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A property of intuitionistic fuzzy implications

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Abstract
In [2], K. Atanassov extended the most popular fuzzy implications to intuitionistic fuzzy implications and studied their properties. This paper aims to exam whether those implications satisfy the property $X \to Y = X \to (X \to Y)$, where X and Y are propositional forms and $\to$ refers to intuitionistic fuzzy implications.

Introduction

Based on the concept of Fuzzy Sets (FSs) and Fuzzy Logic (FL), the concept of Intuitionistic Fuzzy Sets (IFSs) and Intuitionistic Fuzzy Logic(IFL) was introduced by Atanassov in [1]. In FL the truth value of a proposition p is a real number comes from the unit interval [0, 1], which expresses the 'truth degree' of p. In IFL, a 'falsity degree' is added besides the 'truth degree'. Thus the value of p can be expressed by two real numbers come from the unit interval [0, 1] in an ordered couple. These real numbers are denoted by 'truth degree' and 'falsity degree' of p - $\mu(p)$ and $\nu(p)$ respectively. The truth value V is a mapping from S to $[0, 1]^2$, defined as $V(p) = <\mu(p), \nu(p)>$, where S is the set of propositions and p is a proposition in S. The mappings $\mu : [0, 1] \to [0, 1]$ and $\nu : [0, 1] \to [0, 1]$ satisfy $\mu(p) + \nu(p) \leq 1$. Accordingly, the concept of fuzzy implication in FL is extended to intuitionistic fuzzy implication in IFL. In [2], K. Atanassov extended the most popular fuzzy implications to intuitionistic fuzzy implications and studied their properties. And in our recent work, we study for a fuzzy implication $\to$ the property $X \to Y = X \to (X \to Y)$, $\forall (X, Y) \in [0, 1]^2$ in FL. In this paper we will exam for the intuitionistic fuzzy implications whether or not they also satisfy this property. Section 2 will give some preliminaries for the truth value of proposition p and the intuitionistic fuzzy implications. Section 3 will give the proof and the results.

Preliminaries

We will consider the truth values V (X) and V (Y) of two propositional forms X and Y. For these truth values hold:

$$V(X) = <\mu(X), \nu(X)>, V(Y) = <\mu(Y), \nu(Y)>,$$

where:

$$\mu(X), \nu(X), \mu(Y), \nu(Y) \in [0, 1] \Rightarrow V(X), V(Y) \in [0, 1]^2$$

For simplicity, let the truth values V (X) and V (Y), respectively <μ(X), ν(X)> and <μ(Y), ν(Y)> be denoted as <a, b> and <c, d>, where a, b, c, d $\in$ [0, 1]. Some properties are given below:

1. X is a 0 element iff V (X) = <0, 1>;
2. X is a 1 element iff V (X) = <1, 0>;
3. X $\leq^*$ Y iff a $\leq$ b and c $\geq$ d, where $\leq^*$ define an order for all the values of propositions in proposition sets;
4. max(X, Y) = max(<a, b>,<c, d>) =<max(a, c), min(b, d)>,
   min(X, Y) = min(<a, b>,<c, d>) =<min(a, c), max(b, d)>;
5. n(X) = n(<a, b>) =<b, a>, where n denotes the negation in IFL.

	Name	**Form of implication**
1	*Zadeh*	$< \max(b, \min(a,c)), \min(a,d) >$
2	*Gaines-Rescher*	$< 1 - sg(a-c), d.sg(a-c) >$
3	*Gödel*	$< 1-(1-c).sg(a-c), d.sg(a-c) >$
4	*Kleene-Dienes*	$< \max(b,c), \min(a,d) >$
5	*Lukasie-wicz*	$< \min(1,(b+c)), \max(0, a+d-1) >$
6	*Reichen-bach*	$< b+ac, ad >$
7	*Willmott*	$< \min(\max(b,c), \max(a,b), \max(c,d)), \max(\min(a,d), \min(a,b), \min(c,d)) >$
8	*Wu*	$< 1-(1-\min(b,c)).sg(a-c), \max(a,d).sg(a-c).sg(d-b) >$
9	*Klir and Yuan 1*	$< b+a^2c, ab+a^2d >$
10	*Klir and Yuan 2*	$< c.\overline{sg}(1-a)+sg(1-a).(\overline{sg}(1-c)+b.sg(1-c)),$ $d.\overline{sg}(1-a)+a.sg(1-a).sg(1-c) >$
11	*Atanassov 1*	$< 1-(1-c).sg(a-c), d.sg(a-c).sg(d-b) >$
12	*Atanasov 2*	$< 1-(1-\min(b,c)).sg(sg(a-c)+sg(d-b)) - \min(b,c).sg(a-c).sg(d-b),$ $1-(1-\max(a,d)).sg(\overline{sg}(a-c)+\overline{sg}(d-b)) - \min(a,d).\overline{sg}(a-c).\overline{sg}(d-b) >$
13	*Atanassov 3*	$< \max(b,c), 1-\max(b,c) >$
14	*Atanassov and Kolev*	$< b+c-bc, ad >$
15	*Atanassov and Trifonov*	$< 1-(1-c).sg(a-c) - d.\overline{sg}(a-c).sg(d-b), d.sg(d-b) >$

