Completeness of *IFS(X)* as a metric space

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Abstract

The aim of this article is to prove the completeness of the space of intuitionistic fuzzy sets in X, where X can be finite or infinite universe of discourse.
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1 Introduction

Theory of intuitionistic fuzzy sets was introduced by Atanassov [1] as a natural generalization of usual fuzzy sets.

Denote by X a universe of discourse. An intuitionistic fuzzy set $A \subset X$ is represented by two functions: μ_A - the *membership function* and ν_A - the *non-membership function*. In other words

$$A = (\mu_A, \nu_A),$$

where $\mu_A, \nu_A : X \longrightarrow [0, 1]$ are functions satisfying

$$(\forall x \in X)\ (\mu_A(x) + \nu_A(x) \leq 1).$$

The family of all intuitionistic fuzzy sets in X will be denoted by $IFS(X)$.

The difference between two objects will be usually expressed by their distance in some space. Atanassov [2] and Szmidt and Kacprzyk [3] described distances between intuitionistic fuzzy sets. These metrics are generalizations of Hamming and Euclidean distances.

2 Distance in the finite universe

In this section we will assume the finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Atanassov [2] suggested a direct generalization of Hamming and Euclidean distances for intuitionistic fuzzy sets $A, B \in IFS(X)$. Their formulas are listed below:

- the Hamming distance

$$d'(A, B) = \frac{1}{2} \sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \right),$$

- the Euclidean distance

$$e'(A,B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n}((\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)}.$$

Theorem 1 *The family of intuitionistic fuzzy sets $IFS(X)$ in finite universe X is a complete metric space considering the Hamming distance $d'(A,B)$.*

Proof.
First we show, that every Cauchy's sequence of IF-sets is convergent. Then we prove, that the limit is an element of the family $IFS(X)$.
(i) Let $\varepsilon > 0$ and $(A_n)_{n=1}^{\infty}$ be the Cauchy's sequence in $IFS(X)$. Then there exists $n_0 \in N$ such that for any $m, n \geq n_0 \ : d'(A_m, A_n) < \varepsilon$.

Take $x_i \in X$ fixed and denote for any $A, B \in IFS(X)$

$$\varrho_1(A,B) = |\mu_A(x_i) - \mu_B(x_i)|.$$

Then

$$\varrho_1(A_m, A_n) = |\mu_{A_m}(x_i) - \mu_{A_n}(x_i)| \leq d'(A_m, A_n) < \varepsilon,$$

hence $(A_n)_{n=1}^{\infty}$ is the Cauchy's sequence considering the metric $\varrho_1(A,B)$. Since (R, ϱ_1) is complete metric space, then $(A_n)_{n=1}^{\infty}$ has a limit A, i.e.

$$\varrho_1(A_n, A) = |\mu_{A_n}(x_i) - \mu_A(x_i)| < \varepsilon,$$

or

$$\lim_{n\to\infty} \mu_{A_n}(x_i) = \mu_A(x_i)$$

for fixed $x_i \in X$.

If we do the same consideration for every $x_i \in X$, we can define for each $i = 1, \ldots, n$

$$\lim_{n\to\infty} \mu_{A_n}(x_i) = \mu_A(x_i).$$

Similarly, if we denote for any $A, B \in IFS(X)$ and $x_i \in X$ fixed

$$\varrho_2(A,B) = |\nu_A(x_i) - \nu_B(x_i)|,$$

we get

$$\varrho_2(A_n, A) = |\nu_{A_n}(x_i) - \nu_A(x_i)| < \varepsilon.$$

So for every $x_i \in X$, we can define

$$(\forall i = 1, \ldots, n) \ (\lim_{n\to\infty} \nu_{A_n}(x_i) = \nu_A(x_i)).$$

Now we will show, that $A = (\mu_A, \nu_A)$ is the limit of the Cauchy sequence $(A_n)_{n=1}^{\infty}$ in $IFS(X)$.

