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INTUITIONISTIC FUZZY SET ENTROPIES

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Abstract. The concepts of the fuzzy intuitionistic set entropy first proposed in [4] and [5], and then thoroughly discussed in [7] are now complemented with an idea of the intuitionistic entropy which takes into consideration an intuitionistic set description parameter - hesitancy margin.

Key words: fuzzy intuitionistic set, hesitancy margin, entropy, valuation, fuzzy set entropy

1. INTRODUCTORY REMARKS AND BASIC CONCEPTS

For their formal description, mathematical models of systems of different types need enormous amount of information which often turns out to be given in an imprecise (fuzzy) manner. In consequence, new methods are worked out to describe such information and new tools designed to compare mathematical models and select the best one for a given system.
Thus, on one hand, there arise theories created to describe indefiniteness and non-univocality in the materialistic world that surrounds us (e.g. L.A.Zadeh's fuzzy sets theory [10], K.Atanassov's fuzzy intuitionistic sets theory [2]), while on the other hand, special tools are constructed to measure this indeterminacy in practice (e.g. the notions of entropy, energy or correlation).
The paper presents different concepts of intuitionistic fuzzy sets measurements by means of differently defined entropy. Our discussion is going to be limited to the discrete case since in practice we generally deal with finite sets (although frequently with a great number of elements). The continuous case can be the subject of further studies.

Let $U = \{x_1, x_2, \ldots, x_N\}$ be a finite space of consideration.

DEFINITION 1.1.[10] By a *fuzzy set* in space U we shall understand set A defined by a so-called *membership function* $\mu_A : U \rightarrow <0,1>$ of the form

(1) $$A = \{(x, \mu_A(x) : x \in U\}$$

The family of fuzzy sets in *U* is denoted by FS(*U*).

The question of measuring indeterminacy (fuzziness) of such sets was put forth by A.de Luca and S.Termini [6] in 1970s. The quantity that they defined to be used to measure fuzziness of a fuzzy set was required to fulfil the following:

A1) fuzziness is zero if the set is not fuzzy (i.e. $\mu(x) \in \{0,1\}$)
A2) fuzziness is maximal if the set is the fuzziest (i.e. $\mu(x) \equiv ½$)
A3) fuzziness of a "fuzzier" set is greater (where the relation of greater fuzziness is defined as:

$$A \prec\prec B \Leftrightarrow (\mu_A(x) \leq \mu_B(x) \leq \frac{1}{2} \; and \; \mu_A(x) \geq \mu_B(x) \geq \frac{1}{2})$$

In [6] it is suggested that this quantity can be called non-probabilistic entropy of fuzzy set *A*, and is shown that the formula

(2) $$d(A) = H(\mu_A) + H(1 - \mu_A)$$

where

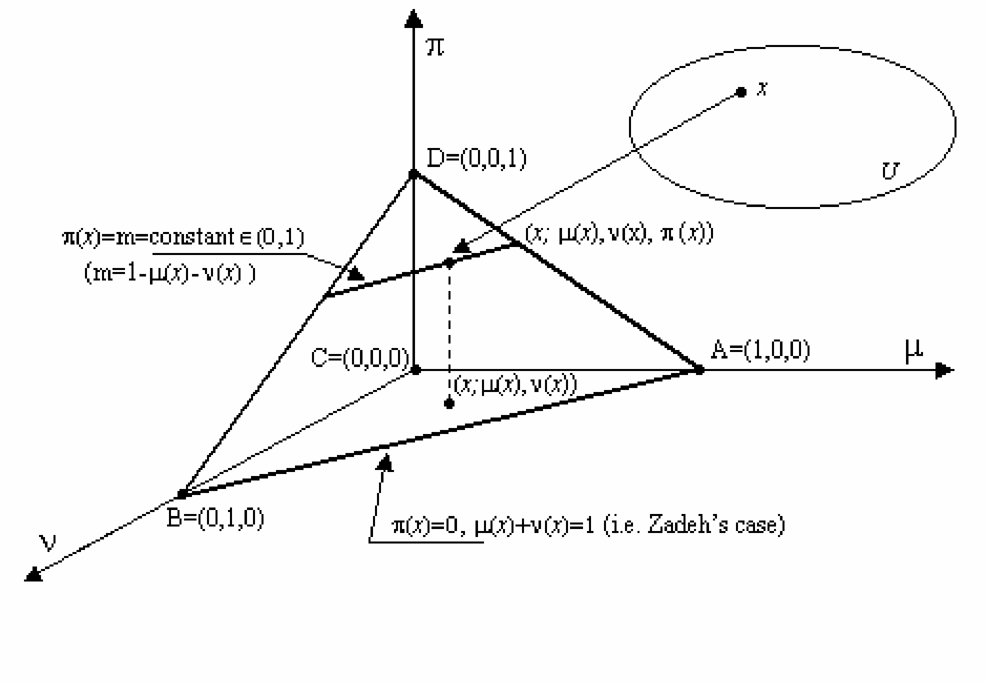
(3) $$H(x) = \begin{cases} -K\sum_{i=1}^{N} x_i \ln x_i & for \;\; x_i \in (0,1) \\ 0 & for \;\; x_i \in \{0,1\} \end{cases}$$