Table 1: List of intuitionistic fuzzy implications.

In FL, a fuzzy implication $\rightarrow$ is a mapping from [0, 1]^2 to [0, 1] satisfying boundary conditions:
0 $\rightarrow$ 0 = 0 $\rightarrow$ 1 = 1 $\rightarrow$ 1 = 1 and 1 $\rightarrow$ 0 = 0. This definition is extended to intuitionistic

fuzzy implication as:

$$<0, 1> \rightarrow <0, 1>=<0, 1> \rightarrow <1, 0>=<1, 0> \rightarrow <1, 0>=<1, 0>$$

and

$$<1, 0> \rightarrow <0, 0>=<0, 0>$$

and in this case the intuitionistic fuzzy implication $\rightarrow$ is a mapping from $[0, 1]^2$ x $[0, 1]^2$ to $[0,1]^2$.

In [2], K. Atanassov showed a table in which he extended the most popular fuzzy implications in FL to intuitionistic fuzzy implications in IFL. In next section, we will exam for these intuitionistic fuzzy implications if they satisfy the property $X \rightarrow Y = X \rightarrow (X \rightarrow Y)$ which we have analyzed for fuzzy implications in FL in our recent work. First we quote the Table 1 from [2].

In this table, sg and $\overline{sg}$ are two functions defined as:

$$sg(x) = \begin{cases} 1, x > 0 \\ 0, x \leq 0 \end{cases} \quad \text{and} \quad \overline{sg}(x) = \begin{cases} 0, x > 0 \\ 1, x \leq 0 \end{cases}$$

Main results

We will check if the listed in the table above implications hold this equality.

Theorem 1: For every two propositional forms X, Y:

a) the implications *1*, *2*, *3*, *4*, *11*, *13* and *15* hold

$$V(X \rightarrow Y) = V(X \rightarrow (X \rightarrow Y)) \quad (1)$$

b) the implications *5*, *7*, *10* and *14* hold

$$V(X \rightarrow Y) \leq V(X \rightarrow (X \rightarrow Y)) \quad (2)$$

c) the implications *8* and *12* hold

$$V(X \rightarrow Y) \geq V(X \rightarrow (X \rightarrow Y)) \quad (3)$$

d) the implications *6* and *9* hold

$$V(X \rightarrow Y) \neq V(X \rightarrow (X \rightarrow Y)) \quad (4)$$

Proof:

I. The form of **Zadeh** implication is:

$$< a,b > \rightarrow < c,d > = < \max(b, \min(a,c)), \min(a,d) >$$

So,

$$< a,b > \rightarrow (< a,b > \rightarrow < c,d >) =$$
$$< a,b > \rightarrow < \max(b, \min(a,c)), \min(a,d) > =$$
$$< \max(b, \min(a, \max(b, \min(a,c)))), \min(a, \min(a,d)) > =$$
$$< \max(b, \min(a, \max(b, \min(a,c)))), \min(a,d) >$$

If $\max(b, \min(a, \max(b, \min(a,c)))) = \max(b, \min(a,c))$ then (1) will be held

1) if $a \leq b \leq c \Rightarrow \max(b, \min(a,c)) = \max(b,a) = b$

$$\max(b, \min(a, \max(b, \min(a,c)))) = \max(b, \min(a, \max(b,a))) = \max(b, \min(a,b))$$
$$= \max(b,a) = b$$

