definition 3.4. *An IFTS X is said to be disconnected if* $A \cup B = (1,0)$ *and* $A \cap B = (0,1)$ *where* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *are non-empty open intuitionistic fuzzy subsets of* X.

Theorem 3.1. *Union of two non-empty separated intuitionistic fuzzy subsets of an IFTS X is disconnected.*

Proof. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two non-empty separated intuitionistic fuzzy subsets of an IFTS X, so $A \cap \bar{B} = (0,1)$ and $\bar{A} \cap B = (0.1)$. Let $G = \bar{B}^C$ and $H = \bar{A}^C$. Then G and H are open and $(A \cup B) \cap G = (1_{\mathrm{A}}, 0)$ and $(A \cup B) \cap H = (1_{\mathrm{B}}, 0)$ are non-empty disjoint IFSs whose union is $A \cup B$. Thus G and H form a disconnection of $A \cup B$. Hence $A \cup B$ is disconnected. □

Theorem 3.2. *Consider* $\mathcal{M} = \{A_i\}$, *where* $A_i = (\mu_{A_i}, \nu_{A_i})$ *is a class of IF-connected subsets of an IFTS X such that no two members of* $\mathcal{M}$ *are separated. Then* $B = \bigcup_i A_i$ *is IF-connected.*

Proof. Assume that B is not IF-connected. Let $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$ are two open IFS of X such that $G \cup H$ is an IF-disconnection of B. Now each $A_i \in \mathcal{M}$ is IF-connected and so is contained in either G or H and disjoint from the other. Since any two members of $A_{i_1}, A_{i_2} \in \mathcal{M}$ are not separated and so $A_{i_1} \cup A_{i_2}$ is IF-connected, hence $A_{i_1} \cup A_{i_2}$ is contained in either G or H and disjoint from the other. Accordingly all the members of $\mathcal{M}$ and hence $B = \bigcup_i A_i$ must be contained in either G or H and disjoint from the other. But this contradicts the fact that $G \cup H$ is an IF-disconnection of B, hence B is IF-connected. □

Theorem 3.3. *Let $G \cup H$ be a disconnection of an IFS $A = (\mu_A, \nu_A)$. Then $A \cap G$ and $A \cap H$ are separated IFSs.*

Proof. Here $A \cap G$ and $A \cap H$ are disjoint, hence we need only to show that each IFS contains no limit point of the other. Let $p_{(m,n)}, m, n \in I$ be a limit point of $A \cap G$ and suppose $p_{(m,n)} \in A \cap H$. Then H is an open IFS containing $p_{(m,n)}$ and so H contains a point of A ∩ G distinct from $p_{(m,n)}$, i.e., $(A \cap G) \cap H \neq (0,1)$. But $(A \cap G) \cap (A \cap H) = (0,1) = (A \cap G) \cap H$. Accordingly $p_{(m,n)} \notin A \cap H$. Similarly if $p_{(m,n)}$ be a limit point of $A \cap H$, then $p_{(m,n)} \notin A \cap G$. Thus $A \cap G$ and $A \cap H$ are separated IFSs. □

Theorem 3.4. *If an IFTS (X, τ) is IF-disconnected and $\tau^* \supseteq \tau$ then (X, τ^*) is also IF-disconnected.*

Proof. Given IFTS (X, τ) is IF-disconnected. Let $A, B \in \tau$ where $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are non-empty open intuitionistic fuzzy subsets of X, then $A \cup B = (1,0)$ and $A \cap B = (0,1)$. Since $\tau^* \supseteq \tau$ and $A, B \in \tau$ then obviously $, B \in \tau^*$, hence $A \cup B = (1,0)$ and $A \cap B = (0,1)$, which implies that (X, τ^*) is also IF-disconnected. □

Definition 3.5. *An IFTS (X, τ) is called*

a) *Intuitionistic fuzzy connected (IFC) (i) if (X, τ) has no proper clopen (clopen means closed-open) IFS.*
b) *IFC (ii) if there do not exist non-empty IFSs A, B in X which are separated and $A \cup B = (1,0)$.*
c) *IFC (iii) if there is no clopen IFS $A >> (0,1)$ which is C1 separated.*
d) *IFC (iv) if there do not exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau \backslash \{(0,1), (1.0)\}$ such that $A \cup B = (r, 0)$ with $0 < r \leq 1$ and $A \cap B = (0,1)$.*
e) *IFC (v) iff for any $\alpha \in I_0$, there exist no non-empty proper subset $H \subseteq X$ such that $\alpha 1_H = \alpha(1_H, 1_{X-H}), \alpha 1_{X-H} = \alpha(1_{X-H}, 1_H) \in \tau$.*
f) *IFC (vi) iff there exist no non-zero Q-separated IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X with $A \cup B = (1,0)$.*

