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A note on the convergence of intuitionistic fuzzy sets

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Abstract In this paper is shown that fuzzy sets may be represented as a limit of an appropriate infinite sequence of intuitionistic fuzzy sets, depending on an operator $F_{\alpha,\beta}$ introduced in [1]. The necessary and sufficient conditions for that are given. The situation looks like the one in the case of irrational and rational real numbers if we agree to make an analogy between proper intuitionistic fuzzy sets and retional numbers from one side, and between fuzzy sets and irrational numbers from the other side.

Also, for the first item, a necessary and sufficient conditions for the convergence of an infinite product and of infinite series are given, which are based on the notion of intuitionistic fuzzy set and on the operator $F_{\alpha,\beta}$ only.

1 Preliminary notes

Let E be a universe, and $A \subset E$, $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ are arbitrary functions such that for every $x \in E$

$$\pi_A(x) \equiv 1 - \mu_A(x) - \nu_A(x) \geq 0 \tag{1}$$

holds.

Following [1] we introduce the set $PIFS$ of all proper intuitionistic fuzzy sets, noting that each element $A \in PIFS$ is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}, \tag{2}$$

and. moreover, there exists at least one $x \in E$, such that the inequality

$$\pi_A(x) > 0 \tag{3}$$

holsd, where π_A is introduced by (1). When for every $x \in E$:

$$\pi_A(x) = 0, \tag{4}$$

the set A from (2) is an ordinary fuzzy set. The class of all fuzzy sets is denoted further by FS.

From the definition of $PIFS$ and FS it is clear that

$$PIFS \cap FS = \emptyset.$$

Let $\alpha, \beta \in (0,1)$ be fixed real numbers and $\alpha + \beta \leq 1$. In [1] is introduced operator $F_{\alpha,\beta} : PIFS \cup FS \to PIFS \cup FS$ given by

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}. \tag{5}$$

The following four assertions are true. We omit their proofs since they are a matter of direct verifications.

Lemma 1: Let $\alpha + \beta < 1$. Then $A \in PIFS$ iff $F_{\alpha,\beta}(A) \in PIFS$.

Lemma 2: If $A \in FS$, then $F_{\alpha,\beta}(A) \in FS$.

Lemma 3: Let $\alpha + \beta < 1$. Then $A \in FS$ iff $F_{\alpha,\beta}(A) \in FS$.

Lemma 4: If $\alpha + \beta = 1$, then $F_{\alpha,\beta}(A) \in FS$.

Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two infinite sequences of real numbers and let

$$0 < \alpha_n + \beta_n < 1, \tag{6}$$

for $n = 1, 2, 3, ...$ In [2] is established that

$$F_{\alpha_n,\beta_n}(F_{\alpha_{n-1},\beta_{n-1}}(...(F_{\alpha_1,\beta_1}(A))...)) = F_{e_n,f_n}(A), \tag{7}$$

where if $n \geq 2$, e_n and f_n are given by:

$$e_n = \alpha_1 + \sum_{k=2}^{n} \alpha_k \prod_{j=1}^{k-1}(1 - \alpha_j - \beta_j), \tag{8}$$

$$f_n = \beta_1 + \sum_{k=2}^{n} \beta_k \prod_{j=1}^{k-1}(1 - \alpha_j - \beta_j), \tag{9}$$

and for $n = 1$ we have $e_1 = \alpha_1,\ f_1 = \beta_1$.

Following [2] again, we consider the very important identity for our further considerations:

$$1 - e_n - f_n = \prod_{k=1}^{n}(1 - \alpha_k - \beta_k), \tag{10}$$

for $n = 1, 2, ...$

In [3] is proved that each one of sequences $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ converges.

Let

$$\lim_{n\to\infty} e_n = e,$$

$$\lim_{n\to\infty} f_n = f.$$

Then (6) and (10) yield $0 < e, f$ and

$$e + f \leq 1. \tag{11}$$

2 Main results

Using (11) we introduce the boundary operator $F_{e,f} : PIFS \cup FS \to PIFS \cup FS$ putting

$$F_{e,f} \equiv \lim_{n\to\infty} (F_{\alpha_n,\beta_n}(F_{\alpha_{n-1},\beta_{n-1}}(...(F_{\alpha_1,\beta_1}(A))...))) = \lim_{n\to\infty} F_{e_n,f_n}(A)$$

$$= F_{\lim\limits_{n\to\infty} e_n, \lim\limits_{n\to\infty} f_n}(A). \tag{12}$$

The main aim of our investigation is to find criterions showing when $F_{e,f}(A) \in PIFS$, or when $F_{e,f}(A) \in FS$. For that purpose we require

Lemma 5 ([4]): The infinite product

$$\prod_{k=1}^{\infty}(1 - \alpha_k - \beta_k) \tag{13}$$

converges iff the series

$$\sum_{k=1}^{\infty}(\alpha_k + \beta_k) \tag{14}$$

converges.