2) if $a \leq c \leq b \Rightarrow \max(b, \min(a,c)) = \max(b,a) = b$

$$\max(b, \min(a, \max(b, \min(a,c)))) = \max(b, \min(a, \max(b,a))) = \max(b, \min(a,b))$$
$$= \max(b,a) = b$$

3) if $b \leq a \leq c \Rightarrow \max(b, \min(a,c)) = \max(b,a) = a$

$$\max(b,\min(a,\max(b,\min(a,c)))) = \max(b,\min(a,\max(b,a))) = \max(b,\min(a,a))$$
$$= \max(b,a) = a$$

4) if $b \le c \le a \Rightarrow \max(b,\min(a,c)) = \max(b,c) = c$

$$\max(b,\min(a,\max(b,\min(a,c)))) = \max(b,\min(a,\max(b,c))) = \max(b,\min(a,c))$$
$$= \max(b,c) = c$$

5) if $c \le a \le b \Rightarrow \max(b,\min(a,c)) = \max(b,c) = b$

$$\max(b,\min(a,\max(b,\min(a,c)))) = \max(b,\min(a,\max(b,c))) = \max(b,\min(a,b))$$
$$= \max(b,a) = b$$

6) if $c \le b \le a \Rightarrow \max(b,\min(a,c)) = \max(b,c) = b$

$$\max(b,\min(a,\max(b,\min(a,c)))) = \max(b,\min(a,\max(b,c))) = \max(b,\min(a,b))$$
$$= \max(b,b) = b$$

So (1) holds for Zadeh implication in IFL.

II. The form of **Gaines-Rescher** implication is:

$$< a,b > \to < c,d > = < 1 - sg(a-c), d.sg(a-c) >$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < 1 - sg(a-c), d.sg(a-c) > =$$
$$< 1 - sg(a-1+sg(a-c)), d.sg(a-c).sg(a-1+sg(a-c)) >$$

1) if $a \le c \Rightarrow < 1 - sg(a-c), d.sg(a-c) > = < 1,0 >$

$$< 1 - sg(a-1+sg(a-c)), d.sg(a-c).sg(a-1+sg(a-c)) > = < 1,0 >$$

2) if $a > c \Rightarrow < 1 - sg(a-c), d.sg(a-c) > = < 0,d >$

$$< 1 - sg(a-1+sg(a-c)), d.sg(a-c).sg(a-1+sg(a-c)) > = < 0,d >$$

So (1) holds for Gaines-Rescher implication in IFL.

III. The form of **Gödel** implication is:

$$< a,b > \to < c,d > = < 1-(1-c).sg(a-c), d.sg(a-c) >$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < 1-(1-c).sg(a-c), d.sg(a-c) > =$$
$$< 1-(1-1+(1-c).sg(a-c)).sg(a-1+(1-c).sg(a-c)),$$
$$d.sg(a-c).sg(a-1+(1-c).sg(a-c)) > =$$
$$< 1-(1-c).sg(a-1+(1-c).sg(a-c)), d.sg(a-c).sg(a-1+(1-c).sg(a-c)) >$$

1) $a \le c \Rightarrow < 1-(1-c).sg(a-c), d.sg(a-c) > = < 1,0 >$

$$< 1-(1-c).sg(a-1+(1-c).sg(a-c)), d.sg(a-c).sg(a-1+(1-c).sg(a-c)) > = < 1,0 >$$

2) $a > c \Rightarrow < 1-(1-c).sg(a-c), d.sg(a-c) > = < c,d >$

$$< 1-(1-c).sg(a-1+(1-c).sg(a-c)), d.sg(a-c).sg(a-1+(1-c).sg(a-c)) > = < c,d >$$

$$< a,b > \to < c,d > = < a,b > \to (< a,b > \to < c,d >)$$

IV. The form of **Kleene-Dienes** implication is:

$$< a,b > \to < c,d > = < \max(b,c), \min(a,d) >$$

On the other hand:

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < \max(b,c), \min(a,d) > =$$

$$< \max(b, \max(b,c)), \min(a, \min(a,d)) >=$$
$$< \max(b,c), \min(a,d) >$$

So (1) is held for Kleene-Dienes implication.

V. The form of **Lukasiewicz** implication is:

$$< a,b > \to < c,d > = < \min(1,(b+c)), \max(0, a+d-1) >$$

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < \min(1,(b+c)), \max(0,a+d-1) > =$$
$$< \min(1, b+\min(1,b+c)), \max(0, a+\max(0,a+d-1)-1) >$$

So, for (2) to be held for Lukasiewicz implication the non-equalities:

$$\min(1,(b+c)) \le \min(1, b+\min(1,b+c))$$

and

$$\max(0,a+d-1) \ge \max(0, a+\max(0,a+d-1)-1)$$

should be realized.