satisfies axioms A1), A2), A3), and $K > 0$ is a constant number.

The concept described above still serves as a model for constructing the theory of entropy for fuzziness and its variants, while axioms A1), A2), A3) are necessary conditions.
In 1983, K.Atanassov in [1] generalized the idea of a fuzzy set to the notion of an intuitionistic fuzzy set.

DEFINITION 1.2. An intuitionistic fuzzy set *A* in space *U* is the structure
(4) $$A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\}$$
where μ_A, ν_A: $U \rightarrow <0,1>$ are such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ and describe the degree

of membership of element x in A and the degree of non-membership of x in A, espectively. The difference $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is then called a *hesitancy margin of membership of x in A*.
The family of intuitionistic fuzzy sets in U is denoted by IFS(U).
Parameter $\pi_A(x) \in <0,1>$ is the element that emphasizes the aspect of intuitionistic fuzziness of such a set. The figure below illustrates the relation between the fuzzy set in the sense of K.Atanassov, the one in the sense of L.A.Zadeh, and the set in the usual sense of G.Cantor's set theory.



The points of triangle plane ABD correspond to the elements of the intuitionistic fuzzy set defined over space U under consideration.

2. INTUITIONISTIC FUZZY SET ENTROPY

In [7], the concept of fuzzy set entropy was first extended onto the family of intuitionistic fuzzy sets. Thus:

DEFINITION 2.1. Arithmetic entropy $D(A)$ of a fuzzy intuitionistic set $A \in$ IFS(U) is the number

(5) $$D(A) = \frac{d(\mu_A) + d(\nu_A)}{2} = \frac{d(\mu_A) + d(1 - \mu_A - \pi_A)}{2}$$

where d is the entropy calculated according to (2) for a fuzzy set in the sense of Zadeh.

It is shown in [7] that $D(A)$ satisfies axioms A1) - A3), which is the model requirement imposed on entropy in [6], and that the following relations are valid:

1) $D(A) = D(A')$

2) $D(A \cup B) = D(A) + D(B) - D(A \cap B)$.

Property 2) means that $D(A)$ is a valuation, i.e. it satisfies the necessary condition to be a entropy measure of a set.

In [7], one can also find another idea of the fuzzy intuitionistic set entropy which proves competitive to that expressed by Definition 2.1.

DEFINITION 2.2. Logical entropy $\widetilde{D}(A)$ of set $A \in$ IFS(U) is the number

(8) $$\widetilde{D}(A) = H(\mu_A) + H(\nu_A) = H(\mu_A) + H(1 - \mu_A - \pi_A)$$

where H is a function given by (3).

Entropy (8) can be written in the form

$$\widetilde{D}(A) = \begin{cases} -K \sum_{i=1}^{N} [\mu_A(x_i) \ln \mu_A(x_i) + \nu_A(x_i) \ln \nu_A(x_i)], & \mu_A, \nu_A \in (0,1) \\ 0, & \mu_A, \nu_A \in \{0,1\} \end{cases}$$

which, in some sense, is a generalization of the formula for Shannon's entropy for (non-fuzzy) random events.