Theorem 3.5. *The following statements are equivalent:*

a) *IFTS (X, τ) is IFC (vi)*
b) *There do not exist two non-zero disjoint closed IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$, where $\max(\mu_A, \mu_B) = 1$.*
c) *There do not exist two non-zero disjoint open IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$, where $\max(\mu_A, \mu_B) = 1$.*
d) *IFTS (X, τ) is IFC (ii)*

Proof. (a) ⇨ (b): Let there exist IFSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau^C$ such that $A \neq B, A \cup B = (1,0)$ and $A \cap B = (0,1)$ then clearly A and B are Q-separated. So that, (X, τ) is not IFC (vi), a contradiction to (a).

(b) ⇨ (c): If $A, B \in \tau$ where $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B), A \cup B = (1,0)$ and $A \cap B = (0,1)$ then A and B closed which contradicts (b).

(c) ⇨ (d): If (X, τ) is not IFC (ii) then there exist $A, B \in I^X - \{(1,0)\}$ such that A, B are separated and $A \cup B = (1,0)$. Now $\exists G, H \in \tau$ such that $A \subseteq G, B \subseteq H$ and $G \cap B = (1,0) = H \cap A$. But then G and H satisfying $G \cap H = (0,1)$ and $G \cup H = (1,0)$ which contradicting (c).

(d) ⇨ (a): If there exist some IFS $A = (1_A, 1_{A^c}) \in \tau \cap \tau^C - \{(0,1),(1,0)\}$, then $A = (1_A, 1_{A^c})$, $A^C = (1_{A^c}, 1_A)$ are two non-zero separated sets with $\max(1_A, 1_{A^c}) = 1$. This contradicts (d). □

Theorem 3.6. *An IFTS (X,τ) is IF-connected if and only if there exists no non-empty IFOS A and B in X such that $A = B^C$.*

Proof. Necessity. Assume that A and B are two IFOSs in X such that $A \neq (0,1) \neq B$ and $A = B^C$, since B is an IFOS which implies that $B^C = A$ is an IFCS and $B \neq (0,1)$ implies that $B^C \neq (1,0)$, i.e., $A \neq (1,0)$. Hence there exists a proper IFS A as $A \neq (0,1)$ and $A \neq (1,0)$, such that A is both IFOS and IFCS. But this is a contradiction that (X,τ) is IF-connected.

Sufficiency. Let (X,τ) is an IFTS and A is both IFOS and IFCS in X such that $(0,1) \neq A \neq (1,0)$. Here $A = B^C$. In this case B is an IFOS and $A \neq (1,0)$. This implies that, $B = A^C \neq (0,1)$, which is a contradiction. Hence, there exist no proper IFS in X which is both IFO and IFC. So, (X,τ) is IF-connected. □

Theorem 3.7. *Let (X,T) be a topological space and (X,τ) be its corresponding IFTS, where $\tau = \{(1_A, 1_{A^c}) : A \in T\}$. Then (X,T) is connected if and only if (X,τ) is IF-connected.*

Proof. Suppose (X,T) is disconnected, so there exist two non-empty subsets A, B of X such that $A \cup B = X, A \cap B = \emptyset$. Since $A, B \in T$ then $1_A = (1_A, 1_{A^c}) \in \tau$ and $1_B = (1_B, 1_{B^c}) \in \tau$.

Now,
$$\begin{aligned} 1_A \cup 1_B &= (1_A, 1_{A^c}) \cup (1_B, 1_{B^c}) \\ &= (1_A \cup 1_B, 1_{A^c} \cap 1_{B^c}) \\ &= (1_{A\cup B}, 1_{A^c \cap B^c}) \\ &= (1_{A\cup B}, 1_{(A\cup B)^c}) \\ &= (1_X, 1_\emptyset) \\ &= (1,0). \end{aligned}$$

Again,
$$\begin{aligned} 1_A \cap 1_B &= (1_A, 1_{A^c}) \cap (1_B, 1_{B^c}) \\ &= (1_A \cap 1_B, 1_{A^c} \cup 1_{B^c}) \\ &= (1_{A\cap B}, 1_{A^c \cup B^c}) \\ &= (1_{A\cap B}, 1_{(A\cap B)^c}) \\ &= (1_\emptyset, 1_X) \\ &= (0,1). \end{aligned}$$

So, (X,τ) is IF-disconnected. Hence (X,T) is connected if (X,τ) is IF-connected.