First we will consider the difference: $\min(1,(b+c)) - \min(1, b+\min(1,b+c))$

1) if $1 \le b+c \Rightarrow \min(1,(b+c)) - \min(1, b+\min(1,b+c)) = 1 - \min(1, b+1) = 0$

2) if $1 > b+c \Rightarrow \min(1,(b+c)) - \min(1, b+\min(1,b+c)) = b+c-\min(1, b+b+c)$

2.1) if $1 \le 2b+c \Rightarrow b+c-\min(1,2b+c) = b+c-1 < 0$

2.2) if $1 > 2b+c \Rightarrow b+c-\min(1,2b+c) = b+c-2b-c = -b \le 0$

So, $\min(1,(b+c)) \le \min(1, b+\min(1,b+c))$.

Second we will consider the difference:

$\max(0,a+d-1) - \max(0, a+\max(0,a+d-1)-1)$

1) if $0 \ge a+d-1 \Rightarrow \max(0,a+d-1) - \max(0, a+\max(0,a+d-1)-1) = 0 - \max(0,a-1) = 0$

2) if $0 < a+d-1 \Rightarrow \max(0,a+d-1) - \max(0, a+\max(0,a+d-1)-1) =$
$a+d-1-\max(0,2a+d-2)$

2.1) if $0 \ge 2a+d-2 \Rightarrow a+d-1-\max(0,2a+d-2) = a+d-1 > 0$

2.2) if $0 < 2a+d-2 \Rightarrow a+d-1-\max(0,2a+d-2) = a+d-1-2a-d+2 = 1-a \ge 0$

$\max(0,a+d-1) \ge \max(0, a+\max(0,a+d-1)-1)$

So (2) is held for Lukasiewicz implication.

VI. The form of **Reichenbach** implication is:

$$< a,b > \to < c,d > = < b+ac, ad >$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < b+ac, ad > =$$
$$< b+a(b+ac), a^2 d > = < b+ab+a^2 c, a^2 d >$$

1) if $a=1, b=0 \Rightarrow b+ac = b+a(b+ac) = c \Rightarrow < b+ac, ad > = < b+a(b+ac), a^2 d >$

So,

$$< a,b > \to < c,d > = < a,b > \to (< a,b > \to < c,d >)$$

2) otherwise we will consider the difference:

$b+ac-b-a(b+ac) = b+ac-b-ab-a^2 c = ac-ab-a^2 c = a(c-ac-b)$

2.1) if $c=1 \Rightarrow 1-(a+b) \ge 0$

2.2) if $c=0 \Rightarrow -b \le 0$

So,

$$< a,b > \to < c,d > \ne < a,b > \to (< a,b > \to < c,d >)$$

So (4) is held for Reichenbach implication in IFL.

VII. The form of **Willmott** implication is:
$< a,b > \to < c,d > = < \min(\max(b,c),\max(a,b),\max(c,d)),\max(\min(a,d),\min(a,b),\min(c,d)) >$
So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < \min(\max(b,c),\max(a,b),\max(c,d)),\max(\min(a,d),\min(a,b),\min(c,d)) >$$
$$< \min(\max(b,\min(\max(b,c),\max(a,b),\max(c,d))),\max(a,b),$$
$$\max(\min(\max(b,c),\max(a,b),\max(c,d)),\max(\min(a,d),\min(a,b),\min(c,d))),$$
$$\max(\min(a,\max(\min(a,d),\min(a,b),\min(c,d)),\min(a,b),$$
$$\min(\min(\max(b,c),\max(a,b),\max(c,d)),\max(\min(a,d),\min(a,b),\min(c,d))) >$$

We replace:

$$x = \min(\max(b,c),\max(a,b),\max(c,d)),$$
$$y = \max(\min(a,d),\min(a,b),\min(c,d))$$

So we will consider the relationship between:
$< x, y >$ and $< \min(\max(b,x),\max(a,b),\max(x,y)),\max(\min(a,y),\min(a,b),\min(x,y)) >$.

The proof is analogical to the proof of **I.**, that is why we will not consider it here. It is proved that for Willmott implication (2) is held.

