NIFS 11 (2005), 1, 1-5

INTUITIONISTIC FIZZY IMPLICATIONS AND MODUS PONENS

Krassimir T. Atanassov

CLBME-Bulgarian Academy of Sciences, P.O. Box 12, Sofia-1113, Bulgaria

e-mail: krat@bas.bg

Some variants of intuitionistic fuzzy implications will be discussed, using as a basis the book [1] by Georg Klir and Bo Yuan. A great number of fuzzy implications have been discussed there. In [2] analogous of these implications for the case of intuitionistic fuzzy logics are given (for intuitionistic fuzzy sets and logics see, e.g., [3]). The axioms from [1] are checked for the intuitionistic fuzzy implications. Let us note these implications by $I(x, y)$. The axioms from [1] are the following.

Axiom 1 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z))$.

Axiom 2 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(z, x) \leq I(z, y))$.

Axiom 3 $(\forall y)(I(0, y) = 1)$.

Axiom 4 $(\forall y)(I(1, y) = y)$.

Axiom 5 $(\forall x)(I(x, x) = 1)$.

Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.

Axiom 7 $(\forall x, y)(I(x, y) = 1 \text{ iff } x \leq y)$.

Axiom 8 $(\forall x, y)(I(x, y) = I(N(y), N(x)))$, where N is an operation for a negation.

Axiom 9 I is a continuous function.

The intuitionistic fuzzy form of the implications from [1] are given below, following [2]. In this case if x is a variable then its truth-value is represented by the ordered couple $V(x) = \langle a, b \rangle$ so that $a, b, a + b \in [0, 1]$, where a and b are degree of validity and of non-validity of x.

Bellow we shall assume that for the above three variables x, y and z there hold the equalities: $V(x) = \langle a, b \rangle, V(y) = \langle c, d \rangle, V(z) = \langle e, f \rangle$.

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [3]) by:

$$x \text{ is an IFT if and only if } a \geq b.$$

When an axiom is valid only in IFT sense, in the Table its will be marked by "*". In some definitions we shall use functions sg and $\overline{sg}$:

$$sg(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

In ordinary intuitionistic fuzzy logic the negation of variable x is $N(x)$ such that

$$V(N(x)) = \langle b, a \rangle.$$

For two variables x and y operation "conjunction" (&) is defined by:

$$V(x\&y) = \langle \min(\mu(x), \mu(y)), \max(\nu(x), \nu(y)) \rangle,$$

TABLE: List of intuitionistic fuzzy implications

Name	Form of implication	Axioms
Zadeh	$\langle \max(b, \min(a, c)), \min(a, d) \rangle$	2,3,4,5*,7*,9
Gaines-Rescher	$\langle 1 - sg(a - c), d.sg(a - c) \rangle$	1,2,3,5
Gödel	$\langle 1 - (1 - c).sg(a - c), d.sg(a - c) \rangle$	1,2,3,4,5,7*
Kleene-Dienes	$\langle \max(b, c), \min(a, d) \rangle$	1,2,3,4,5*,6,8,9
Lukasiewicz	$\langle \min(1, b + c), \max(0, a + d - 1) \rangle$	1,2,3,4,5*,8,9
Reichenbach	$\langle b + ac, ad \rangle$	2,3,4,5*,9
Willmott	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d)) \rangle$	3*,4*,5*,8,9
Wu	$\langle 1 - (1 - \min(b, c)).sg(a - c),$ $\max(a, d).sg(a - c).sg(d - b) \rangle$	1,2,3,5
Klir and Yuan 1	$\langle b + a^2c, ab + a^2d \rangle$	2,3,4,5*
Klir and Yuan 2	$\langle c.\overline{sg}(1 - a) + sg(1 - a).(\overline{sg}(1 - c) + b.sg(1 - c)),$ $d.\overline{sg}(1 - a) + a.sg(1 - a).sg(1 - c) \rangle$	2,3,4

In [2] is proved the correctness of the above definitions of implications.

As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0\rangle$.

Theorem 1 Each of the implications from the Table satisfy Modus Ponens in the case of tautology.

Proof: Let $V(x) = \langle 1, 0\rangle$, i.e., $a = 1, b = 0$, and let $V(I(x, y)) = \langle 1, 0\rangle$. For example, for Second Klir and Yuan's implication we shall obtain that

$$\langle 1, 0\rangle = \langle c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-a) + b.sg(1-c)), d.\overline{sg}(1-a) + a.sg(1-a).sg(1-c)\rangle$$

$$= \langle c, d\rangle.$$

Therefore, $c = 1$ and $d = 0$.