Remark 1: Let for $n = 1, 2, 3, ...$

$$b_n \equiv \prod_{k=1}^{n}(1 - \alpha_k - \beta_k). \tag{15}$$

Then $b_n \in (0, 1)$ and $\{b_n\}_{n=1}^{\infty}$ is a decreasing sequence. Hence $\{b_n\}_{n=1}^{\infty}$ converges.

If

$$b = \lim_{n\to\infty} b_n,$$

then

$$0 \le b < 1.$$

The infinity product (13) is introduced by

$$\prod_{k=1}^{\infty}(1 - \alpha_k - \beta_k) = \lim_{n\to\infty} b_n = b. \tag{16}$$

When $b \neq 0$ we say that the infinite product (13) coverges to b. When $b = 0$ we say that the product (13) diverges.

From (10), when n tends to $+\infty$, we obtain

$$1 - e - f = \prod_{k=1}^{\infty}(1 - \alpha_k - \beta_k). \tag{17}$$

The equality (17) implies the following result.

Lemma 6: The equality

$$e + f = 1 \tag{18}$$

holds iff the infinite product (16) diverges.

Let $A \in FS$. Then $F_{e,f}(A) \in FS$, since Lemma 2 holds. That is way we will further assume that $A \in PIFS$.

Below we give the main result of the paper.

Theorem 1: Let $A \in PIFS$. Then $F_{e,f}(A) \in FS$ iff the infinite product (16) diverges (i.e., iff

$$b = \lim_{n\to\infty} b_n = 0,$$

see (15)).

Proof Let (16) diverge. Then (18) holds, because of Lemma 6. Therefore, (18) and Lemma 4 imply $F_{e,f}(A) \in FS$.

Let $F_{e,f}(A) \in FS$ and let us assume that (16) converges. Then Lemma 6 yields

$$e + f < 1.$$

The last inequality and Lemma 3 imply $A \in FS$. But that contradicts to $A \in PIFS$, since $FS \cap PIFS = \emptyset$. Hence (16) diverges. The Theorem is proved.

Ciorollary 1: Let $A \in PIFS$. If the series $\sum_{n=1}^{\infty} b_n$ converges, then $F_{e,f}(A) \in FS$, where b_n is given by (15).

Proof Obviously,

$$b = \lim_{n\to\infty} b_n = 0,$$

since $\sum_{n=1}^{\infty} b_n$ converges. Therefore, (16) diveregs and it remains only to apply Theorem 1. The assertion is proved.

Lemma 5 and Theorem 1 yield

Theorem 2: Let $A \in PIFS$. Then $F_{e,f}(A) \in FS$ iff the series $\sum_{k=1}^{\infty}(\alpha_k + \beta_k)$ diverges to $+\infty$.

Theorem 1 and Theorem 2 may be represented in the forms:

Theorem 3: Let $A \in PIFS$. Then $F_{e,f}(A) \in PIFS$ iff the infinite product (16) converges.

Theorem 4: Let $A \in PIFS$. Then $F_{e,f}(A) \in PIFS$ iff the series $\sum_{k=1}^{\infty}(\alpha_k + \beta_k)$ converges.

Corollary 2: Let $A \in PIFS$. If $F_{e,f}(A) \in PIFS$ then

$$\lim_{n\to\infty}(\alpha_n + \beta_n) = 0.$$

Remark 2 Let there exist $k \in \{1, 2, 3, ...\}$ such that

$$\alpha_k + \beta_k = 1.$$

Then, according to Lemma 4, we obtain that

$$F_{\alpha_k,\beta_k}(F_{\alpha_{k-1},\beta_{k-1}}(...(F_{\alpha_1,\beta_1}(A))...))) \in FS.$$

Hence for $n \geq k$ we have

$$F_{e_n,f_n}(A) \in FS,$$

since Lemma 2, holds. Hence

$$F_{e,f}(A) \in FS.$$

The above remark makes the requirement (6) meaningful.

3 Criterions for convergence and divergence of infinite products and series

Let $\{u_k\}_{k=1}^{\infty}$ be an infinite sequence of real numbers belonging to the interval $(0,1)$ and let us consider the infinite product

$$\prod_{n=1}^{n}(1-u_n) \tag{19}$$

and the infinite series

$$\sum_{n=1}^{n} u_n. \tag{20}$$

As a corollary of Theorems 1 - 4 we obtain the following criterions.

Criterion 1: The product (19) diverges iff for arbitrary chosen $A \in PIFS$ and for arbitrary chosen decomposition of the kind

$$u_n = \alpha_n + \beta_n,\ (n = 1,2,3,...) \tag{21}$$

with α_n, β_n positive real numbers, it is fulfilled that $F_{e,f}(A) \in FS$, where

$$e = \lim_{n\to\infty} e_n,$$

$$f = \lim_{n\to\infty} f_n,$$

and e_n, f_n are given by (8) and (9).