VIII. The form of **Wu** implication is:

$$< a,b > \to < c,d > = < 1-(1-\min(b,c)).sg(a-c),\max(a,d).sg(a-c).sg(d-b) >$$
$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < 1-(1-\min(b,c)).sg(a-c),\max(a,d).sg(a-c).sg(d-b) >$$
$$< 1-(1-\min(b,1-(1-\min(b,c)).sg(a-c)).sg(a-1-(1-\min(b,c)).sg(a-c))),$$
$$\max(a,\max(a,d).sg(a-c).sg(d-b)).sg(a-1+(1-\min(b,c)).sg(a-c)).$$
$$sg(\max(a,d).sg(a-c).sg(d-b)-b) >$$

1) if $a \le c \Rightarrow sg(a-c) = 0$

$$< 1-(1-\min(b,c)).sg(a-c),0 > = < 1,0 >$$
$$< 1-(1-\min(b,1-(1-\min(b,c)).sg(a-c)).sg(a-1-(1-\min(b,c)).sg(a-c))),$$
$$\max(a,\max(a,d).sg(a-c).sg(d-b)).sg(a-1+(1-\min(b,c)).sg(a-c)).$$
$$sg(\max(a,d).sg(a-c).sg(d-b)-b) > = < 0,0 >$$

So,

$$< a,b > \to < c,d > > < a,b > \to (< a,b > \to < c,d >).$$

2) $a > c \Rightarrow sg(a-c) = 1$

$$< 1-(1-\min(b,c)).sg(a-c),\max(a,d).sg(a-c).sg(d-b) > =$$
$$< \min(b,c),\max(a,d).sg(d-b) >$$
$$< 1-(1-\min(b,1-(1-\min(b,c)).sg(a-c)).sg(a-1-(1-\min(b,c)).sg(a-c))),$$
$$\max(a,\max(a,d).sg(a-c).sg(d-b)).sg(a-1+(1-\min(b,c)).sg(a-c)).$$
$$sg(\max(a,d).sg(a-c).sg(d-b)-b) > = < 0,\max(a,\max(a,d).sg(d-b)).$$
$$sg(a-1+(1-\min(b,c))).sg(\max(a,d).sg(d-b)-b) >$$

2.1) if $d \le b \Rightarrow sg(d-b) = 0$

$$< \min(b,c),\max(a,d).sg(d-b) > = < \min(b,c),0 >$$

$$< 0, \max(a, \max(a,d).sg(d-b)).sg(a-1+(1-\min(b,c))).$$
$$sg(\max(a,d).sg(d-b)-b) >=< 0,0 >$$

So,

$$< a,b >\to< c,d >\geq< a,b >\to (< a,b >\to< c,d >) .$$

2.2) if $d > b \Rightarrow sg(d-b) = 1$

$$< \min(b,c), \max(a,d).sg(d-b) >=< \min(b,c), \max(a,d) >$$
$$< 0, \max(a, \max(a,d).sg(d-b)).sg(a-1+(1-\min(b,c))).$$
$$sg(\max(a,d).sg(d-b)-b) >=$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >$$

2.2.1) if $b \leq c \Rightarrow a > b, b < d$

$$< \min(b,c), \max(a,d) >=< b, \max(a,d) >$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >=$$
$$< 0, \max(a,d).sg(\max(a,d)-b) >$$

2.2.1.1) if $a \leq d$

$$< \min(b,c), \max(a,d) >=< b, \max(a,d) >=< b,d >$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >=$$
$$< 0, \max(a,d).sg(\max(a,d)-b) >=< 0,d >$$

So,

$$< a,b >\to< c,d >\geq< a,b >\to (< a,b >\to< c,d >) .$$

2.2.1.2) if $a > d$

$$< \min(b,c), \max(a,d) >=< b, \max(a,d) >=< b,a >$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >=$$
$$< 0, \max(a,d).sg(\max(a,d)-b) >=< 0,a >$$

So,

$$< a,b >\to< c,d >\geq< a,b >\to (< a,b >\to< c,d >) .$$

2.2.2) if $b > c \Rightarrow c < d, a > c$

$$< \min(b,c), \max(a,d) >=< c, \max(a,d) >$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >=$$
$$< 0, \max(a,d).sg(\max(a,d)-b) >$$