Conversely, suppose (X,τ) is IF-disconnected. Since $1_A, 1_B \in \tau$ so $mm, 0)$ and $1_A \cap 1_B = (0,1)$, then we can write $1_A \cup 1_B = (1,0)$

$$\begin{aligned} &\Rightarrow (1_A, 1_{A^c}) \cup (1_B, 1_{B^c}) = (1,0) \\ &\Rightarrow (1_A \cup 1_B, 1_{A^c} \cap 1_{B^c}) = (1,0) \\ &\Rightarrow (1_{A\cup B}, 1_{A^c \cap B^c}) = (1,0) \end{aligned}$$

So, $1_{A\cup B} = 1 = 1_X \Rightarrow 1^{-1}(0,1] = A \cup B = X$.

Again, $1_A \cap 1_B = (0,1)$.

$$\begin{aligned} &\Rightarrow (1_A, 1_{A^c}) \cap (1_B, 1_{B^c}) = (0,1) \\ &\Rightarrow (1_A \cap 1_B, 1_{A^c} \cup 1_{B^c}) = (0,1) \\ &\Rightarrow (1_{A\cap B}, 1_{A^c \cup B^c}) = (0,1). \end{aligned}$$

So, $1_{A\cap B} = 0 = 1_{\emptyset} \Rightarrow 1^{-1}(0,1] = A \cap B = \emptyset$. Hence (X,T) is disconnected.
So, (X,T) is connected if and only if (X,τ) is IF-connected. □

Theorem 3.8. *If (X,τ) and (Y,δ) are IF-connected space then $(X\times Y, \tau\times\delta)$ is also IF-connected.*

Proof. Consider $(X\times Y,\tau\times\delta)$ is not IF-connected then $\exists A, B \in \tau\times\delta$ such that $A\cup B=(1,0)$ and $A\cap B=(0,1)$. Since $A,B\in\tau\times\delta$ then $A=C\times D$ and $B=E\times F$, where $C=(\mu_C,\nu_C)$, $E=(\mu_E,\nu_E)\in\tau$ and $D=(\mu_D,\nu_D), F=(\mu_F,\nu_F)\in\delta$. Now $C\times D=(\mu_C \overset{\times}{\underset{\cdot}{}} \mu_D, \nu_C \underset{\times}{\overset{\cdot}{}} \nu_D)$, where $\left(\mu_C \overset{\times}{\underset{\cdot}{}} \mu_D\right)(x,y)=\min(\mu_C(x),\mu_D(y))$ and $(\nu_C \underset{\times}{\overset{\cdot}{}} \nu_D)(x,y)=\max\big(\nu_C(x),\nu_D(y)\big)$, $\forall (x,y)\in \tau\times\delta$. Similarly, $E\times F=(\mu_E \overset{\times}{\underset{\cdot}{}} \mu_F, \nu_E \underset{\times}{\overset{\cdot}{}} \nu_F)$.
Now, $A\cup B=(1,0) \Rightarrow (C\times D)\cup(E\times F)=(1,0)$
$\Rightarrow \left(\mu_C \overset{\times}{\underset{\cdot}{}} \mu_D, \nu_C \underset{\times}{\overset{\cdot}{}} \nu_D\right) \cup \left(\mu_E \overset{\times}{\underset{\cdot}{}} \mu_F, \nu_E \underset{\times}{\overset{\cdot}{}} \nu_F\right)=(1,0)$
$\Rightarrow (\min\big(\mu_C(x),\mu_D(y)\big)\cup\min\big(\mu_E(x),\mu_F(y)\big), \max\big(\nu_C(x),\nu_D(y)\big)\cap\max\big(\nu_E(x),\nu_F(y)\big))=(1,0)$
i.e., $\min\big(\mu_C(x),\mu_D(y)\big)\cup\min\big(\mu_E(x),\mu_F(y)\big)=1$
$\Rightarrow$ Either, $\min\big(\mu_C(x),\mu_D(y)\big)=1$ or, $\min\big(\mu_E(x),\mu_F(y)\big)=1$
$\Rightarrow$ Either $\mu_C(x)=1, \mu_D(y)=1$ or, $\mu_E(x)=1, \mu_F(y)=1$
For, $\max\big(\nu_C(x),\nu_D(y)\big)\cap\max\big(\nu_E(x),\nu_F(y)\big)=0$
$\Rightarrow \max\big(\nu_C(x),\nu_D(y)\big)=0$ and $\max\big(\nu_E(x),\nu_F(y)\big)=0$
$\Rightarrow \nu_C(x)=0, \nu_D(y)=0, \nu_E(x)=0, \nu_F(y)=0$

<u>Case I:</u> Suppose $\mu_C(x)=1, \mu_D(y)=1$.
Then $C\cup E=(\mu_C,\nu_C)\cup(\mu_E,\nu_E)=(\mu_C\cup\mu_E, \nu_C\cap\nu_E)=(1,0)$ as $\mu_C(x)=1$.