Let $\varepsilon > 0$ and $x_i \in X$. Then there exist

$$n_1 \in N \text{ such that for any } n \geq n_1 \qquad |\mu_{A_n}(x_i) - \mu_A(x_i)| < \tfrac{\varepsilon}{n} \ ,$$

$$n_2 \in N \text{ such that for any } n \geq n_2 \qquad |\nu_{A_n}(x_i) - \nu_A(x_i)| < \tfrac{\varepsilon}{n} \ .$$

Denote $n_0 = \max\{n_1, n_2\}$. Then for any $n \geq n_0$ is

$$d'(A_n, A) = \frac{1}{2}\sum_{i=1}^{n}(|\mu_{A_n}(x_i) - \mu_A(x_i)| + |\nu_{A_n}(x_i) - \nu_A(x_i)|) < \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\varepsilon}{n} + \frac{\varepsilon}{n}\right) = \frac{1}{2}\left(n\frac{2\varepsilon}{n}\right) = \varepsilon.$$

(ii) Finally we prove, that $A = (\mu_A, \nu_A) \in IFS(X)$. Let $A_n \in IFS(X)$, $n = 1, 2, \ldots$, then for every $x_i \in X$:

$$\mu_{A_n}(x_i) + \nu_{A_n}(x_i) \leq 1.$$

After a limit transition we get

$$\mu_A(x_i) + \nu_A(x_i) = \lim_{n\to\infty} \mu_{A_n}(x_i) + \lim_{n\to\infty} \nu_{A_n}(x_i) = \lim_{n\to\infty} (\mu_{A_n}(x_i) + \nu_{A_n}(x_i)) \leq 1.$$

Theorem 2 *The family of intuitionistic fuzzy sets $IFS(X)$ in finite universe X is a complete metric space considering the Euclidean distance $e'(A, B)$.*

Proof.
(i) Let $\varepsilon > 0$ and $x_i \in X$, $(i = 1, \ldots, n)$. Denote for any $A, B \in IFS(X)$

$$\varrho_1(A, B) = \sqrt{\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2}\ .$$

Let $(A_n)_{n=1}^{\infty}$ be the Cauchy's sequence in $IFS(X)$. Then for $\frac{\varepsilon}{\sqrt{2}} > 0$ there exists $n_0 \in N$ such that for any $m, n \geq n_0$: $e'(A_m, A_n) < \frac{\varepsilon}{\sqrt{2}}$.

Then

$$\varrho_1(A_m, A_n) \leq \sqrt{\sum_{i=1}^{n}((\mu_{A_m}(x_i) - \mu_{A_n}(x_i))^2 + (\nu_{A_m}(x_i) - \nu_{A_n}(x_i))^2)} =$$

$$= \sqrt{2}\sqrt{\frac{1}{2}\sum_{i=1}^{n}((\mu_{A_m}(x_i) - \mu_{A_n}(x_i))^2 + (\nu_{A_m}(x_i) - \nu_{A_n}(x_i))^2)} < \sqrt{2}\frac{\varepsilon}{\sqrt{2}} = \varepsilon\ .$$

Since ϱ_1 is the Euclidean metric in R^n and (R^n, ϱ_1) is complete metric space, then the Cauchy sequence $(A_n)_{n=1}^{\infty}$ converges to A:

$$\varrho_1(A_n, A) < \varepsilon.$$

Hence

$$\varepsilon > \sqrt{\sum_{i=1}^{n}(\mu_{A_n}(x_i) - \mu_A(x_i))^2} \geq \sqrt{(\mu_{A_n}(x_i) - \mu_A(x_i))^2} = |\mu_{A_n}(x_i) - \mu_A(x_i)|,$$

and

$$\lim_{n\to\infty} \mu_{A_n}(x_i) = \mu_A(x_i) \qquad \forall i = 1, \ldots, n.$$

Similarly for

$$\varrho_2(A, B) = \sqrt{\sum_{i=1}^{n}(\nu_A(x_i) - \nu_B(x_i))^2}$$

we get

$$\lim_{n\to\infty} \nu_{A_n}(x_i) = \nu_A(x_i) \qquad \forall i = 1, \dots, n.$$