2.2.2.1) if $a \leq d$

$$< \min(b,c), \max(a,d) >=< c, \max(a,d) >=< c,d >$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >=$$
$$< 0, \max(a,d).sg(\max(a,d)-b) >=< 0,d >$$

So,

$$< a,b >\to< c,d >\geq< a,b >\to (< a,b >\to< c,d >) .$$

2.2.2.2) if $a > d, d > b \Rightarrow a > b$

$$< \min(b,c), \max(a,d) >=< c, \max(a,d) >=< c,a >$$
$$< 0, \max(a,d).sg(a-\min(b,c)).sg(\max(a,d)-b) >=$$
$$< 0, \max(a,d).sg(\max(a,d)-b) >=< 0,a >$$

So,

$$< a,b >\to< c,d >\geq< a,b >\to (< a,b >\to< c,d >) .$$

So Wu implication holds (3).

IX. The form of **Klir and Yuan 1** implication is:

$$< a,b > \to < c,d > = < b + a^2c, ab + a^2d >$$

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < b + a^2c, ab + a^2d > =$$
$$< b + a^2(b + a^2c), ab + a^2(ab + a^2d) > = < b + a^2b + a^4c, ab + a^3b + a^4d >$$

We will consider the difference:

$$b + a^2c - (b + a^2b + a^4c) \Rightarrow b + a^2c - b - a^2b - a^4c = a^2c - a^2b - a^4c = a^2(c - a^2c - b)$$

1) if $c = 1 \Rightarrow a^2(c - a^2c - b) = a^2(1 - a^2 - b) \geq 0$

2) if $c = 0 \Rightarrow a^2(c - a^2c - b) = a^2(-b) \leq 0$

So for Klir and Yuan 1 implication is held

(4) $< a,b > \to < c,d > \neq < a,b > \to (< a,b > \to < c,d >)$.

X. The form of **Klir and Yuan 2** implication is:

$$< a,b > \to < c,d > = < c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)),$$
$$d.\overline{sg}(1-a) + a.sg(1-a).sg(1-c) >$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)), d.\overline{sg}(1-a) + a.sg(1-a).sg(1-c) > =$$
$$< (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c))).\overline{sg}(1-a) +$$
$$sg(1-a).(\overline{sg}(1 - (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)))) +$$
$$b.sg(1 - (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c))))),$$
$$(d.\overline{sg}(1-a) + a.sg(1-a).sg(1-c)).\overline{sg}(1-a) +$$
$$a.sg(1-a).sg(1 - (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)))) >$$

So we will consider the relationship between

$< c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)),\ d.\overline{sg}(1-a) + a.sg(1-a).sg(1-c) >$ and

$$< (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c))).\overline{sg}(1-a) +$$
$$sg(1-a).(\overline{sg}(1 - (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)))) +$$
$$b.sg(1 - (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c))))),$$
$$(d.\overline{sg}(1-a) + a.sg(1-a).sg(1-c)).\overline{sg}(1-a) +$$
$$a.sg(1-a).sg(1 - (c.\overline{sg}(1-a) + sg(1-a).(\overline{sg}(1-c) + b.sg(1-c)))) >$$

The proof is analogical to the proof of **VIII.,** that is why we will not consider it here. It is proved that for Klir and Yuan 2 implication (2) is held:

$$< a,b > \to < c,d > \leq < a,b > \to (< a,b > \to < c,d >)$$

XI. The form of **First Atanassov** implication is:

$$< a,b > \to < c,d > = < 1 - (1-c).sg(a-c), d.sg(a-c).sg(d-b) >$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < 1 - (1-c).sg(a-c), d.sg(a-c).sg(d-b) > =$$

$$<1-(1-(1-(1-c).sg(a-c))).sg(a-(1-(1-c).sg(a-c))),$$
$$d.sg(a-c).sg(d-b).sg(a-(1-(1-c).sg(a-c))).sg(d.sg(a-c).sg(d-b)-b)>=$$
$$<1-(1-c).sg(a-c).sg(a-1+(1-c).sg(a-c)),$$
$$d.sg(a-c).sg(d-b).sg(a-1+(1-c).sg(a-c)).sg(d.sg(a-c).sg(d-b)-b)>$$

1) if $a \le c \Rightarrow sg(a-c)=0$

$$<1-(1-c).sg(a-c), d.sg(a-c).sg(d-b)>=<1,0>$$
$$<1-(1-c).sg(a-c).sg(a-1+(1-c).sg(a-c)),$$
$$d.sg(a-c).sg(d-b).sg(a-1+(1-c).sg(a-c)).sg(d.sg(a-c).sg(d-b)-b)>=<1,0>$$