The other checks for this and for the next theorems are similar.

Theorem 2 None of the implications from the Table satisfy Modus Ponens in the case of IFT.

Theorem 3 For every one of the implications from the Table and for every two variables x and y, $I(x\&I(x, y), y)$ is an IFT.

Proof: We shall check the first Klir and Yuan's implication. The other checks are similar, but this one is the most complex.

$$V(I(x\&I(x, y), y)) = V(I(\langle a, b\rangle\&I(\langle a, b\rangle, \langle c, d\rangle), \langle c, d\rangle))$$

$$= V(I(\langle a, b\rangle\&\langle b + a^2c, ab + a^2d\rangle), \langle c, d\rangle))$$

$$= V(I(\langle\min(a, b + a^2c), \max(b, ab + a^2d)\rangle, \langle c, d\rangle))$$

$$= \langle\max(b, ab+a^2d)+\min(a, b+a^2c)^2.c, \min(a, b+a^2c).\max(b, ab+a^2d)+\min(a, b+a^2c)^2.d\rangle$$

Let

$$A = \max(b, ab+a^2d)+\min(a, b+a^2c)^2.c-\min(a, b+a^2c).\max(b, ab+a^2d)-\min(a, b+a^2c)^2.d.$$

We shall study two cases.

Case 1: $a \leq b + a^2c$. Then

$$b \geq a - a^2c = a - a^2 + a^2.(1-c) \geq a - a^2 + a^2d,$$

i.e.,

$$a^2d \le b - a + a^2. \qquad (*)$$

For A we obtain

$$A = \max(b, ab + a^2d) + a^2c - a.\max(b, ab + a^2d) - a^2d$$

$$= \max(b, ab + a^2d).(1 - a) + a^2.(c - d)$$

If $c \ge d$, obviously, $A \ge 0$. Let $c < d$. Then from $a \le b + a^2c$ and $\max(b, ab + a^2d) \ge b$, it follows that

$$A \ge b.(1 - a) + a^2.(c - d) = b - ab + a^2c - a^2d$$

(from (*))

$$\ge b - ab + a^2c - a^2d - b + a - a^2 = -ab + a^2c + a - a^2$$

$$\ge a - ab - a^2 = a.(1 - a - b) \ge 0.$$

Case 2: $a > b + a^2c$. Then

$$A = \max(b, ab + a^2d) + (b + a^2c)^2.c - (b + a^2c).\max(b, ab + a^2d) - (b + a^2c)^2.d$$

$$= \max(b, ab + a^2d).(1 - b - a^2c) + (b + a^2c)^2.(c - d).$$

Because $1 - b - a^2c \ge 1 - a - b \ge 0$, if $c \ge d$, then, $A \ge 0$. Let $c < d$. Then from $a \ge \min(a, b + a^2c)$ it follows that

$$A = \max(b, ab+a^2d)+\min(a, b+a^2c)^2.c-\min(a, b+a^2c).\max(b, ab+a^2d)-\min(a, b+a^2c)^2.d$$

$$= \max(b, ab + a^2d)(1 - b - a^2c) + \min(a, b + a^2c)^2(c - d)$$

$$\ge (ab + a^2d).(1 - b - a^2c) + a^2.c - a^2d$$

$$= ab + a^2d - ab^2 - a^2bd - a^3bc - a^4cd + a^2c - a^2d$$

$$= ab(1 - b - ad) + a^2c.(1 - ab - a^2d)$$

$$\ge ab(1 - b - a) + a^2c.(1 - b - a) \ge 0.$$

Therefore, in all cases $A \ge 0$, i.e. for the first Klir and Yuan's implication the expression $I(x \& I(x, y), y)$ is an IFT.

References:

[1] Klir G. and Bo Yuan, *Fuzzy Sets and Fuzzy Logic.* Prentice Hall, New Jersey, 1995.

[2] Atanassov K., B. Papadopoulos and A. Syropoulos, Variants of intuitionistic fuzzy implications. submitted to *International Journal of Approximate Reasonong.*

[3] Atanassov, K. *Intuitionistic Fuzzy Sets.* Springer Physica-Verlag, Heidelberg, 1999.