We prove that $A = (\mu_A, \nu_A)$ is the limit of the Cauchy sequence $(A_n)_{n=1}^{\infty}$.
Let $\varepsilon > 0$ and $x_i \in X$. Then there exist

$$n_1 \in N \text{ such that for any } n \geq n_1 \qquad |\mu_{A_n}(x_i) - \mu_A(x_i)| < \tfrac{\varepsilon}{\sqrt{n}} \ ,$$

$$n_2 \in N \text{ such that for any } n \geq n_2 \qquad |\nu_{A_n}(x_i) - \nu_A(x_i)| < \tfrac{\varepsilon}{\sqrt{n}} \ .$$

Denote $n_0 = \max\{n_1, n_2\}$. Then for any $n \geq n_0$ is

$$e'(A_n, A) = \sqrt{\frac{1}{2}\sum_{i=1}^{n} ((\mu_{A_m}(x_i) - \mu_{A_n}(x_i))^2 + (\nu_{A_m}(x_i) - \nu_{A_n}(x_i))^2)} <$$

$$< \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{\varepsilon}{\sqrt{n}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right)} = \sqrt{\frac{1}{2}n\left(\frac{2\varepsilon^2}{n}\right)} = \varepsilon \ .$$

(ii) The proof of $A = (\mu_A, \nu_A) \in IFS(X)$ is the same as in the part (ii) of proof of previous theorem.

3 Distance in the infinite universe

Our aim is to define distance of intuitionistic fuzzy sets $A, B \in IFS(X)$, where X doesn't need to be a finite universe of discourse. Then we show, that every Cauchy's sequence of intuitionistic fuzzy sets in $IFS(X)$ is convergent with a limit from this space. In other words - the space $IFS(X)$ is complete also in the infinite case of X, considering the given distance.

Definition 1 *Let $(X, \mathcal{S}, P)$ be a probability space. Let $IFS(X)$ be the set of all intuitionistic fuzzy sets $A = (\mu_A, \nu_A)$, where μ_A, ν_A are $\mathcal{S}$-measurable. For any two intuitionistic fuzzy sets $A, B \in IFS(X)$ we define the Hamming distance by following:*

$$\bar{d}(A, B) = \int_X (|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|)\, dP \ .$$

Theorem 3 *The family of intuitionistic fuzzy sets $IFS(X)$ is a complete metric space considering the Hamming distance $\bar{d}(A, B)$.*

Proof.
(i) Existence. Let $(A_n)_{n=1}^{\infty}$ be the Cauchy's sequence in $IFS(X)$. This means, that for any $\varepsilon > 0$ there exists $n_0 \in N$ such that for any $m, n \geq n_0$:

$$\bar{d}(A_m, A_n) = \int_X (|\mu_{A_m}(x) - \mu_{A_n}(x)| + |\nu_{A_m}(x) - \nu_{A_n}(x)|)\, dP < \varepsilon,$$

for any $x \in X$.

Hence $(\mu_{A_n})_{n=1}^{\infty}$, $(\nu_{A_n})_{n=1}^{\infty}$ are Cauchy's sequences in $L^1(X,\mathcal{S},P)$, which is a complete space. Then there exist measurable functions $\mu_A, \nu_A \in L^1(X,\mathcal{S},P)$ such that

$$\lim_{n\to\infty} \int |\mu_{A_n}(x) - \mu_A(x)| dP = 0 \ ,$$

$$\lim_{n\to\infty} \int |\nu_{A_n}(x) - \nu_A(x)| dP = 0 \ .$$

So we have

$$0 = \lim_{n\to\infty} \int |\mu_{A_n}(x) - \mu_A(x)| + |\nu_{A_n}(x) - \nu_A(x)| dP = \lim_{n\to\infty} \bar{d}(A_n, A),$$

which means, that $A = (\mu_A, \nu_A)$ is the limit of Cauchy's sequence $(A_n)_{n=1}^{\infty}$.