So,

$$<a,b>\to<c,d>=<a,b>\to(<a,b>\to<c,d>)$$

2) if $a > c \Rightarrow sg(a-c)=1$

$$<1-(1-c).sg(a-c), d.sg(a-c).sg(d-b)>=<c, d.sg(d-b)>$$
$$<1-(1-c).sg(a-c).sg(a-1+(1-c).sg(a-c)),$$
$$d.sg(a-c).sg(d-b).sg(a-1+(1-c).sg(a-c)).sg(d.sg(a-c).sg(d-b)-b)>=$$
$$<c, d.sg(d-b).sg(d.sg(d-b)-b)>$$

We will consider the difference:

$$d.sg(d-b)-d.sg(d-b).sg(d.sg(d-b)-b)=d.sg(d-b)(1-sg(d.sg(d-b)-b))=0$$

Therefore, for First Atanassov implication (1) is held.

XII. The form of **Second Atanassov** implication is:

$$<a,b>\to<c,d>=<1-(1-\min(b,c)).sg(sg(a-c)+sg(d-b))-\min(b,c).sg(a-c).sg(d-b),$$
$$1-(1-\max(a,d)).sg(\overline{sg}(a-c)+\overline{sg}(d-b))-\min(a,d).\overline{sg}(a-c).\overline{sg}(d-b)>$$

We replace:

$$x=1-(1-\min(b,c)).sg(sg(a-c)+sg(d-b))-\min(b,c).sg(a-c).sg(d-b)$$
$$y=1-(1-\max(a,d)).sg(\overline{sg}(a-c)+\overline{sg}(d-b))-\max(a,d).\overline{sg}(a-c).\overline{sg}(d-b)$$

So:

$$<a,b>\to(<a,b>\to<c,d>)=$$
$$<1-(1-\min(b,x)).sg(sg(a-x)+sg(y-b))-\min(b,x).sg(a-x).sg(y-b),$$
$$1-(1-\max(a,y)).sg(\overline{sg}(a-x)+\overline{sg}(y-b))-\max(a,y).\overline{sg}(a-x).\overline{sg}(y-b)>$$

The proof is analogical to **I.** So we will not consider it here. It is proved that for Second Atanassov implication (3) is held:

$$<a,b>\to<c,d>\ge<a,b>\to(<a,b>\to<c,d>)$$

XIII. The form of **Third Atanassov** implication is:

$$<a,b>\to<c,d>=<\max(b,c),1-\max(b,c)>$$

So,

$$<a,b>\to(<a,b>\to<c,d>)=$$
$$<a,b>\to<\max(b,c),1-\max(b,c)>=$$
$$<\max(b,\max(b,c)),1-\max(b,\max(b,c))>=<\max(b,c),1-\max(b,c)>$$

So for Third Atanassov implication (1) is held.

XIV. The form of **Atanassov and Kolev** implication is:

$$<a,b>\to<c,d>=<b+c-bc,ad>$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < b+c-bc, ad > =$$
$$< b+b+c-bc-b(b+c-bc), a^2 d > = < 2b+c-2bc-b^2+b^2 c, a^2 d >$$

We will consider the two differences:

$b+c-bc-(2b+c-2bc-b^2+b^2c)$ and $ad-a^2d$.

1) $b+c-bc-(2b+c-2bc-b^2+b^2c) = b(c-1)(1-b) \le 0$

2) $ad-a^2d = ad(1-a) \ge 0$

So (2) is held for Atanassov and Kolev implication.

XV. The form of **Atanassov and Trifonov** implication is:

$$< a,b > \to < c,d > = < 1-(1-c).sg(a-c)-d.\overline{sg}(a-c).sg(d-b), d.sg(d-b) >$$

So,

$$< a,b > \to (< a,b > \to < c,d >) =$$
$$< a,b > \to < 1-(1-c).sg(a-c)-d.\overline{sg}(a-c).sg(d-b), d.sg(d-b) >$$

We replace:

x = $1-(1-c).sg(a-c)-d.\overline{sg}(a-c).sg(d-b)$,

y = $d.sg(d-b)$.

and the proof is analogical to **VIII.**

So Atanassov and Trifonov implication holds (1).

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