Criterion 2: The product (19) converges iff for arbitrary chosen $A \in PIFS$ and for arbitrary chosen decomposition (21) with α_n, β_n - positive real numbers, it is fulfilled that $F_{e,f}(A) \in PIFS$.

Criterion 3: The series (20) diverges to $+\infty$ iff for arbitrary chosen $A \in PIFS$ and for arbitrary chosen decomposition (21) with α_n, β_n - positive real numbers, it is fulfilled that $F_{e,f}(A) \in FS$.

Criterion 4: The series (20) converges iff for arbitrary chosen $A \in PIFS$ and for arbitrary chosen decomposition (21) with α_n, β_n - positive real numbers, it is fulfilled that $F_{e,f}(A) \in PIFS$.

4 Some results concerning $F_{\alpha,\beta}$

Let $\lambda, \theta \in (0,1)$ be fixed real numbers and $\lambda + \theta \leq 1$. Putting

$$\alpha_n = \lambda^n,\ \beta_n = \theta^n,\ n = 1,2,3,...$$

We note that in [3] are proved the estimations:

$$e_n \leq e = \lim_{n\to\infty} e_n \leq \varphi(\lambda,\theta),$$

$$f_n \leq f = \lim_{n\to\infty} f_n \leq \psi(\lambda,\theta),$$

$n = 1,2,3,...$, where

$$\varphi(\lambda,\theta) = \frac{\lambda}{1-\lambda}(1-\lambda(\lambda+\theta)),$$

$$\psi(\lambda,\theta) = \frac{\theta}{1-\theta}(1-\theta(\lambda+\theta)).$$

The following assertion shows that the operator $F_{\varphi(\lambda,\theta),\psi(\lambda,\theta)}$ is well defined, at least when

$$\lambda \leq \frac{1}{2},\ \theta \leq \frac{1}{2},$$

then the inequalities:

$$0 < \varphi(\lambda,\theta) < 1, \tag{22}$$

$$0 < \psi(\lambda,\theta) < 1, \tag{23}$$

$$\varphi(\lambda,\theta) + \psi(\lambda,\theta) \leq 1 \tag{24}$$

hold.
Proof we have

$$\varphi(\lambda,\theta) \leq 2\lambda(1-\lambda(\lambda+\theta)) = 2\lambda - 2\lambda^2.(\lambda+\theta) < 2\lambda \leq 1$$

and (22) is proved.

Analogically, one may prove (23).

To prove (24), we will note that

$$\varphi(\lambda+\theta) + \psi(\lambda+\theta) = -2 + (1-\lambda-\theta).(\frac{1}{1+\lambda} + \frac{1}{1+\theta}) + (\lambda+\theta)(\lambda+\theta+2).$$

Hence (24) is equivalent to the inequality

$$(1-x).(\frac{1}{1-\lambda} + \frac{1}{1-\theta}) + x(x+2) - 3 \leq 0, \tag{25}$$

where $x = \lambda + \theta$.

We rewrite (25) in the form

$$(1-x).(\frac{1}{1-\lambda} + \frac{1}{1-\theta}) \leq (1-x)(x+3). \tag{26}$$

When $x = 1$, (26) is true. Since, $\lambda + \theta \leq 1$, we have $x \leq 1$.

Let $x < 1$. Then $1 - x > 0$ and (26) yields

$$\frac{1}{1-\lambda} + \frac{1}{1-\theta} \leq x + 3. \tag{27}$$

Now, from (27) and from the representations

$$\frac{1}{1-\lambda} = 1 + \lambda + \lambda^2 + ...,$$

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + ...,$$

$$\lambda^2 + \lambda^3 + ... = \frac{\lambda^2}{1-\lambda},$$

$$\theta^2 + \theta^3 + ... = \frac{\theta^2}{1-\theta},$$

we obtain the inequality

$$\frac{\lambda^2}{1-\lambda} + \frac{\theta^2}{1-\theta} \leq 2, \tag{28}$$

which we will prove.

From

$$1-\lambda \geq \frac{1}{2},\ 1-\theta \geq \frac{1}{2}$$

we obtain

$$\frac{\lambda^2}{1-\lambda} + \frac{\theta^2}{1-\theta} \leq 2(\lambda^2+\theta^2) \leq 2((\frac{1}{2})^2 + (\frac{1}{2})^2) = 1$$

and (28) and the Lemma are proved.

Therefore, we may write

$$F_{e,f}(A) \sim F_{\varphi(\lambda,\theta),\psi(\lambda,\theta)}(A)$$

for and arbitrary $A \in PIFS \cup FS$.

References

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