Moreover

$$0 = \lim_{n\to\infty} \int (|\mu_{A_n}(x) - \mu_A(x)| + |\nu_{A_n}(x) - \nu_A(x)|) dP \geq \lim_{n\to\infty} \int |\mu_{A_n}(x) + \nu_{A_n}(x) - (\mu_A(x) + \nu_A(x))| dP \geq 0,$$

hence

$$\lim_{n\to\infty} \int |\mu_{A_n}(x) + \nu_{A_n}(x) - (\mu_A(x) + \nu_A(x))| dP = 0.$$

(ii) Completeness. Denote $\mu_{A_n}(x) + \nu_{A_n}(x) = f_n(x)$ and $\mu_A(x) + \nu_A(x) = f(x)$ for any $x \in X$. From the definition of intuitionistic fuzzy sets we get

$$0 \leq f_n(x) \leq 1$$

and the last equality in step (i) can be written as

$$\lim_{n\to\infty} \int |f_n(x) - f(x)| dP = 0.$$

Let $B = \{x; f(x) < 0\}$, then for all $x \in B: \ f_n(x) \geq 0 > f(x)$ and

$$\lim_{n\to\infty} \int_B |f_n(x) - f(x)| dP = \lim_{n\to\infty} \int_B (f_n(x) - f(x)) dP = 0.$$

Immediately we get

$$0 \leq \lim_{n\to\infty} \int_B f_n(x) dP = \int_B f(x) dP.$$

Since $f(x) < 0$ in B, then $P(B) = 0$.

Let $C = \{x; f(x) > 1\}$, then for all $x \in C: \ f_n(x) \leq 1 < f(x)$ and

$$\lim_{n\to\infty} \int_C |f_n(x) - f(x)| dP = \lim_{n\to\infty} \int_C (f(x) - f_n(x)) dP = 0.$$

Hence

$$\int_C f(x) dP = \lim_{n\to\infty} \int_C f_n(x) dP \leq \int_C 1 dP = P(C)$$

and also

$$\int_C (f(x) - \chi_C(x))dP = \int_C f(x)dP - P(C) \leq 0.$$

Since for every $x \in C$ we have $f(x) - \chi_C(x) > 0$, then $P(C) = 0$.

We have proved, that $0 \leq f(x) \leq 1$ $\forall x \in X$. This means that

$$f(x) = \mu_A(x) + \nu_A(x) \leq 1 \qquad \forall x \in X,$$

so the Cauchy's sequence $(A_n)_{n=1}^{\infty}$ has a limit $A \in IFS(X)$ considering the distance $\bar{d}(A, B)$.

Definition 2 *Let $a, b \in R$, $a \leq b$. Let $(X, \mathcal{S}, P)$ be a probability space. Let $IFS(X)$ be the set of all intuitionistic fuzzy sets $A = (\mu_A, \nu_A)$, where μ_A, ν_A are $\mathcal{S}$-measurable with integrable quadrate. For any two intuitionistic fuzzy sets $A, B \in IFS(X)$ we define the Euclidean distance by following:*

$$\bar{e}(A, B) = \sqrt{\int_a^b (\mu_A(x) - \mu_B(x))^2 dP + \int_a^b (\nu_A(x) - \nu_B(x))^2 dP}\ .$$

Theorem 4 *The family of intuitionistic fuzzy sets $IFS(X)$ is a complete metric space considering the Euclidean distance $\bar{e}(A, B)$.*

Proof.
(i) Existence. Let $(A_n)_{n=1}^{\infty}$ be the Cauchy's sequence in $IFS(X)$. This means, that for any $\varepsilon > 0$ there exists $n_0 \in N$ such that for any $m, n \geq n_0$:

$$\bar{e}(A_m, A_n) = \sqrt{\int_a^b (\mu_{A_m}(x) - \mu_{A_n}(x))^2 dP + \int_a^b (\nu_{A_m}(x) - \nu_{A_n}(x))^2 dP} < \varepsilon,$$

for any $x \in X$.