<u>Case II:</u> Suppose $\mu_E(x)=1, \mu_F(y)=1$.
Then $D\cup F=(\mu_D,\nu_D)\cup(\mu_F,\nu_F)=(\mu_D\cup\mu_F, \nu_D\cap\nu_F)=(1,0)$ as $\mu_F(y)=1$.
Again, $A\cap B=(0,1) \Rightarrow (C\times D)\cap(E\times F)=(0,1)$
$\Rightarrow \left(\mu_C \overset{\times}{\underset{\cdot}{}} \mu_D, \nu_C \underset{\times}{\overset{\cdot}{}} \nu_D\right) \cap \left(\mu_E \overset{\times}{\underset{\cdot}{}} \mu_F, \nu_E \underset{\times}{\overset{\cdot}{}} \nu_F\right)=(1,0)$
$\Rightarrow (\min\big(\mu_C(x),\mu_D(y)\big)\cap\min\big(\mu_E(x),\mu_F(y)\big), \max\big(\nu_C(x),\nu_D(y)\big)\cup\max\big(\nu_E(x),\nu_F(y)\big))=(0,1)$,
i.e., $\min\big(\mu_C(x),\mu_D(y)\big)\cap\min\big(\mu_E(x),\mu_F(y)\big)=0$
$\Rightarrow \min\big(\mu_C(x),\mu_D(y)\big)=0$ and $\min\big(\mu_E(x),\mu_F(y)\big)=0$
$\Rightarrow$ Either $\mu_C(x)=0$, or $\mu_D(y)=0$ and either $\mu_E(x)=0$ or $\mu_F(y)=0$
Again, for, $\max\big(\nu_C(x),\nu_D(y)\big)\cup\max\big(\nu_E(x),\nu_F(y)\big)=1$
$\Rightarrow$Either $\max\big(\nu_C(x),\nu_D(y)\big)=1$ or, $\max\big(\nu_E(x),\nu_F(y)\big)=1$
$\Rightarrow$ Either $\nu_C(x)=1$ or $\nu_D(y)=1$, or, either $\nu_E(x)=1$ or $\nu_F(y)=1$

<u>Case III:</u> Suppose $\mu_C(x)=0$, or $\mu_D(y)=0$ and $\nu_C(x)=1$.
Then $C\cap E=(\mu_C,\nu_C)\cap(\mu_E,\nu_E)=(\mu_C\cap\mu_E, \nu_C\cup\nu_E)=(0,1)$.
<u>Case IV:</u> Suppose $\mu_E(x)=0$ or $\mu_F(y)=0$ and $\nu_F(y)=1$.
Then $D\cap F=(\mu_D,\nu_D)\cap(\mu_F,\nu_F)=(\mu_D\cap\mu_F, \nu_D\cup\nu_F)=(0,1)$.

So, (X,τ) and (Y,δ) are not connected, hence if (X,τ) and (Y,δ) are IF-connected then $(X\times Y,\tau\times\delta)$ is IF-connected. ☐

Definition 3.6. *An IFTS (X,τ) is said to be totally IF-connected if for each pair of IFP $p_{\alpha,\beta}, q_{\rho,\theta} \in X$, there exists a disconnection $G \cup H$ of X with $p_{\alpha,\beta} \in G$ and $q_{\rho,\theta} \in H$.*

Theorem 3.9. *The continuous image of a totally IF-disconnected space is totally IF-disconnected.*

Proof. Let $f:(X,\tau)\to(Y,\delta)$ be a continuous function from an IFTS (X,τ) to (Y,δ). Consider $x_{\alpha,\beta}, y_{r,s}$ be two IFP in $Y=f(X)$. Since f is continuous $f^{-1}(x_{\alpha,\beta})$ and $f^{-1}(y_{r,s})$ are IFP in X. If (X,τ) is totally IF-disconnected then there exists a disconnection $G\cup H$ of X where $f^{-1}(x_{\alpha,\beta}) \in G=(\mu_G,\nu_G)$ and $f^{-1}(y_{r,s}) \in H=(\mu_H,\nu_H)$. Since $f^{-1}(x_{\alpha,\beta}) \in G \Rightarrow x_{\alpha,\beta} \in f(G)$ and $f^{-1}(y_{r,s}) \in H \Rightarrow y_{r,s} \in f(H)$. Again $G\cup H$ is a disconnection of X such that $G\cup H=(1,0)$ and $\cap H=(0,1)$.