Clearly, axioms A1) - A3) are satisfied , as well as the conditions for symmetry and valuation

(9) $$\widetilde{D}(A) = \widetilde{D}(A'),$$

(10) $$\widetilde{D}(A \cup B) = \widetilde{D}(A) + \widetilde{D}(B) - \widetilde{D}(A \cap B).$$

Both the concepts described above treat the entropy of an intuitonistic fuzzy set in a stiff way - as a real number. However, it seems reasonable to consider a softer case, like it is done by R.Yager with the probability of a fuzzy event [9], and define entropy more flexibly taking into account an intuitionistic element in the form of the hesitancy margin. A sample of such an approach is the concept of the fuzzy event probability in [8] where it is viewed as a number from the interval <0,1> determined by so called minimal and maximal probabilities. Hence

DEFINITION 2.3 Intuitionistic entropy of set $A \in$ IFS(U) is a quantity $\hat{D}(A)$ from the interval

(11) $$\hat{D}(A) \in < D_{\min}(A), D_{\max}(A) >$$

where

(12) $$D_{\min}(A) = H(\mu_A) + H(1-\mu_A) = -K\sum_{i=1}^{N} S(\mu_A(x_i))$$

and

$$S(x) = \begin{cases} x\ln x + (1-x)\ln(1-x), & x \in (0,1) \\ 0, & x \in \{0,1\} \end{cases}$$

is Shannon's function, while

$$D_{\max}(A) = D_{\min}(A) + H(\pi_A) + H(1-\pi_A).$$

3. NUMERICAL EXAMPLE

Let $U = \{x_1, x_2, x_3, x_4\}$ be a set of four exams to be taken by students in a coming examination period, and let $A = \{(x, \mu_A(x), \nu_A(x), \pi_A(x)\}$ be a fuzzy intuitionistic set of "difficult" exams in U:

A={(x_1; 0.8, 0.1, 0.1), (x_2; 0.7, 0.1, 0.2), (x_3; 0.5, 0.4, 0.1), (x_4; 0.4, 0.4, 0.3)}.

Then, arithmetic entropy $D(A)$ (in (5)), logical entropy $\widetilde{D}(A)$ (in (8)), and intuitionistic entropy (in (11)-(13)) are, respectively, equal to

$D(A) \approx$ K·2.237,

$\widetilde{D}(A) \approx$ K·2.336,

$\hat{D}(A) \in < D_{\min}(A), D_{\max}(A) >$ and $D_{\min}(A) \approx K \cdot 2.477$ and $D_{\max}(\mathrm{A}) \approx K \cdot 4{,}1284$.

$D_{\min}(A)$ may be interpreted as the lowest guaranteed entropy of set $A \in$ IFS(U), while $D_{\max}(A)$ is then the highest entropy represented by A.

In the calculations above, K is an arbitrary positive constant and may be used, for instance, as a normalization factor. The value of K and the choice of the definition of entropy depends on a particular real situation. What should be considered essential is that, if intuitionism is reduced to fuzziness in Zadeh's sense, each of the concepts above is brought down to one and the same quantity, defined by the non-probabilistic fuzzy set entropy (2), whose value is equal to that of $D_{\min}(A)$.

It is worth noting that the concepts above differ from the entropy of the fuzzy set $A_{\mathrm{FS}\,0.5} \in$ FS(U) resulting from the transformation of $A \in$ IFS(U) by means of the procedure of "the best approximation" of an intuitionistic set with a fuzzy one [3]. Consider $A \in$ IFS(U) for which $A_{\mathrm{FS}\,0.5} \in$ FS(U) is of the form

(14) $A_{FS0.5} = \{ (x, \mu_A(x) + 0.5\pi_A(x))\}$

which, in our example, is

$$A_{FS\,0.5}=\{(x_1, 0.85), (x_2, 0.8), (x_3, 0.55), (x_4, 0.5)\}$$

and finally $d(A_{FS\,0.5}) \approx K\cdot 2.3044$.
The result obtained differs from the one expected for the intuitionistic entropy of $A \in \mathrm{IFS}(U)$ as, for example, the middle of the interval $<D_{min}(A), D_{max}(A)>$.

4. FINAL REMARKS

Different definitions of fuzzy intuitionistic set entropy discussed in the paper clearly evidence the fact that the notion has not obtained its final shape yet and a model solution is still being searched for. While the approach presented in [7], i.e. the logical and arithmetical, are well grounded formally and satisfy A. de Luca and S.Termini's axiom system [6], the intuitionistic approach obviously exhibits greater flexibility and to a considerable extent involves the element of intuitionistic fuzziness. Being insufficiently elaborated, the intuitionistic approach might be considered not to be promising enough. Thus it seems most advisable to further analyze its properties and compare it with other concepts of the intuitionistic fuzzy set entropy.

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