Hence $(\mu_{A_n})_{n=1}^{\infty}$, $(\nu_{A_n})_{n=1}^{\infty}$ are Cauchy's sequences in $L^2(a, b)$, which is a complete space. Then there exist measurable functions $\mu_A, \nu_A \in L^2(a, b)$ such that

$$\lim_{n\to\infty} \sqrt{\int_a^b (\mu_{A_n}(x) - \mu_A(x))^2 dP} = 0\ ,$$

$$\lim_{n\to\infty} \sqrt{\int_a^b (\nu_{A_n}(x) - \nu_A(x))^2 dP} = 0\ .$$

Then

$$0 = \lim_{n\to\infty} \sqrt{\int_a^b (\mu_{A_n}(x) - \mu_A(x))^2 dP} + \sqrt{\int_a^b (\nu_{A_n}(x) - \nu_A(x))^2 dP} \geq$$

$$\geq \lim_{n\to\infty} \sqrt{\int_a^b (\mu_{A_n}(x) - \mu_A(x))^2 dP + \int_a^b (\nu_{A_n}(x) - \nu_A(x))^2 dP} \geq 0.$$

We have proved that

$$\lim_{n\to\infty} \bar{e}(A_n, A) = \lim_{n\to\infty} \sqrt{\int_a^b (\mu_{A_n}(x) - \mu_A(x))^2 dP + \int_a^b (\nu_{A_n}(x) - \nu_A(x))^2 dP} = 0.$$

(ii) Completeness. From step (i) we have

$$0 \leq \lim_{n\to\infty} \sqrt{\int_a^b ((\mu_A(x) - \mu_{A_n}(x)) + (\nu_A(x) - \nu_{A_n}(x)))^2 dP} \leq$$

$$\leq \lim_{n\to\infty} \left(\sqrt{\int_a^b (\mu_A(x) - \mu_{A_n}(x))^2 dP} + \sqrt{\int_a^b (\nu_A(x) - \nu_{A_n}(x))^2 dP} \right) = 0.$$

Then $\lim_{n\to\infty} \sqrt{\int_a^b ((\mu_A(x) - \mu_{A_n}(x)) + (\nu_A(x) - \nu_{A_n}(x)))^2 dP} = 0.$

If we denote $\mu_{A_n}(x) + \nu_{A_n}(x) = f_n(x)$ and $\mu_A(x) + \nu_A(x) = f(x)$ for any $x \in X$, then

$$0 \leq f_n(x) \leq 1$$

and by previous

$$\lim_{n\to\infty} \sqrt{\int_a^b (f(x) - f_n(x))^2 dP} = 0,$$

hence

$$\lim_{n\to\infty} \int_a^b (f(x) - f_n(x))^2 dP = 0.$$

Let $B = \{x; f(x) < 0\}$, then for all $x \in B:\ f_n(x) \geq 0 > f(x)$ and

$$0 \leq \int_B (f(x))^2 \leq \lim_{n\to\infty} \int_B (f_n(x) - f(x))^2 dP \leq \lim_{n\to\infty} \int_a^b (f_n(x) - f(x))^2 dP = 0.$$

Immediately we get

$$\int_B (f(x))^2 = 0.$$

Since $(f(x))^2 > 0$ in B, then $P(B) = 0$.

Let $C = \{x; f(x) > 1\}$, then for all $x \in C:\ f_n(x) \leq 1 < f(x)$ and

$$0 = \lim_{n\to\infty} \int_a^b (f(x) - f_n(x))^2 dP \geq \lim_{n\to\infty} \int_C (f(x) - f_n(x))^2 dP \geq \lim_{n\to\infty} \int_C (f(x) - 1)^2 dP \geq 0.$$

Hence

$$\int_C (f(x) - 1)^2 dP = 0.$$

Since for every $x \in C$ we have $(f(x) - 1)^2 > 0$, then $P(C) = 0$.

We have proved, that

$$0 \leq f(x) = \mu_A(x) + \nu_A(x) \leq 1 \qquad \forall x \in X,$$

so the Cauchy's sequence $(A_n)_{n=1}^{\infty}$ has a limit $A \in IFS(X)$ considering the distance $\bar{e}(A, B)$.

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