Here, $G\cup H=(1,0)\Rightarrow(\mu_G,\nu_G)\cup(\mu_H,\nu_H)=(1,0)$

$\Rightarrow(\mu_G\cup\mu_H,\nu_G\cap\nu_H)=(1,0)$

And $G\cap H=(0,1)\Rightarrow(\mu_G,\nu_G)\cap(\mu_H,\nu_H)=(1,0)$

$\Rightarrow(\mu_G\cap\mu_H,\nu_G\cup\nu_H)=(1,0)$.

So, $f(G)=(f(\mu_G),f(\nu_G))$ and $f(H)=(f(\mu_H),f(\nu_H))$ gives

$$\begin{aligned} f(G)\cup f(H) &= (f(\mu_G),f(\nu_G))\cup(f(\mu_H),f(\nu_H)) \\ &= (f(\mu_G)\cup f(\mu_H), f(\nu_G)\cap f(\nu_H)) \\ &= ((\mu_G\cup\mu_H)(f^{-1}(x)),(\nu_G\cap\nu_H)(f^{-1}(x))) \\ &= (1,0). \end{aligned}$$

And
$$\begin{aligned} f(G)\cap f(H) &= (f(\mu_G),f(\nu_G))\cap(f(\mu_H),f(\nu_H)) \\ &= (f(\mu_G)\cap f(\mu_H), f(\nu_G)\cup f(\nu_H)) \\ &= ((\mu_G\cap\mu_H)(f^{-1}(x)),(\nu_G\cup\nu_H)(f^{-1}(x))) \\ &= (0,1). \end{aligned}$$

So, $Y=f(X)$ is totally IF-disconnected. ☐

Definition 3.7. *An IFTS (X,τ) is T_1-space if $\forall$ IF-singleton $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta}\neq y_{m,n}$, then $\exists A=(\mu_A,\nu_A), B=(\mu_B,\nu_B)\in\tau$ such that $x_{\alpha,\beta}\in A, y_{m,n}\notin A$ and $x_{\alpha,\beta}\notin B$, $y_{m,n}\in B$.*

Theorem 3.10. *Every IF- T_1 space is totally IF-disconnected space.*

Proof. Let (X,τ) be an IFTS and also IF- T_1 space. If $x_{\alpha,\beta}, y_{m,n}\in X$ with $x_{\alpha,\beta}\neq y_{m,n}$ then $\exists A=(\mu_A,\nu_A), B=(\mu_B,\nu_B)\in\tau$ such that $x_{\alpha,\beta}\in A, y_{m,n}\notin A$ and $x_{\alpha,\beta}\notin B, y_{m,n}\in B$.

Now

$$x_{\alpha,\beta}\in A=(\mu_A,\nu_A)\Rightarrow\mu_A(x)\geq\alpha,\nu_A(x)\leq\beta,$$

$$x_{\alpha,\beta}\notin B=(\mu_B,\nu_B)\Rightarrow\mu_B(x)<\alpha,\nu_B(x)>\beta,$$

$$y_{m,n}\notin A=(\mu_A,\nu_A)\Rightarrow\mu_A(y)<m,\nu_A(y)>n,$$

$$y_{m,n}\in B=(\mu_B,\nu_B)\Rightarrow\mu_B(y)\geq m,\nu_B(y)\leq n.$$

So, $(A\cup B)(x)=(\mu_A\cup\mu_B,\nu_A\cap\nu_B)>(\alpha,\beta),(A\cap B)(x)=(\mu_A\cap\mu_B,\nu_A\cup\nu_B)<(\alpha,\beta)$ and $(A\cup B)(y)=(\mu_A\cup\mu_B,\nu_A\cap\nu_B)>(m,n),(A\cap B)(y)=(\mu_A\cap\mu_B,\nu_A\cup\nu_B)<(m,n)$. Hence $A\cup B$ is a disconnection of X, so (X,τ) is totally IF-disconnected. ☐

4 Conclusion

The results presented in this paper indicate that many of the basic concepts in general topology can readily to extend to intuitionistic fuzzy topological spaces. Although the theory of intuitionistic fuzzy set is still in embryonic stage, it shows promise of having wide applications

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